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On certain bounds for the divisor function

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Abstract: We offer various bounds for the divisor function $d(n)$, in terms of n, or other arithmetical functions.
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1 Introduction

For a positive integer $n > 1$, let $d(n)$ denote the number of all positive divisors of n. Our aim in what follows will be to offer bounds for $d(n)$ in terms of functions of n, or other arithmetic functions, such as the Euler totient function $\varphi(n)$ or $\sigma(n)$, the sum of divisors of n. We will use also the function $\omega(n)$ denoting number of distinct prime factors of n.

If $n = p_1^{a_1} \cdots p_r^{a_r}$ (p_i distinct primes, a_i positive integers) is the prime factorization of n, then it is well-known that

$$d(n) = (a_1 + 1) \cdots (a_r + 1), \tag{1}$$

where $a_i \geq 1$ $(i = 1, 2, \ldots, r)$; so (1) immediately gives

$$2 \leq 2^{\omega(n)} \leq d(n). \tag{2}$$

A classical upper bound for $d(n)$ is

$$d(n) < 2\sqrt{n} \quad (n > 1), \tag{3}$$

while it is also known that (see [6])

$$d(n) < \sqrt{n} \ \text{ for } \ n \geq 1262. \tag{4}$$

If n is composite, then it is well-known that [8]

$$\varphi(n) \leq n - \sqrt{n}, \tag{5}$$

where $\varphi(n)$ denotes the Euler totient function. Thus, from (4) and (5) we get

$$\varphi(n) \leq n - \sqrt{n} < n - d(n) \ \text{ for } \ n \geq 1262,$$

composite, so we get

$$d(n) < \sqrt{n} < n - \varphi(n) \ \text{ for such values of } \ n. \tag{6}$$

In [7] it has been shown that

$$d(n) < 4\sqrt[3]{n}, \ n > 1 \tag{7}$$

and this clearly improves (3) of (4) for sufficiently large values of n.

In what follows, we will offer other bounds which are more precise for certain values of n.

2 Main results

Theorem 1. *One has*

$$2 \leq \frac{\omega(n)\sigma(n) + \varphi(n)}{n} \leq d(n) \leq \frac{\sigma(n) - (\sqrt{n} - 1)^2}{\sqrt{n}} < \frac{\sigma(n)}{\sqrt{n}} \tag{8}$$

for $n \geq 2$, where the second inequality holds true for $n \neq 6$.

Proof. As $\omega(n) \geq 1$, the first inequality of (8) follows by $\sigma(n) \cdot \omega(n) + \varphi(n) \geq \sigma(n) + \varphi(n) \geq 2n$. This last inequality follows, e.g., by $\sigma(n) \geq \psi(n)$, where

$$\psi(n) = n \cdot \prod_{p|n} \left(1 + \frac{1}{p}\right)$$

denotes the Dedekind arithmetical function. By the algebraic inequality

$$\prod_{i=1}^{r}(1 + x_i) + \prod_{i=1}^{r}(1 - x_i) \geq 2$$

for $x_i \in (0, 1)$ the result immediately follows (by letting $x_i = \dfrac{1}{p_i}$). For the second inequality of (8), use the following result of the author [4]:

$$\omega(n)\sigma(n) + \varphi(n) \leq nd(n) \tag{9}$$

for any $n \geq 2$, $n \neq 6$; with equality only for $n = 10$ or $n = \mathrm{prime}$.

Finally, the third inequality of (8) is a consequence of another result by the author [3]:

$$\sigma(n) \geq n + 1 + \sqrt{n} \cdot [d(n) - 2]. \tag{10}$$

This follows by $n + 1 - 2\sqrt{n} = (\sqrt{n} - 1)^2$, and simple computations. The last inequality of (8) is trivial, but we note that it improves the classical result $d(n) < \frac{\sigma(n)}{\sqrt{n}}$, i.e.,

$$\sigma(n) > d(n) \cdot \sqrt{n} \quad (n > 1), \tag{11}$$

due to Sivaramakrishnan and Venkataraman. □

Remark 1. *Particularly, we get the new inequality*

$$\frac{\omega(n)\sigma(n) + \varphi(n)}{\sqrt{n}} \leq \frac{\sigma(n) - (\sqrt{n} - 1)^2}{\sqrt{n}} \quad (n \neq 6),$$

or written equivalently :

$$\sigma(n) \cdot [\sqrt{n} - \omega(n)] \geq \varphi(n) + \sqrt{n} \cdot (\sqrt{n} - 1)^2 \tag{12}$$

for $n \neq 6$; $n \geq 2$.

Theorem 2. *One has*

$$\sigma(n) \geq \sqrt{d(n)[d(n) - 1] \cdot n + \sigma_2(n)} \geq [d(n) - 1]\sqrt{n} + \sqrt{\frac{\sigma_2(n)}{d(n)}} \tag{13}$$

for any $n \geq 2$; *where* $\sigma_2(n)$ *denotes the sum of squares of divisors of* n.

Proof. Let a_i $(i = \overline{1, n})$ be positive real numbers, and let A_n, respectively, G_n denote their arithmetic, respectively, geometric means of (a_i). Let $Q_n = \sqrt{\frac{1}{n} \sum_{i=1}^{n} a_i^2}$. In paper [5] is proved among others the inequalities:

$$n^2 \cdot A_n^2 \geq n \cdot [(n - 1) \cdot G_n^2 + Q_n^2] \geq [(n - 1) \cdot G_n + Q_n]^2. \tag{14}$$

Let $n = d(k)$, where $k \geq 2$; and $a_i = d_i$, where $1 = d_1 < d_2 < \cdots < d_n$ are the distinct divisors of k. Then, it is immediate that $A_n = \frac{\sigma(k)}{d(k)}$, $G_n = (d_1 \cdots d_n)^{1/n} = \sqrt{k}$ which is well-known, see e.g. [6]), $(Q_n)^2 = \frac{\sigma_2(k)}{d(k)}$. After some elementary computations, from (14), we get inequality (13), where the inequality (13) is considered for k (in place of n.) □

Remark 2. *From the first inequality of* (13) *we get*

$$d(n) \cdot [d(n) - 1] \leq \frac{\sigma^2(n) - \sigma_2(n)}{n}. \tag{15}$$

We now consider extensions of inequalities of type (7). *We will use the method of [7]. One has the following similar inequalities.*

Theorem 3. *One has*

$$d(n) < k_1 \cdot \sqrt[4]{n}, \text{ where } k_1 \approx 9.1118\ldots, \tag{16}$$
$$d(n) < k_2 \cdot \sqrt[5]{n}, \text{ where } k_2 \approx 33.4725\ldots, \tag{17}$$
$$d(n) < k_3 \cdot \sqrt[6]{n}, \text{ where } k_3 \approx 188.7496\ldots, \tag{18}$$
$$d(n) < k_4 \cdot \sqrt[7]{n}, \text{ where } k_4 \approx 2539.882\ldots, \tag{19}$$

for any $n \geq 2$.

The following auxiliary result will be used:

Lemma 1. *One has*

$$(a+1) \cdot p^{-a/k} \leq M_k(p), \tag{20}$$

where

$$M_k(p) = \frac{k}{\log p} \cdot p^{(\log p - k)/k \log p} \tag{21}$$

for all $a \geq 1$, $p \geq 2$; $k \geq 2$.

Proof of Lemma 1. Let us consider the real variable function $f(a) = (a+1) \cdot p^{-a/k}$. This function has a derivative

$$f'(a) = p^{-a/k} \cdot \left[1 - \frac{(a+1)\log p}{k}\right] = 0$$

if and only if $a = a_0 = \dfrac{k}{\log p} - 1$. If is immediate that a_0 is maximum point to f, the maximum being attained at a_0. We get that $f(a) \leq f(a_0) = M_k(p)$, so inequality (21) follows.

As $a + 1 = d(p^a)$, we get from (21) that

$$d(p^a) \leq p^{a/k} \cdot M_k(p). \tag{22}$$

This completes the proof of the Lemma. □

Proof of Theorem 3. Now, remark that $p^a > (a+1)^4$ follows by $p^a \geq 17^a$ if $17^a > (a+1)^4$. Since $17^1 > 2^4 = 16$, by an easy induction argument the inequality holds true for any $a \geq 1$. As

$$n = \prod_{p \leq 13} p^a \cdot \prod_{p \geq 17} p^a,$$

we get that $d(n) \leq M_4(2)M_4(3)M_4(5)M_4(7)M_4(11)M_4(13) \cdot \prod_{p \geq 2} p^{a/4} = k_4 \cdot n^{1/4}$. Now, by a computer we get that $M_4(2) = 2.524\ldots$, $M_4(3) = 1,762\ldots$, $M_4(5) = 1.367\ldots$, $M_4(7) = 1.230\ldots$, $M_4(11) = 1.117\ldots$, $M_4(13) = 1.089\ldots$ and we get that $k_4 \approx 9.11\ldots$, so inequality (16) follows.

For the proof of (17) we will use the inequality $37^{a/5} > a + 1$, and proceed in the same manner, as in the case $k = 4$.

A computer computation gives us

$$\prod_{2 \leq p \leq 31} M_5(p) \approx 33.4725\ldots,$$

and the inequality (17) follows.

The proofs of (18) and (19) can be derived in the same manner, and we omit the details. □

Remark 3. *If n is odd, the above inequalities can be further sharpened. For example, when $k=3$, (when $M_3(2) = 1$) we get*

$$d(n) < (1.842\ldots)\sqrt[3]{n} < 2\sqrt[3]{n} \ \ (n \text{ odd}), \tag{23}$$

and for $k = 4$ and 5,

$$d(n) < 4 \cdot \sqrt[4]{n} \ \ (n \text{ odd}), \tag{24}$$

$$d(n) < 11 \cdot \sqrt[5]{n} \ \ (n \text{ odd}). \tag{25}$$

Remark 4. *The method of Theorem 3 can be used also for $k = 2$, and as $M_2(2) \cdot M_2(3) = 1.7413\ldots$ we get*

$$d(n) < (1.75)\sqrt{n} \ \ (n \geq 2) \tag{26}$$

and

$$d(n) < (1.16)\sqrt{n} \ \ (n \text{ odd}) \tag{27}$$

which further improve relation (3)*.*

Lemma 2. *Let $k, l > 0$ and $a_i \geq 1$ $(i = 1, 2, \ldots, n)$. Then*

$$n \cdot \sum_{i=1}^{n} a_i^{k+1} - \sum_{i=1}^{n} a_i^k \cdot \sum_{i=1}^{n} a_i^l \geq \sum_{i=1}^{n} a_i^k \cdot \sum_{i=1}^{n} \frac{1}{a_i^l} - n \cdot \sum_{i=1}^{n} a_i^{k-l} \tag{28}$$

Proof. We will use the classical Chebysheff inequality

$$\sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i - n \cdot \sum_{i=1}^{n} x_i y_i \leq 0, \tag{29}$$

for the sequences (x_i) and (y_i) having the property $(x_i - x_j) \cdot (y_i - y_j) \geq 0$ for all $i, j \in \{1, 2, \ldots, n\}$. Let $x_i = a_i^k$, $y_i = a_i^l + \frac{1}{a_i^l}$. Remark that $y_i - y_j = a_i^l + \frac{1}{a_i^l} - a_j^l - \frac{1}{a_j^l} = (a_i^l \cdot a_j^l) \cdot (a_i^l - a_j^l) \cdot (a_i^l \cdot a_j^l - 1)$ and as $(a_i^k - a_j^k) \cdot (a_i^l - a_j^l) \geq 0$ and $a_i^l \cdot a_j^l - 1 \geq 0$ by $a_i \geq 1$, $a_j \geq 1$, inequality (29) can be applied. After simple computations, we get relation (28). □

Remark 5. *For $k = l$ we get from* (28)

$$n \cdot \sum_{i=1}^{n} a_i^{2k} - \left(\sum_{i=1}^{n} a_i^k\right)^2 \geq \left(\sum_{i=1}^{n} a_i^k\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{a_i^k}\right) - n^2 \geq 0, \tag{30}$$

where the last inequality is well-known.

Lemma 3. *Let $(a_i), (b_i), (i=1, 2, \ldots, n)$ be two sequences with the property $(a_i - a_j)(b_i - b_j) \leq 0$ $(i, j \in \{1, 2, \ldots, n\})$. Then one has the inequalities*

$$\sum_{i=1}^{n} a_i b_i \leq \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}, \tag{31}$$

and

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{n \cdot \sum_{i=1}^{n} a_i^2 b_i^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2}. \tag{32}$$

Proof. First remark that, the converse of inequality (29) is true now, so the first inequality of (31); and also the second inequality of (32) (applied to $x_i = a_i^2$, $y_i = b_i^2$) follows. The classical inequality $\left(\sum_{i=1}^{n} x_i\right)^2 \le n \cdot \sum_{i=1}^{n} y_i^2$ applied first to $x_i = a_i$, then $x_i = b_i$ gives the second inequality of (31), while applying it to $x_i = a_i b_i$ we get the first inequality of (32). □

Remark 6. *Both inequalities offer a refinement of the classical Cauchy inequality*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Theorem 4. *For any $m \ge 2$ and $k, l > 0$ one has*

$$d(m) \cdot [\sigma_{k+l}(m) + \sigma_{k-l}(m)] \ge \sigma_k(m) \cdot \sigma_l(m) \cdot \left(1 + \frac{1}{m^l}\right), \tag{33}$$

where $\sigma_a(m)$ denotes the sum of a-th powers of divisors of m.

Proof. Let $1 = d_1 < d_2 < \cdots < d_n = m$ be the distinct divisors of m, where $n = d(m)$. Applying Lemma 2 for $a_i = d_i$, after simple computations we get relation (33) by remarking that $\sigma_{-l}(n) = \sigma_l(n)/n^l$. □

Remark 7. *For $k = l$ we get from* (33)

$$d(m) \cdot [\sigma_{2k}(m) + d(m)] \ge (\sigma_k(m))^2 \cdot \left(1 + \frac{1}{m^k}\right). \tag{34}$$

Theorem 5. *For $m \ge 2$ one has*

$$m^l \cdot \sigma_{k-l}(m) \le \frac{1}{d(m)} \cdot \sigma_k(m) \cdot \sigma_l(m) \le \sqrt{\sigma_{2k}(m) \cdot \sigma_{2l}(m)}, \tag{35}$$

and

$$m^l \cdot \sigma_{k-l}(m) \le m^l \sqrt{d(m) \cdot \sigma_{2k-2l}(m)} \le \sqrt{\sigma_{2k}(m) \cdot \sigma_{2l}(m)}. \tag{36}$$

Proof. Apply relation (31) to $a_i = d_i^k$ and $b_i = (n/d_i)^l$ where $1 \le d_1 < \cdots < d_n = m$ are the distinct divisors of m. Remark that $1 \le d_1^k < \cdots < d_n^k$, while $\left(\frac{m}{d_1}\right)^l \ge \left(\frac{m}{d_2}\right)^l \ge \cdots \ge \left(\frac{m}{d_n}\right)^l$, so Lemma 2 can be applied. After simple computations, inequality (34) follows. The proof of (35) follows in the same manner from relation (32). □

Remark 8. *For $k = l$ we get*

$$m^k \cdot (m) \le \frac{(\sigma_k(m))^2}{d(m)} \le \sigma_{2k}(m), \tag{37}$$

$$m^k \cdot d(m) \le m^k \cdot d(m) \le \sigma_{2k}(m), \tag{38}$$

where the first inequality of (38) *is trivial.*

Remark 9. *For $l = k-1$, we get*

$$m^{k-1} \cdot \sigma(m) \leq \frac{1}{d(m)} \cdot \sigma_k(m) \cdot \sigma_{k-1}(m) \leq \sqrt{\sigma_{2k}(m)\sigma_{2k-2}(m)}, \tag{39}$$

$$m^{k-1} \cdot \sigma(m) \leq m^{k-1} \cdot \sqrt{d(m) \cdot \sigma_{2k-2}(m)} \leq \sqrt{\sigma_{2k}(m)\sigma_{2k-2}(m)}. \tag{40}$$

The first inequalities of (39) *and* (40) *provide:*

$$\frac{\sigma^2(m)}{\sigma_2(m)} \leq d(m) \leq \frac{\sigma_k(m) \cdot \sigma_{k-1}(m)}{m^{k-1} \cdot \sigma(m)}. \tag{41}$$

Finally, we prove:

Theorem 6. *For $n \geq 0$, k arbitrary one has*

$$\frac{\sigma_{kd(m)+1}(m)}{\sigma_k(m)} \geq m^{kd(m)/2}; \tag{42}$$

$$\sigma_k(m) \cdot \sigma_{k(d(m)-1)}(m) \leq [d(m)-1] \cdot \sigma_{kd(m)}(m) + d(m) \cdot m^{kd(m)/2}; \tag{43}$$

$$\sigma_{k \cdot d(m)}(m) + d(m) \cdot [d(m)-1] m^{kd(m)/2} \geq m^{kd(m)/2} \cdot \frac{(\sigma_k(m))^2}{m^k}. \tag{44}$$

Proof. For the proof of (42) we will use the Faiziev's inequality (see [2]):

$$a_1^{n+1} + \cdots + a_n^{n+1} \geq (a_1 \cdots a_n) \cdot (a_1 + \cdots + a_n). \tag{45}$$

Put $a_i = d_i^k$ (where $1 \leq d_1 < \cdots < d_n = m$ the distinct divisors of m. As it is well-known that $d_1 \cdots d_n = m^{d(m)/2}$, the left side of (45) is $\sigma_{kd(m)+1}(m)$, so relation (42) follows.

For the proof of (43) we will use J. Surányi's inequality (see [1])

$$(a_1 + \cdots + a_n) \cdot (a_1^{n-1} + \cdots + a_n^{n-1}) \leq (n-1) \cdot (a_1^n + \cdots + a_n^n) + na_1 \cdots a_n, \tag{46}$$

while for the proof of (44) we will apply V. Cârtoaje's inequality [1]

$$a_1^m + \cdots + a_n^n + n(n-1)a_1 \cdots a_n \geq a_1 \cdots a_n(a_1 + \cdots + a_n) \cdot \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right). \tag{47}$$

The proofs are similar to that of (42), and we omit the details. □

Remark 10. *For $k = 1$ we get from* (44)*:*

$$d(m) \cdot [d(m)-1] \geq \frac{(\sigma(m))^2}{m} - \sigma_{d(m)}(m). \tag{48}$$

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