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Melham's sums for some Lucas polynomial sequences

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Abstract: A Lucas polynomial sequence is a pair of generalized polynomial sequences that satisfy the Lucas recurrence relation. Special cases include Fibonacci polynomials, Lucas polynomials, and Balancing polynomials. We define the (a, b)-type Lucas polynomial sequences and prove that their Melham's sums have some interesting divisibility properties. Results in this paper generalize the original Melham's conjectures.

Keywords: Lucas polynomial sequence, Fibonacci sequence, Lucas sequence, Melham's conjectures.
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1 Introduction

Let F_n and L_n be the n-th Fibonacci and n-th Lucas numbers, which are defined by the recurrence relations

$$F_{n+1} = F_n + F_{n-1} \quad \text{and} \quad L_{n+1} = L_n + L_{n-1},$$

respectively, for $n \geq 1$ with initial values $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$.

Melham [8] made two conjectures on sums involving odd powers of certain Fibonacci numbers and Lucas numbers with even index.

Conjecture 1. *Let $m \geq 1$ be a positive integer. Then the sum*

$$L_1 L_3 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$$

can be expressed as $(F_{2n+1} - 1)^2 M_{2m-1}(F_{2n+1})$, where $M_{2m-1}(x)$ is a polynomial of degree $2m-1$ with integer coefficients.

Conjecture 2. *Let $m \geq 0$ be a positive integer. Then the sum*

$$L_1 L_3 \cdots L_{2m+1} \sum_{k=1}^{n} L_{2k}^{2m+1}$$

can be expressed as $(L_{2n+1} - 1) N_{2m}(L_{2n+1})$, where $N_{2m}(x)$ is a polynomial of degree $2m$ with integer coefficients.

Ozeki [9] was the first one to give an expression for the sum $\sum_{k=1}^{n} F_{2k}^{2m+1}$ as a polynomial in power of F_{2n+1}. Subsequently, Prodinger [11] did the same thing independently and obtained more summation formulae of similar type. Indeed, they gave the following so-called Ozeki–Prodinger identity:

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \sum_{\ell=0}^{m} \frac{(-1)^{m+\ell}}{5^{m-\ell}} F_{2n+1}^{2\ell+1} \sum_{j=\ell}^{m} \frac{1}{L_{2j+1}} \binom{2m+1}{m-j} \frac{2j+1}{2\ell+1} \binom{j+\ell}{j-\ell}$$
$$- \frac{1}{5^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \frac{F_{2j+1}}{L_{2j+1}}.$$

In 2004, Wiemann and Cooper [16] proved that the summation

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \frac{F_{2j+1}}{L_{2j+1}}$$

is a multiple of 5^m. Hence, the second term of Ozeki-Prodinger identity is an integer. Sun et al. [12] gave proof for Melham's two conjectures. In fact, in a spirit of Wang and Zhang's previous work [13], they proved the polynomial version of Melham's conjectures as follows.

Theorem 1.1 ([12])**.** *Let $F_n(x), L_n(x)$ be the n-th Fibonacci polynomial and the n-th Lucas polynomial, respectively. For any positive integers m and n, the Melham's sum*

$$L_1(x) L_3(x) \cdots L_{2m+1}(x) \sum_{k=1}^{n} F_{2k}^{2m+1}(x)$$

can be expressed as $(F_{2n+1}(x) - 1)^2 P_{2m-1}(x, F_{2n+1}(x))$, where $P_{2m-1}(x, y)$ is a polynomial in two variables x and y with integer coefficients and of degree $2m-1$ in y.

Theorem 1.2 ([12, 13]). *For any positive integers m and n, the Melham's sum*

$$L_1(x)L_3(x)\cdots L_{2m+1}(x)\sum_{k=1}^{n} L_{2k}^{2m+1}(x)$$

can be expressed as $(L_{2n+1}(x)-1)Q_{2m}(x, L_{2n+1}(x))$, where $Q_{2m}(x,y)$ is a polynomial in two variables x and y with integer coefficients and of degree $2m$ in y.

In 2019, Chen and Wang [3] used a different approach to conclude that

$$L_1(x)L_3(x)\cdots L_{2n+1}(x)\sum_{k=1}^{n} F_{2k}^{2m+1}(x) \equiv 0 \bmod (F_{2n+1}(x)-1)^2.$$

The above congruence with $x=1$ also gives a partial answer of the Conjecture 1.

For any two non-zero polynomials $p(x)$ and $q(x)$ with integer coefficients, we introduce the Lucas polynomial sequence $W_n(x)$ [4, 5] as the sequence of polynomials satisfying the Lucas recurrence relation

$$W_n(x) = p(x)W_{n-1}(x) + q(x)W_{n-2}(x) \text{ for } n \geq 2,$$

with the initials $W_0(x)=0$, and $W_1(x)=1$. Its companion sequence $w_n(x)$ was defined in [4,5] by satisfying the same recurrence relation with slightly different initials

$$w_0(x) = 2, \text{ and } w_1(x) = p(x).$$

A (a,b)-type Lucas polynomial sequence is a pair of generalized polynomials $V_n^{(a,b)}(x) := V_n(x)$, and $v^{(a,b)}(x) := v_n(x)$ which is defined as $p(x)=ax$ and $q(x)=b$ for integers a,b with $a^2+b^2 \neq 0$. That is, they satisfy recurrence relations

$$V_n(x) = (ax)V_{n-1}(x) + bV_{n-2}(x), \quad v_n(x) = (ax)v_{n-1}(x) + bv_{n-2}(x) \text{ for } n \geq 2, \tag{1}$$

with initials $V_0(x)=0, V_1(x)=1, v_0(x)=2$, and $v_1(x)=ax$, respectively. For simplicity, we denote the $(a,1)$-type Lucas polynomial sequence by $W_n(x)$ and $w_n(x)$, and the $(a,-1)$-type Lucas polynomial sequence by $\overline{W}_n(x)$ and $\overline{w}_n(x)$. It is worth noting that many important polynomial sequences satisfy the recurrence relation (1), such as Fibonacci polynomial, Lucas polynomial, Pell polynomial, and so on. See Table 1 on the next page.

Our main results are listed as follows.

Theorem 1.3. *For any positive integers m and n, the Melham's sum for the (a,b)-type Lucas polynomial sequence*

$$v_1(x)v_3(x)\cdots v_{2m+1}(x)\sum_{k=1}^{n} V_{2k}^{2m+1}(x)$$

is divisible by $(V_{2n+1}(x)-1)$ only if $|b|=1$.

Theorem 1.4. *For any positive integers m and n, the Melham's sum*

$$w_1(x)w_3(x)\cdots w_{2m+1}(x)\sum_{k=1}^{n} W_{2k}^{2m+1}(x)$$

can be expressed as $(ax)^m(W_{2n+1}(x)-1)^2\widetilde{H}_{2m-1}(x, W_{2n+1}(x))$, where $\widetilde{H}_{2m-1}(x,y)$ is a polynomial in two variables x and y with integer coefficients and of degree $2m-1$ in y.

Table 1. Special cases of Lucas polynomials

Polynomials	(a,b)	**Initial values**
Fibonacci $F_n(x)$	$(a,b)=(1,1)$	$F_0(x)=0, F_1(x)=1$
Lucas $L_n(x)$	$(a,b)=(1,1)$	$L_0(x)=2, L_1(x)=x$
Pell $P_n(x)$	$(a,b)=(2,1)$	$P_0(x)=0, P_1(x)=1$
Pell–Lucas $Q_n(x)$	$(a,b)=(2,1)$	$Q_0(x)=2, Q_1(x)=2x$
Chebyshev of the first kind $T_n(x)$	$(a,b)=(2,-1)$	$T_0(x)=1, T_1(x)=x$
Chebyshev of the second kind $U_n(x)$	$(a,b)=(2,-1)$	$U_0(x)=1, U_1(x)=2x$
Balancing $B_n(x)$ [10]	$(a,b)=(6,-1)$	$B_0(x)=0, B_1(x)=1$
Lucas-balancing $C_n(x)$ [10]	$(a,b)=(6,-1)$	$C_0(x)=1, C_1(x)=3x$
Dickson $D_n(x,\alpha)$ [2]	$(a,b)=(1,-\alpha)$	$D_0(x,\alpha)=2$, $D_1(x,\alpha)=x$
Fermat $\mathcal{F}_n(x)$ [14]	$(a,b)=(3,-2)$	$\mathcal{F}_0(x)=0, \mathcal{F}_1(x)=1$
Fermat–Lucas $f_n(x)$ [15]	$(a,b)=(3,-2)$	$f_0(x)=2, f_1(x)=3x$

Theorem 1.5. *For any positive integers m and n, the Melham's sum*

$$(a^2x^2-4)^{m+1}\overline{W}_1(x)\overline{W}_3(x)\cdots\overline{W}_{2m+1}(x)\sum_{k=1}^{n}\overline{W}_{2k}^{2m+1}(x)$$

can be expressed as $(\overline{w}_{2n+1}(x)-ax)^2\widetilde{M}_{2m-1}(x,\overline{w}_{2n+1}(x))$, where $\widetilde{M}_{2m-1}(x,y)$ is a polynomial in two variables x and y with integer coefficients and of degree $2m-1$ in y.

An outline of this paper is as follows. In Section 2, we derive basic properties of (a,b)-type Lucas polynomials and derive the expansions of sum involving odd powers of (a,b)-type Lucas polynomials with even index. This paper also gives some analog results of Wiemann and Cooper in [16]. We prove Theorem 1.3 in Section 3, and then discuss the case $b=1$ and prove Theorem 1.4 in Section 4. In light of expressions $W_{mn}(x)$ (respectively, $w_{mn}(x)$) as a polynomial in $W_n(x)$ (respectively, polynomial in $w_n(x)$) for odd m, we derive the Ozeki–Prodinger-like identities for some Lucas polynomial sequences. The discussion of the case $b=-1$ and the proof of Theorem 1.5 will be given in Section 5. Some remarks are made in the concluding section.

2 Preliminaries

Let a and b be integers. According to the recurrence relation (1), we define the negative index of (a,b)-type Lucas polynomial sequences as below:

$$V_{-n}(x)=\frac{(-1)^{n+1}V_n(x)}{b^n},\quad\text{and}\quad v_{-n}(x)=\frac{(-1)^n v_n(x)}{b^n},\quad (b\neq 0) \tag{2}$$

for any positive integer n.

Let $\alpha(x)=(ax+\sqrt{a^2x^2+4b})/2$ and $\beta(x)=(ax-\sqrt{a^2x^2+4b})/2$ be the roots of the characteristic polynomial $\lambda^2-ax\lambda-b=0$ such that $\alpha(x)+\beta(x)=ax$ and $\alpha(x)\beta(x)=-b$. It is easy to obtain $\Delta(x) := \alpha(x) - \beta(x) = \sqrt{a^2x^2+4b}$. Also it is easy to derive, for $n \geq 1$, the Binet formula:

$$V_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \quad v_n(x) = \alpha^n(x) + \beta^n(x).$$

Note that

$$\alpha^n(x) = \frac{v_n(x) + \Delta(x)V_n(x)}{2}, \quad \beta^n(x) = \frac{v_n(x) - \Delta(x)V_n(x)}{2}.$$

Proposition 2.1. *For integers m and n, we have*

(a) $V_m(x)v_n(x) = V_{m+n}(x) + (-b)^n V_{m-n}(x)$.

(b) $\Delta^2(x)V_m(x)V_n(x) = v_{m+n}(x) - (-b)^n v_{m-n}(x)$.

(c) $v_m(x)v_n(x) = v_{m+n}(x) + (-b)^n v_{m-n}(x)$.

(d) $\frac{d}{dx}v_n(x) = anV_n(x)$.

(e) *For $a \neq 0$, $V_{2n+1}\left(i\sqrt{x^2+\frac{4b}{a^2}}\right) = \frac{(-1)^n v_{2n+1}(x)}{ax}$, where $i^2 = -1$.*

Proof. Part **(a)**, **(b)**, and **(c)** follows easily by the Binet formula and (2). For part **(d)**, note that

$$\frac{d}{dx}\alpha(x) = \frac{a}{\sqrt{a^2x^2+4b}}\alpha(x), \quad \frac{d}{dx}\beta(x) = -\frac{a}{\sqrt{a^2x^2+4b}}\beta(x).$$

So we have

$$\frac{d}{dx}v_n(x) = \frac{an}{\sqrt{a^2x^2+4b}}\left[\alpha^n(x) - \beta^n(x)\right] = anV_n(x).$$

For part **(e)**, let $y = i\sqrt{x^2+\frac{4b}{a^2}}$. If $a > 0$, we note that

$$\alpha(y) = i\alpha(x), \text{ and } \beta(y) = -i\beta(x).$$

Thus, we obtain $v_{2n+1}(y) = (-1)^n i\Delta(x)V_{2n+1}(x)$, or equivalently,

$$V_{2n+1}\left(i\sqrt{y^2+\frac{4b}{a^2}}\right) = \frac{(-1)^n v_{2n+1}(y)}{ay}$$

The proof is similar when $a < 0$, so we omit here. □

It is also easy to obtain the following relation between V and v-polynomials,

$$v_m(x) = V_{m+1}(x) + bV_{m-1}(x),$$

for any integer m. For instance, one can prove this by induction on m.

Proposition 2.2. *There are explicit formulae for $V_n(x)$ and $v_n(x)$:*

$$V_n(x) = \sum_{\ell=0}^{\lfloor\frac{(n-1)}{2}\rfloor} \binom{n-\ell-1}{\ell}(ax)^{n-2\ell-1}b^\ell, \tag{3}$$

and

$$v_n(x) = \sum_{\ell=0}^{\lfloor\frac{n}{2}\rfloor} \frac{n}{n-\ell}\binom{n-\ell}{\ell}(ax)^{n-2\ell}b^\ell, \; n \neq 0. \tag{4}$$

Proof. By part **(d)** of Proposition 2.1, it is sufficient to show the explicit formula for $v_n(x)$. We prove by induction on n. For $n = 1$, the formula holds obviously. Suppose that it is true for $n < k$ with $k \geq 2$. If k is even, say $k = 2t$, by (1) we have

$$\begin{aligned}
v_{2t}(x) &= (ax)v_{2t-1}(x) + bv_{2t-2}(x) \\
&= \sum_{\ell=0}^{t-1} \frac{2t-1}{2t-1-\ell}\binom{2t-1-\ell}{\ell}(ax)^{2t-2\ell}b^{\ell} + \sum_{\ell=0}^{t-1} \frac{2t-2}{2t-2-\ell}\binom{2t-2-\ell}{\ell}(ax)^{2t-2-2\ell}b^{\ell+1} \\
&= (ax)^{2t} + \sum_{\ell=1}^{t-1} \frac{2t}{2t-\ell}\binom{2t-\ell}{\ell}(ax)^{2t-2\ell}b^{\ell} + 2b^t = \sum_{\ell=0}^{t} \frac{2t}{2t-\ell}\binom{2t-\ell}{\ell}(ax)^{2t-2\ell}b^{\ell}.
\end{aligned}$$

Here we have used the combinatorial identity:

$$\frac{2t-1}{2t-1-\ell}\binom{2t-1-\ell}{\ell} + \frac{2t-2}{2t-1-\ell}\binom{2t-1-\ell}{\ell-1} = \frac{2t}{2t-\ell}\binom{2t-\ell}{\ell}.$$

This proves the explicit formula for $v_k(x)$ when k is even. Similarly, the formula holds when $k = 2t-1$ is odd. So our proof is done by induction. □

For integers $m \geq 0$ and $n \geq 1$, by the Binet formula for $V_n(x)$, we have

$$\begin{aligned}
V_n^{2m+1}(x) &= \left[\frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}\right]^{2m+1} \\
&= \frac{1}{\Delta^{2m+1}(x)} \sum_{j=0}^{2m+1} (-1)^{j+1}\binom{2m+1}{j}\alpha^{jn}(x)\beta^{(2m+1-j)n}(x) \\
&= \frac{1}{\Delta^{2m}(x)} \sum_{j=0}^{m} (-1)^{(n+1)j} b^{nj}\binom{2m+1}{j}\frac{\alpha^{(2m+1-2j)n}(x) - \beta^{(2m+1-2j)n}(x)}{\alpha(x) - \beta(x)} \\
&= \frac{1}{(a^2x^2+4b)^m} \sum_{j=0}^{m} (-1)^{(n+1)j} b^{nj}\binom{2m+1}{j} V_{(2m+1-2j)n}(x).
\end{aligned}$$

This implies

$$\sum_{k=1}^{n} V_{2k}^{2m+1}(x) = \frac{1}{(a^2x^2+4b)^m} \sum_{j=0}^{m} (-1)^{m-j}\binom{2m+1}{m-j} \sum_{k=1}^{n} b^{2k(m-j)} V_{(2j+1)2k}(x). \tag{5}$$

Similarly, we have

$$\sum_{k=1}^{n} v_{2k}^{2m+1}(x) = \sum_{j=0}^{m} \binom{2m+1}{m-j} \sum_{k=1}^{n} b^{2k(m-j)} v_{(2j+1)2k}(x). \tag{6}$$

The next two propositions deal with a divisibility of $V-v$ type convolution which is motivated by a result in [16].

Proposition 2.3 ([16])**.** *For any non-negative integer m, we let $f_m(y)$ be a polynomial in y defined by*

$$f_m(y) = (y+1)(y^3+1)\cdots(y^{2m+1}+1)\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\frac{y^{2j+1}-1}{y^{2j+1}+1}.$$

Then $f_m(y)$ has a polynomial factor $(y-1)^{2m+1}$.

We can say further that $f_m(y)$ has another polynomial factor $(y+1)^m$. To see this, for each j, we let $\pi_j(y) = (y+1)\cdots(y^{2j-1}+1)(y^{2j+1}-1)(y^{2j+3}+1)\cdots(y^{2m+1}+1)$ and note that $(y+1) \mid (y^{2j+1}+1)$ since $2j+1$ is odd and

$$(y+1)^m \mid \pi_j(y),$$

for all $m \geq 1$. And $f_m(y) = \sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\pi_j(y)$ implies that $(y+1)^m$ is a polynomial factor of $f_m(y)$.

Lemma 2.1 (See also Lemma 3 in [16]). *Let m and k be integers with $k < m$. Then*

$$\sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}h(j) = 0,$$

where $h(j)$ is any polynomial of odd degree $2k+1$ and $h(i) = -h(2m-i+1)$ for $i = 0, 1, \ldots, m$.

Proof. Let Δ denote the forward-difference operator. That is, $\Delta h(x) = h(x+1) - h(x)$. We have

$$\Delta^{2m+1}h(0) = \sum_{j=0}^{2m+1}(-1)^{j+1}\binom{2m+1}{j}h(j) = 2\sum_{j=0}^{m}(-1)^{j+1}\binom{2m+1}{j}h(j).$$

The second equality holds by the hypothesis of the polynomial $h(j)$. However, $\Delta^{2m+1}h(0) = 0$ since $h(j)$ is of degree $2k+1 < 2m+1$. Thus, we conclude our assertion. $\square$

Proposition 2.4. *For a non-negative integer m, we let $g_m(y)$ be a polynomial in y defined by*

$$g_m(y) = (y-1)(y^3-1)\cdots(y^{2m+1}-1)\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\frac{y^{2j+1}+1}{y^{2j+1}-1}.$$

Then $g_m(y)$ has a polynomial factor $(y-1)^m(y+1)^{2m+1}$.

Proof. It is clear that $g_m(y)$ has a polynomial factor $(y-1)^m$. (See the paragraph after Proposition 2.3.) Let $h_m(y) := \sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}(y^{2j+1}+1)/(y^{2j+1}-1)$. We rewrite $h_m(y)$ as

$$h_m(y) = \sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\left(1 + 2(y^{2j+1}-1)^{-1}\right).$$

We prove that $g_m(y)$ has a polynomial factor $(y+1)^{2m+1}$ by claiming that

$$\left.\frac{d^p}{dy^p}h_m(y)\right|_{y=-1} = 0 \quad \text{for } p = 0, 1, \ldots, 2m. \tag{7}$$

Clearly $h_m(-1) = 0$ and this proves (7) when $p = 0$. Now we apply a result given by Leslie [7]: For any n times continuously differentiable function $g(x)$, we have

$$\frac{d^n}{dx^n}\left[\frac{1}{g(x)}\right] = \sum_{k=1}^{n}(-1)^k\binom{n+1}{k+1}\frac{1}{g^{k+1}(x)}\left[\frac{d^n}{dx^n}g^k(x)\right].$$

Then we have

$$\begin{aligned}\frac{d^p}{dy^p}\left(\frac{1}{y^{2j+1}-1}\right) &= \sum_{k=1}^{p}(-1)^k\binom{p+1}{k+1}\frac{1}{(y^{2j+1}-1)^{k+1}}\frac{d^p}{dy^p}(y^{2j+1}-1)^k \\ &= \sum_{k=1}^{p}(-1)^k\binom{p+1}{k+1}\frac{1}{(y^{2j+1}-1)^{k+1}}\frac{d^p}{dy^p}\left(\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}y^{(2j+1)r}\right) \\ &= \sum_{k=1}^{p}\binom{p+1}{k+1}\frac{1}{(y^{2j+1}-1)^{k+1}}\sum_{r=1}^{k}(-1)^r\binom{k}{r}p!\binom{(2j+1)r}{p}y^{(2j+1)r-p}.\end{aligned}$$

So we have for $p \geq 1$

$$\begin{aligned}\frac{d^p}{dy^p}h_m(y)\bigg|_{y=-1} &= \sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\frac{d^p}{dy^p}\left(\frac{2}{y^{2j+1}-1}\right)\bigg|_{y=-1} \\ &= \sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\sum_{k=1}^{p}\frac{(-1)^{k+p+1}}{2^k}\binom{p+1}{k+1}\sum_{r=1}^{k}\binom{k}{r}p!\binom{(2j+1)r}{p}.\end{aligned}$$

Following [16], we let $a_{p,r} := \sum_{k=r}^{p}(-1)^{k+r}2^{p-k}\binom{p+1}{k+1}\binom{k}{r}$, and have (see Lemma 5 in [16])

$$\sum_{r=1}^{p}(-1)^r a_{p,r}r^{2t} = 0, \tag{8}$$

for two positive integers p and t with $p \geq 2t$.

We need the expansion

$$p!\binom{x}{p} = \sum_{\ell=0}^{p}s(p,\ell)x^\ell,$$

where $s(p,\ell)$ is the signed Stirling number of the first kind.

Now we should rewrite the above evaluation as

$$\begin{aligned}\frac{d^p}{dy^p}h_m(y)\bigg|_{y=-1} &= \sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\sum_{r=1}^{p}\sum_{k=r}^{p}\frac{(-1)^{k+p+1}}{2^k}\binom{p+1}{k+1}\binom{k}{r} \\ &\qquad \times\sum_{\ell=0}^{p}s(p,\ell)((2j+1)r)^\ell \\ &= \frac{(-1)^{p+1}}{2^p}\sum_{\ell=0}^{p}s(p,\ell)\sum_{r=1}^{p}(-1)^r a_{p,r}r^\ell\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}(2j+1)^\ell.\end{aligned}$$

This expression vanishes by dividing the summation into three cases by $\ell = 0$, $\ell \geq 1$ odd, and $\ell \geq 1$ even. The result is zero when $\ell = 0$ since $s(p,0) = 0$. When ℓ is a odd number, by Lemma 2.1 since $\ell \leq 2m-1$ ($\ell \leq p \leq 2m$ and ℓ is odd), we see that the inner sum

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}(2j+1)^\ell = \sum_{j=0}^{m}(-1)^j\binom{2m+1}{j}(2m-2j+1)^\ell = 0$$

When ℓ is an even number, say $\ell = 2t$ for some integer $t \geq 1$, we just apply (8).

Therefore, we have showed that $\frac{d^p}{dy^p}h_m(y)\Big|_{y=-1} = 0$ for $p = 0, 1, \ldots, 2m$ and complete the proof. $\square$

We remark that all coefficients of both $f_m(y)$ and $g_m(y)$ are even for $m \geq 1$. It is because of

$$\begin{aligned} f_m(y) \equiv g_m(y) &\equiv \sum_{j=0}^{m} \binom{2m+1}{m-j} (y+1)(y^3+1)\cdots(y^{2m+1}+1) \\ &= 4^m (y+1)(y^3+1)\cdots(y^{2m+1}+1) \equiv 0 \pmod{2}. \end{aligned}$$

3 Proof of Theorem 1.3

Assume that $b = 0$. Then $V_n(x) = (ax)^{n-1}$ and $v_n(x) = (ax)^n$. So it is obvious to have $V_{2n+1}(x) - 1 = (ax)^{2n} - 1$ which can not divide $v_{2m+1}(x) \sum_{k=1}^{n} V_{2k}^{2m+1}(x)$ completely. For example, letting $n = m = 1$, we see that

$$(a^2x^2 - 1) \nmid (ax)^7.$$

For $|b| > 1$, suppose that

$$(V_{2n+1}(x) - 1) \mid v_1(x)v_3(x)\cdots v_{2m+1}(x) \sum_{k=1}^{n} V_{2k}^{2m+1}(x),$$

and write $v_1(x)v_3(x)\cdots v_{2m+1}(x) \sum_{k=1}^{n} V_{2k}^{2m+1}(x) = (V_{2n+1}(x) - 1)\, h(x)$, with $h(x) \in \mathbb{Z}[x]$. We consider the whole identity under modulo $|b|$ and have

$$v_1(x)v_3(x)\cdots v_{2m+1}(x) \sum_{k=1}^{n} V_{2k}^{2m+1}(x) \equiv (V_{2n+1}(x) - 1)\, h(x) \pmod{|b|}.$$

It can not happen based on the discussion in the previous paragraph.

Remark. *If $b = -1$, for our convenience, we consider the Balancing polynomial $B_n(x)$ and Lucas-Balancing polynomial $C_n(x)$ [10]. Assume that $n = m = 1$, we get*

$$B_3(x) - 1 = (36x^2 - 1) \nmid 4C_1(x)C_3(x)B_2^3(x) = 648x^5(12x^2 - 1).$$

Altogether, in view of discussion in the next section, Theorem 1.3 may be rephrased as the Melham's sum

$$v_1(x)v_3(x)\cdots v_{2m+1}(x) \sum_{k=1}^{n} V_{2k}^{2m+1}(x)$$

is divisible by $(V_{2n+1}(x) - 1)$ if and only if $b = 1$.

4 The case $b = 1$

Throughout this section, we assume that $b = 1$. Recall that, the $(a, 1)$-type Lucas polynomial sequences are denoted by $\{W_n(x)\}_{n\in\mathbb{Z}}$ and $\{w_n(x)\}_{n\in\mathbb{Z}}$. We begin with a simple lemma.

Lemma 4.1. *For any positive integers n and m, we have*

$$\sum_{k=1}^{n} W_{2km}(x) = \begin{cases} \dfrac{W_{(2n+1)m}(x) - W_m(x)}{w_m(x)} & \text{if } m \text{ is odd;} \\ \dfrac{w_{(2n+1)m}(x) - w_m(x)}{(a^2x^2+4)W_m(x)} & \text{if } m \text{ is even,} \end{cases}$$

$$\sum_{k=1}^{n} W_{(2k-1)m}(x) = \begin{cases} \dfrac{W_{2nm}(x)}{w_m(x)} & \text{if } m \text{ is odd;} \\ \dfrac{w_{2nm}(x) - 2}{(a^2x^2+4)W_m(x)} & \text{if } m \text{ is even,} \end{cases}$$

and

$$\sum_{k=1}^{n} w_{2km}(x) = \begin{cases} \dfrac{w_{(2n+1)m}(x)}{w_m(x)} - 1 & \text{if } m \text{ is odd;} \\ \dfrac{W_{(2n+1)m}(x)}{W_m(x)} - 1 & \text{if } m \text{ is even,} \end{cases}$$

$$\sum_{k=1}^{n} w_{(2k-1)m}(x) = \begin{cases} \dfrac{w_{2nm}(x) - 2}{w_m(x)} & \text{if } m \text{ is odd;} \\ \dfrac{W_{2nm}(x)}{W_m(x)} & \text{if } m \text{ is even.} \end{cases}$$

Proof. Let $\alpha(x) = (ax + \sqrt{a^2x^2+4})/2$ and $\beta(x) = (ax - \sqrt{a^2x^2+4})/2$. Using the Binet formula and noting that $\alpha(x)\beta(x) = -1$, we have

$$\begin{aligned} &\sum_{k=1}^{n} W_{2km}(x) = \sum_{k=1}^{n} \frac{\alpha^{2km}(x) - \beta^{2km}(x)}{\alpha(x) - \beta(x)} \\ =&\frac{1}{\sqrt{a^2x^2+4}} \left[\frac{\alpha^{2m(n+1)}(x) - \alpha^{2m}(x)}{\alpha^{2m}(x) - 1} - \frac{\beta^{2m(n+1)}(x) - \beta^{2m}(x)}{\beta^{2m}(x) - 1} \right] \\ =&\frac{W_{2m(n+1)}(x) - W_{2mn}(x) - W_{2m}(x)}{w_{2m}(x) - w_0(x)}. \end{aligned}$$

The first identity follows by Proposition 2.1 and the others can be proved similarly. □

From (5), it follows by Lemma 4.1 that

$$\sum_{k=1}^{n} W_{2k}^{2m+1}(x) = \frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} \frac{(-1)^{m-j}}{w_{2j+1}(x)} \binom{2m+1}{m-j} W_{(2n+1)(2j+1)}(x) - C_m(x), \tag{9}$$

where

$$C_m(x) = \frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \frac{W_{2j+1}(x)}{w_{2j+1}(x)}.$$

Now we need another important result, which is in the spirit of Jennings' theorem [6].

Lemma 4.2. *For any non-negative integers n and q, we have*

$$\begin{aligned} W_{(2q+1)n}(x) &= \sum_{\ell=0}^{q} (-1)^{n(q+\ell)} \frac{2q+1}{2\ell+1} \binom{q+\ell}{q-\ell} (a^2x^2+4)^{\ell} W_n^{2\ell+1}(x) \\ &= W_n(x) \sum_{\ell=0}^{q} (-1)^{(n+1)(q+\ell)} \binom{q+\ell}{q-\ell} w_n^{2\ell}(x), \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
w_{(2q+1)n}(x) = & \sum_{\ell=0}^{q}(-1)^{(n+1)(q+\ell)}\frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell}w_n^{2\ell+1}(x) \\
& = w_n(x)\sum_{\ell=0}^{q}(-1)^{n(q+\ell)}\binom{q+\ell}{q-\ell}(a^2x^2+4)^{\ell}W_n^{2\ell}(x).
\end{aligned} \tag{11}
$$

Proof. Let $p = 2q+1$ be an odd integer and

$$
y = \alpha^n(x), \quad z = \beta^n(x)
$$

and note that $yz = (-1)^n$. Write

$$
\frac{W_{pn}(x)}{W_n(x)} = \frac{\alpha^{pn}(x) - \beta^{pn}(x)}{\alpha^n(x) - \beta^n(x)} = y^{p-1} + y^{p-2}z + \cdots + yz^{p-2} + z^{p-1}.
$$

Now we have

$$
\begin{aligned}
\frac{W_{pn}(x)}{W_n(x)} = & \left(y^{p-1} + \frac{1}{y^{p-1}}\right) + (-1)^n\left(y^{p-3} + \frac{1}{y^{p-3}}\right) + \cdots \\
& + (-1)^{n\cdot\frac{p-3}{2}}\left(y^2 + \frac{1}{y^2}\right) + (-1)^{n\cdot\frac{p-1}{2}}
\end{aligned}
$$

since $p = 2q+1$ is odd.

We need the following two identities:

$$
\begin{aligned}
& (y^{2q} + y^{-2q}) - (y^{2q-2} + y^{-2q+2}) + \cdots + (-1)^{q-1}(y^2 + y^{-2}) + (-1)^q \\
= & \sum_{\ell=0}^{q}(-1)^{q+\ell}\frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell}(y+y^{-1})^{2\ell},
\end{aligned} \tag{12}
$$

and

$$
(y^{2q} + y^{-2q}) + (y^{2q-2} + y^{-2q+2}) + \cdots + (y^2 + y^{-2}) + 1 = \sum_{\ell=0}^{q}\frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell}(y-y^{-1})^{2\ell}. \tag{13}
$$

By identity (12) we have for n is odd that

$$
\frac{W_{(2q+1)n}(x)}{W_n(x)} = \sum_{\ell=0}^{q}(-1)^{q+\ell}(a^2x^2+4)^{\ell}\frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell}W_n^{2\ell}(x).
$$

When n is even, identity (13) gives

$$
\frac{W_{(2q+1)n}(x)}{W_n(x)} = \sum_{\ell=0}^{q}(a^2x^2+4)^{\ell}\frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell}W_n^{2\ell}(x).
$$

So the first expression of (10) follows. Differentiating both sides of the first expression of (11) with respect to x, and by the part **(d)** of Proposition 2.1, we obtain the second expression of (10).

Now for (11), consider

$$
\begin{aligned}
\frac{w_{pn}(x)}{w_n(x)} & = y^{p-1} - y^{p-2}z + y^{p-3}z^2 - \cdots - yz^{p-2} + z^{p-1} \\
& = \left(y^{2q} + y^{-2q}\right) + (-1)^{n+1}\left(y^{2q-2} + y^{-2q+2}\right) + \cdots \\
& \quad + (-1)^{(n+1)(q-1)}\left(y^2 + y^{-2}\right) + (-1)^{(n+1)q}.
\end{aligned}
$$

When n is odd, the identity (13) gives

$$\frac{w_{(2q+1)n}(x)}{w_n(x)} = \sum_{\ell=0}^{q} \frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell} w_n^{2\ell}(x).$$

When n is even, by (12) we have

$$\frac{w_{(2q+1)n}(x)}{w_n(x)} = \sum_{\ell=0}^{q} (-1)^{q+\ell}\frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell} w_n^{2\ell}(x).$$

Therefore we obtain the first identity of (11).

In addition, we apply the following two identities

$$(y^{2q}+y^{-2q})+(y^{2q-2}+y^{-2q+2})+\cdots+(y^2+y^{-2})+1=\sum_{\ell=0}^{q}(-1)^{q+\ell}\binom{q+\ell}{q-\ell}(y+y^{-1})^{2\ell}, \tag{14}$$

and

$$\begin{aligned}&(y^{2q}+y^{-2q})-(y^{2q-2}+y^{-2q+2})+\cdots+(-1)^{q-1}(y^2+y^{-2})+(-1)^q\\ =&\sum_{\ell=0}^{q}\binom{q+\ell}{q-\ell}(y-y^{-1})^{2\ell},\end{aligned} \tag{15}$$

to the expression of $w_{(2q+1)n}(x)/w_n(x)$ to yield

$$w_{(2q+1)n}(x) = w_n(x)\sum_{\ell=0}^{q}(-1)^{n(q+\ell)}\binom{q+\ell}{q-\ell}(a^2x^2+4)^{\ell}W_n^{2\ell}(x). \qquad \square$$

Remark. *If we apply identities (14) and (15) to the expression of $W_{(2q+1)n}(x)\ /W_n(x)$ in the proof of Lemma 4.2, we also get the second expression of (10). By the way, all identities (12), (13), (14) and (15) can be proved easily by induction on q.*

Taking $n=1$ into Lemma 4.2, it immediately infer to the following.

Corollary 4.1. *For any non-negative integer j, we have*

$$W_{2j+1}(x) = \sum_{\ell=0}^{j}(-1)^{j+\ell}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}(a^2x^2+4)^{\ell}, \tag{16}$$

and

$$w_{2j+1}(x) = (ax)\sum_{\ell=0}^{j}(-1)^{j+\ell}\binom{j+\ell}{j-\ell}(a^2x^2+4)^{\ell}. \tag{17}$$

In particular, $W_{2j+1}(2i/a) = (-1)^j(2j+1)$ and $w_{2j+1}(2i/a) = (-1)^j 2i$ with $i^2=-1$.

We remark here that, from Lemma 4.2, expansions of $W_{2j+1}(x)$ and $w_{2j+1}(x)$ in (ax) can be derived, namely

$$W_{2j+1}(x) = \sum_{\ell=0}^{j}\binom{j+\ell}{j-\ell}(ax)^{2\ell},$$

and

$$w_{2j+1}(x) = \sum_{\ell=0}^{j} \frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}(ax)^{2\ell+1},$$

respectively. Also these two identities can be derived from Proposition 2.2. Another way is to use part **(e)** of Proposition 2.1. By (16) we have

$$W_{2j+1}(x)\left(i\sqrt{x^2+\frac{4}{a^2}}\right) = \sum_{\ell=0}^{j}(-1)^j\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}(ax)^{2\ell}.$$

Thus, by the part **(e)** of Proposition 2.1, it gives the expansion of $w_{2j+1}(x)$ in (ax).

Corollary 4.2. *Let the* $(a,1)$*-type* W *sequence be* $\{W_n\}_{n\geq 0} := \{W_n(1)\}_{n\geq 0}$. *For any two odd primes* p, q *and a positive integer* n, *we have*

$$W_{pn} \equiv \left(\frac{a^2+4}{p}\right) W_n \pmod{p},$$

where $\left(\frac{*}{p}\right)$ *is the Legendre symbol and*

$$W_{pq} \equiv W_p W_q \pmod{pq}.$$

Proof. In light of identity (10), the coefficient of the right hand side divides by $p = 2q+1$ for $\ell = 0$ to $q-1$. Hence, by the Euler's criterion, we have

$$W_{pn} \equiv (a^2+4)^{\frac{p-1}{2}} W_n^p \equiv \left(\frac{a^2+4}{p}\right) W_n \pmod{p}.$$

Letting $n = 1$ and $n = q$ into the above congruence, we have

$$W_p \equiv \left(\frac{a^2+4}{p}\right) \pmod{p} \text{ and } W_{pq} \equiv \left(\frac{a^2+4}{p}\right) W_q \pmod{p},$$

respectively. Thus, $W_{pq} \equiv W_p W_q \pmod{p}$. By symmetry, $W_{pq} \equiv W_p W_q \pmod{q}$. □

Let $\{w_n(1)\}_{n\geq 0}$ be the $(a,1)$-type w sequence. Similarly, by identity (11), we conclude that

$$w_{pn} \equiv w_n \pmod{p},$$

for any odd prime p and any positive integer n.

Back to the proof of Lemma 4.2, we note that

$$\frac{W_{2qn}(x)}{W_n(x)} = \frac{y^{2q}-z^{2q}}{y-z} = y^{2q-1} + y^{2q-2}z + \cdots + yz^{2q-2} + z^{2q-1},$$

and

$$\frac{(\sqrt{a^2x^2+4})W_{2qn}(x)}{w_n(x)} = \frac{y^{2q}-z^{2q}}{y+z} = y^{2q-1} - y^{2q-2}z + y^{2q-3}z^2 - \cdots + yz^{2q-2} - z^{2q-1},$$

with $y = \alpha^n(x), z = \beta^n(x)$ and $yz = (-1)^n$. Now we need two slightly different formulae given by

$$
\begin{aligned}
&(y^{2q-1}+y^{-2q+1})+(y^{2q-3}+y^{-2q+3})+\cdots+(y^{3}+y^{-3})+(y+y^{-1})\\
&=\sum_{\ell=1}^{q}(-1)^{q+\ell}\binom{q+\ell-1}{2\ell-1}(y+y^{-1})^{2\ell-1},
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
&(y^{2q-1}-y^{-2q+1})-(y^{2q-3}-y^{-2q+3})+\cdots+(-1)^{q-1}(y-y^{-1})\\
&=\sum_{\ell=1}^{q}\binom{q+\ell-1}{2\ell-1}(y-y^{-1})^{2\ell-1}.
\end{aligned}
\tag{19}
$$

Both identities (18) and (19) follow easily by induction on q.

Repeating the same argument, we have

$$
\begin{aligned}
W_{2qn}(x)&=W_n(x)\sum_{\ell=1}^{q}(-1)^{(n+1)(q+\ell)}\binom{q+\ell-1}{2\ell-1}w_n^{2\ell-1}(x)\\
&=w_n(x)\sum_{\ell=1}^{q}(-1)^{n(q+\ell)}\binom{q+\ell-1}{2\ell-1}(a^2x^2+4)^{\ell-1}W_n^{2\ell-1}(x)
\end{aligned}
\tag{20}
$$

Combining (20) with identity (10), it leads to the divisible relation

$$W_m(x)\mid W_n(x),$$

for any positive integers m,n with $m\mid n$.

Theorem 4.1. *We have Ozeki–Prodinger-like identities for the* $(a,1)$*-type Lucas polynomial sequences:*

(i)

$$
\begin{aligned}
\sum_{k=1}^{n}W_{2k}^{2m+1}(x)=&\sum_{\ell=0}^{m}\frac{(-1)^{m+\ell}}{(a^2x^2+4)^{m-\ell}}W_{2n+1}^{2\ell+1}(x)\sum_{j=\ell}^{m}\frac{1}{w_{2j+1}(x)}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}\\
&-\frac{1}{(a^2x^2+4)^m}\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\frac{W_{2j+1}(x)}{w_{2j+1}(x)}\\
=&\frac{W_{2n+1}(x)}{(a^2x^2+4)^m}\sum_{\ell=0}^{m}w_{2n+1}^{2\ell}(x)\sum_{j=\ell}^{m}\frac{(-1)^{m-j}}{w_{2j+1}(x)}\binom{2m+1}{m-j}\binom{j+\ell}{j-\ell}\\
&-\frac{1}{(a^2x^2+4)^m}\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\frac{W_{2j+1}(x)}{w_{2j+1}(x)}.
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\sum_{k=1}^{n}w_{2k}^{2m+1}(x)=&\sum_{\ell=0}^{m}w_{2n+1}^{2\ell+1}(x)\sum_{j=\ell}^{m}\frac{1}{w_{2j+1}(x)}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}-4^m\\
=&\ w_n(x)\sum_{\ell=0}^{m}(-1)^{\ell}(a^2x^2+4)^{\ell}W_{2n+1}^{2\ell}(x)\sum_{j=\ell}^{m}\frac{(-1)^j}{w_{2j+1}(x)}\binom{2m+1}{m-j}\binom{j+\ell}{j-\ell}-4^m.
\end{aligned}
$$

Proof. From identity (9) and by Lemma 4.2, we have

$$\sum_{k=1}^{n} W_{2k}^{2m+1}(x) = \sum_{j=0}^{m} \frac{1}{w_{2j+1}(x)} \binom{2m+1}{m-j} \sum_{\ell=0}^{j} \frac{(-1)^{m+\ell}}{(a^2x^2+4)^{m-\ell}} \frac{2j+1}{2\ell+1} \binom{j+\ell}{j-\ell} W_{2n+1}^{2\ell+1}(x)$$
$$- \frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \frac{W_{2j+1}(x)}{w_{2j+1}(x)}.$$

The first term of the right hand side can be rewritten as

$$\sum_{\ell=0}^{m} \frac{(-1)^{m+\ell}}{(a^2x^2+4)^{m-\ell}} W_{2n+1}^{2\ell+1}(x) \sum_{j=\ell}^{m} \frac{1}{w_{2j+1}(x)} \binom{2m+1}{m-j} \frac{2j+1}{2\ell+1} \binom{j+\ell}{j-\ell}.$$

If we start with identity (9) and use identity (10) to expand $W_{(2n+1)(2j+1)}(x)$ as a polynomial in power of $w_n(x)$, the second expression of $\sum_{k=1}^{n} W_{2k}^{2m+1}(x)$ follows.

Similar proof for (ii), we omit here. □

Remark. *There are certainly formulae of power sums involving* $(a,1)$*-type Lucas polynomial sequences (e.g. for* $W_{2k-1}^{2m}(x), W_{2k}^{2m}(x), W_{2k-1}^{2m}(x), w_{2k-1}^{2m+1}(x), w_{2k}^{2m}(x), w_{2k-1}^{2m}(x)$*), and they can be derived easily through the same way.*

Let $H_{2m+1}(x,y)$ be a polynomial in two variables x and y with the degree $2m+1$ of y defined by

$$H_{2m+1}(x,y) = \sum_{\ell=0}^{m} \frac{(-1)^{m+\ell}}{(a^2x^2+4)^{m-\ell}} y^{2\ell+1} \sum_{j=\ell}^{m} \frac{1}{w_{2j+1}(x)} \binom{2m+1}{m-j} \frac{2j+1}{2\ell+1} \binom{j+\ell}{j-\ell} - C_m(x), \tag{21}$$

with

$$C_m(x) = \frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \frac{W_{2j+1}(x)}{w_{2j+1}(x)}.$$

By Theorem 4.1, we obtain an expansion for the sum $\sum_{k=1}^{n} W_{2k}^{2m+1}(x)$ in power of $W_{2n+1}(x)$. In other words, $\sum_{k=1}^{n} W_{2k}^{2m+1}(x) = H_{2m+1}(x, W_{2n+1}(x))$.

Substituting $\alpha(x)/\beta(x)$ for y in Proposition 2.3, we conclude that

$$(a^2x^2+4)^m \mid w_1(x)w_3(x)\cdots w_{2m+1}(x) \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \frac{W_{2j+1}(x)}{w_{2j+1}(x)}.$$

This implies that $w_1(x)w_3(x)\cdots w_{2m+1}(x)C_m(x)$ is a polynomial with integer coefficients.

Recall the fact that $f_m(y)$ has another polynomial factor $(y+1)^m$. (See the paragraph after Proposition 2.3.) Taking $y = \alpha(x)/\beta(x)$, we obtain

$$(ax)^m \mid w_1(x)w_3(x)\cdots w_{2m+1}(x) \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \frac{W_{2j+1}(x)}{w_{2j+1}(x)}. \tag{22}$$

Lemma 4.3. *Let m be a positive integer. For $\ell = 1, 2, \ldots, m$, we have*

$$\sum_{j=0}^{\ell} (-1)^j \binom{2m+1}{j} \binom{2m-j-\ell}{2m-2\ell} p(j; \ell, m) = 0,$$

where $p(j; \ell, m) := p(j)$ is a polynomial in j of odd degree less than $2\ell + 1$ and $p(i) = -p(2m - i + 1)$ for $i = 0, 1, \ldots, m$.

Proof. Denote $b(j)$ by

$$b(j) = \binom{2m-j-\ell}{2m-2\ell} = \frac{(2m-j-\ell)(2m-j-\ell-1)\cdots(\ell-j+1)}{(2m-2\ell)!}$$

and note that $b(\ell+1) = b(\ell+2) = \cdots = b(m) = 0$. Thus, we rewrite the desired identity as

$$\sum_{j=0}^{m} (-1)^j \binom{2m+1}{j} h(j; \ell, m) = 0,$$

where $h(j; \ell, m) = b(j)p(j; \ell, m)$. Now $h(j; \ell, m)$ meets all conditions listed in Lemma 2.1 and the desired identity follows by Lemma 2.1. $\square$

Remark. *One may compare the above result with Lemma 2.6 in [12]. Unfortunately, we feel that the statement of Lemma 2.6 in [12] is wrong. The correct version requests an extra condition on the polynomial $p(j)$ as mentioned in Lemma 4.3.*

We are now at the stage to give a proof of Theorem 1.4.

Proof of Theorem 1.4. First of all, according to identity (9), we show that

$$(ax)^m \;\Big|\; \frac{w_1(x)w_3(x)\cdots w_{2m+1}(x)}{(a^2x^2+4)^m} \sum_{j=0}^{m} \frac{(-1)^{m-j}}{w_{2j+1}(x)} \binom{2m+1}{m-j} W_{(2n+1)(2j+1)}(x).$$

It, together with (22), shows that $(ax)^m$ is a polynomial factor of the Melham's sum in Theorem 1.4. Let

$$\pi_j(w, x) := w_1(x) \cdots w_{2j-1}(x) w_{2j+3}(x) \cdots w_{2m+1}(x)$$

and rewrite the above right hand side as

$$\frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} (-1)^{m-j} \pi_j(w, x) \binom{2m+1}{m-j} W_{(2n+1)(2j+1)}(x).$$

Notice that $(ax)^m \mid \pi_j(w, x)$ for $j = 1, 2, \ldots, m$ and our assertion follows.

Let $H_{2m+1}(x, y)$ be defined by (21). In order to prove the Melham's sum can be divisible by $(W_{2n+1}(x) - 1)^2$, we claim that the polynomial $H_{2m+1}(x, y)$ and its derivative with respect to y both vanish at $y = 1$. Taking $y = 1$ into $H_{2m+1}(x, y)$, we obtain

$$
\begin{aligned}
&H_{2m+1}(x,1)\\
&= \sum_{\ell=0}^{m} \frac{(-1)^{m+\ell}}{(a^2x^2+4)^{m-\ell}} \sum_{j=\ell}^{m} \frac{1}{w_{2j+1}(x)} \binom{2m+1}{m-j} \frac{2j+1}{2\ell+1} \binom{j+\ell}{j-\ell}\\
&\quad - \frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \frac{W_{2j+1}(x)}{w_{2j+1}(x)}\\
&= \frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} \frac{(-1)^m}{w_{2j+1}(x)} \binom{2m+1}{m-j} \left[\sum_{\ell=0}^{j} (-1)^\ell \frac{2j+1}{2\ell+1} \binom{j+\ell}{j-\ell} (a^2x^2+4)^\ell \right]\\
&\quad - \frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} (-1)^{m-j} \binom{2m+1}{m-j} \frac{W_{2j+1}(x)}{w_{2j+1}(x)}\\
&= 0.
\end{aligned}
$$

The last equality holds in view of (16).

Next, by (17), we obtain

$$
\begin{aligned}
&\left. \frac{\partial}{\partial y} H_{2m+1}(x,y) \right|_{y=1}\\
&= \sum_{\ell=0}^{m} \frac{(-1)^{m+\ell}}{(a^2x^2+4)^{m-\ell}} \sum_{j=\ell}^{m} \frac{2j+1}{w_{2j+1}(x)} \binom{2m+1}{m-j} \binom{j+\ell}{j-\ell}\\
&= \frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} \frac{(-1)^m(2j+1)}{w_{2j+1}(x)} \binom{2m+1}{m-j} \left[\sum_{\ell=0}^{j} (-1)^\ell \binom{j+\ell}{j-\ell} (a^2x^2+4)^\ell \right]\\
&= \frac{1}{(a^2x^2+4)^m} \sum_{j=0}^{m} \frac{(-1)^{m+j}(2j+1)}{ax} \binom{2m+1}{m-j}\\
&= 0.
\end{aligned}
$$

The last step follows due to Lemma 2.1 with $h(j) = 2m-2j+1$ and $k=0$. Therefore, the sum $w_1(x)w_3(x)\cdots w_{2m+1}(x) \sum_{k=1}^{n} W_{2k}^{2m+1}(x)$ has a polynomial factor $(W_{2n+1}(x)-1)^2$.

To see that $\widetilde{H}_{2m-1}(x,y)$ is a polynomial with integer coefficients, it remains to show that

$$
w_{2\ell+1}(x)w_{2\ell+3}(x)\cdots w_{2m+1}(x) \sum_{j=\ell}^{m} \frac{(a^2x^2+4)^{\ell-m}}{w_{2j+1}(x)} \binom{2m+1}{m-j} \frac{2j+1}{2\ell+1} \binom{j+\ell}{j-\ell},
$$

is a polynomial with integer coefficients for $\ell = 0,1,\ldots,m$. Let

$$
T_{\ell,m}(x) = \sum_{j=m-\ell}^{m} \frac{1}{w_{2j+1}(x)} \binom{2m+1}{m-j} \frac{2j+1}{2m-2\ell+1} \binom{j+m-\ell}{j-m+\ell}.
$$

It is equivalent to show that $w_{2m-2\ell+1}(x)w_{2m-2\ell+3}(x)\cdots w_{2m+1}(x)T_{\ell,m}(x)$ has a polynomial factor $(a^2x^2+4)^\ell$ for $\ell = 0,1,\ldots,m$.

The case $\ell=0$ is trivial. For $\ell=1$, we have

$$
T_{1,m}(x) = \frac{2m+1}{w_{2m+1}(x)} + \frac{2m+1}{w_{2m-1}(x)},
$$

and $T_{1,m}(2i/a) = 0$ since $w_{2m+1}(2i/a) = (-1)^m 2i$ by (17), where $i^2 = -1$. This implies that $w_{2m-1}(x)w_{2m+1}(x)T_{1,m}(x)$ has a polynomial factor a^2x^2+4. In the following we claim that $T_{\ell,m}(2i/a) = 0$ for $2 \le \ell \le m$ and $\frac{d^p}{dx^p}T_{p+1,m}(x)\big|_{x=\frac{2i}{a}} = 0$ for $p = 1, 2, \ldots, m-1$.

By the definition of $T_{\ell,m}(x)$ we have

$$\begin{aligned}
&T_{\ell,m}(2i/a)\\
&= \frac{1}{2i(2m-2\ell-1)} \sum_{j=m-\ell}^{m} (-1)^j \binom{2m+1}{m-j}(2j+1)\binom{j+m-\ell}{j-m+\ell}\\
&= \frac{(-1)^m}{2i(2m-2\ell+1)} \sum_{j=0}^{\ell} (-1)^j \binom{2m+1}{j}\binom{2m-j-\ell}{2m-2\ell}(2m-2j+1).
\end{aligned}$$

Hence, by Lemma 4.3 with $h(j) = 2m-2j+1$, we have that $T_{\ell,m}(2i/a) = 0$ for $2 \le \ell \le m$.

Now apply a result of Leslie [7] (see page 8 in this paper), we obtain

$$\frac{d^p}{dx^p}\left[\frac{1}{w_{2j+1}(x)}\right]\Bigg|_{x=\frac{2i}{a}} = \sum_{k=1}^{p}(-1)^k\binom{p+1}{k+1}\frac{1}{(2i^{2j+1})^{k+1}}\left[\frac{d^p}{dx^p}w_{2j+1}^k(x)\right]\Bigg|_{x=\frac{2i}{a}}.$$

For two positive integers n, p with $n \ge p$, by (4) we note that

$$\begin{aligned}
\frac{d^p}{dx^p}w_n(x)\Big|_{x=\frac{2i}{a}} &= \sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n}{n-j}\binom{n-j}{j}p!\binom{n-2j}{p}a^{n-2j}x^{n-2j-p}\Bigg|_{x=\frac{2i}{a}}\\
&= a^p n(p-1)!\, i^{n-p}\binom{n+p-1}{2p-1}.
\end{aligned} \tag{23}$$

To see why the last step holds, let

$$A_p(n) := \sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{n}{n-j}\binom{n-j}{j}p!\binom{n-2j}{p}(2i)^{n-2j-p},$$

we note that $A_n(n) = n!$ and

$$A_p(n) = \frac{2i(2p+1)}{n^2-p^2}A_{p+1}(n).$$

It would implies that

$$A_p(n) = \frac{2i(2p+1)}{n^2-p^2}\cdot\frac{2i(2p+3)}{n^2-(p+1)^2}\cdots\frac{2i(2n-1)}{n^2-(n-1)^2}A_n(n),$$

or

$$A_p(n) = n(p-1)!\, i^{n-p}\binom{n+p-1}{2p-1}.$$

In light of the identity

$$\begin{aligned}
w_{2j+1}^k(x) &= \left[\alpha^{2j+1}(x)+\beta^{2j+1}(x)\right]^k\\
&= \sum_{r=0}^{k}\frac{1}{2}\binom{k}{r}\left[\alpha^{(2j+1)r}(x)\beta^{(2j+1)(k-r)}(x)+\alpha^{(2j+1)(k-r)}(x)\beta^{(2j+1)r}(x)\right]\\
&= \sum_{r=0}^{k}\frac{(-1)^r}{2}\binom{k}{r}w_{(2j+1)(k-2r)}(x),
\end{aligned}$$

we have

$$
\begin{aligned}
&\frac{d^p}{dx^p}\left[\frac{1}{w_{2j+1}(x)}\right]\bigg|_{x=\frac{2i}{a}}\\
&=\sum_{k=1}^{p}(-1)^k\binom{p+1}{k+1}\frac{1}{(2i^{2j+1})^{k+1}}\sum_{r=0}^{k}\frac{(-1)^r}{2}\binom{k}{r}\left[\frac{d^p}{dx^p}w_{(2j+1)(k-2r)}(x)\right]\bigg|_{x=\frac{2i}{a}}\\
&=a^p\sum_{k=1}^{p}(-1)^{k+j}\binom{p+1}{k+1}\frac{(2j+1)(p-1)!}{2^{k+2}i^{p+1}}\sum_{r=0}^{k}\binom{k}{r}(k-2r)\binom{(2j+1)(k-2r)+p-1}{2p-1}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\frac{d^p}{dx^p}T_{p+1,m}(x)\bigg|_{x=\frac{2i}{a}}\\
&=\sum_{j=m-p-1}^{m}\binom{2m+1}{m-j}\frac{2j+1}{2m-2p-1}\binom{j+m-p-1}{j-m+p+1}\left[\frac{d^p}{dx^p}\frac{1}{w_{2j+1}(x)}\right]\bigg|_{x=\frac{2i}{a}}\\
&=a^p\sum_{j=m-p-1}^{m}(-1)^j\binom{2m+1}{m-j}\frac{(2j+1)^2}{2m-2p-1}\binom{j+m-p-1}{j-m+p+1}\\
&\qquad\times\sum_{k=1}^{p}(-1)^k\binom{p+1}{k+1}\frac{(p-1)!}{2^{k+2}i^{p+1}}\\
&\qquad\times\sum_{r=0}^{k}\binom{k}{r}(k-2r)\binom{(2j+1)(k-2r)+p-1}{2p-1},
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{d^p}{dx^p}T_{p+1,m}(x)\bigg|_{x=\frac{2i}{a}}=&a^p\sum_{k=1}^{p}\sum_{r=0}^{k}(-1)^k\binom{p+1}{k+1}\binom{k}{r}\frac{(k-2r)(p-1)!}{2^{k+2}i^{p+1}(2m-2p-1)}\\
&\times\sum_{j=m-p-1}^{m}(-1)^j\binom{2m+1}{m-j}\binom{j+m-p-1}{j-m+p+1}\\
&\times\binom{(2j+1)(k-2r)+p-1}{2p-1}(2j+1)^2.
\end{aligned}
$$

Our assertion follows if we can prove that the inner sum vanishes. That is,

$$
\sum_{j=m-p-1}^{m}(-1)^j\binom{2m+1}{m-j}\binom{j+m-p-1}{j-m+p+1}\binom{(2j+1)(k-2r)+p-1}{2p-1}(2j+1)^2=0,
$$

for $p=1,2,\dots,m-1$, or equivalently,

$$
\sum_{j=0}^{p+1}(-1)^{m-j}\binom{2m+1}{j}\binom{2m-j-p-1}{2m-2p-2}H(j;p,m)=0, \tag{24}
$$

where

$$
H(j;p,m):=H(j)=\binom{(2m-2j+1)(k-2r)+p-1}{2p-1}(2m-2j+1)^2.
$$

It is routine that one expresses $H(j)$ as a product of integers and check that $H(i)=-H(2m-i+1)$ for $i=0,1,\dots,m$. Also we note that $H(j)$ is a polynomial in j of degree $2p+1<2(p+1)+1$, an odd number. Thus, (24) holds by Lemma 4.3. □

Another consideration leads to the extension of Melham's Conjecture 2 or Theorem 1.2.

Theorem 4.2. *For any non-negative integers n and m, the Melham's sum*

$$w_1(x)w_3(x)\cdots w_{2m+1}(x)\sum_{k=1}^{n} w_{2k}^{2m+1}(x)$$

can be expressed as $(ax)^m(w_{2n+1}(x)-ax)\widetilde{S}_{2m}(x,w_{2n+1}(x))$, where $\widetilde{S}_{2m}(x,y)$ is a polynomial in two variables x and y with integer coefficients and of degree $2m$ in y.

Proof. The fact that $(ax)^m$ is a polynomial factor of the Melham's sum

$$w_1(x)w_3(x)\cdots w_{2m+1}(x)\sum_{k=1}^{n} w_{2k}^{2m+1}(x)$$

follows similarly by viewing the proof of our Theorem 1.4. We note that this polynomial factor $(ax)^m$ actually dues to the part of the product $w_1(x)w_3(x)\cdots\ w_{2m+1}(x)$.

In view of **(ii)** in Theorem 4.1, we let

$$S_{2m+1}(x,y)=\sum_{\ell=0}^{m} y^{2\ell+1}\sum_{j=\ell}^{m}\frac{1}{w_{2j+1}(x)}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}-4^m.$$

Then by (4),

$$\begin{aligned} S_{2m+1}(x,ax) &= \sum_{\ell=0}^{m}\sum_{j=\ell}^{m}\frac{(ax)^{2\ell+1}}{w_{2j+1}(x)}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}-4^m \\ &= \sum_{j=0}^{m}\frac{1}{w_{2j+1}(x)}\binom{2m+1}{m-j}\left[\sum_{\ell=0}^{j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}(ax)^{2\ell+1}\right]-4^m \\ &= \sum_{j=0}^{m}\binom{2m+1}{m-j}-4^m. \end{aligned}$$

From this and the binomial theorem, we see $S_{2m+1}(x,ax)=0$ and this implies the sum

$$w_1(x)w_3(x)\cdots w_{2m+1}(x)\sum_{k=1}^{n} w_{2k}^{2m+1}(x)$$

has a polynomial factor $(w_{2n+1}(x)-ax)$.

To see that $\widetilde{S}_{2m}(x,y)$ is a polynomial with integer coefficients, it is suffices to show

$$w_{2\ell+1}(x)w_{2\ell+3}(x)\cdots w_{2m+1}(x)\sum_{j=\ell}^{m}\frac{1}{w_{2j+1}(x)}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}$$

is an integer for $0\le\ell\le m$. This is clear since

$$\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}=2\binom{j+\ell+1}{j-\ell}-\binom{j+\ell}{j-\ell} \tag{25}$$

is an integer. □

Corollary 4.3. *Let $Q_n(x)$ be n-th Pell–Lucas polynomial and the Melham's sum for $Q_n(x)$ be define by $Q(n,m;x)=Q_1(x)Q_3(x)\cdots Q_{2m+1}(x)\sum_{k=1}^{n}Q_{2k}^{2m+1}(x)$. Then for any positive integers n and m, we have the quotient $Q(n,m;x)/(Q_{2n+1}(x)-2x)$ is an integer polynomial in x.*

One can substitute the Melham's sum in Theorem 4.2 with

$$w_1(x)w_3(x)\cdots w_{2m+1}(x)\sum_{k=0}^{n} w_{2k}^{2m+1}(x).$$

The only slightly difference is that the value k begins with zero under the summation sign. It is easy to derive that

$$\sum_{k=0}^{n} w_{2k}^{2m+1}(x) = \sum_{\ell=0}^{m} w_{2n+1}^{2\ell+1}(x)\sum_{j=\ell}^{m}\frac{1}{w_{2j+1}(x)}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell} + 4^m.$$

Thus, we conclude a kind of variation of Theorem 4.2.

Theorem 4.3. *For two non-negative integer n, m, the sum*

$$w_1(x)w_3(x)\cdots w_{2m+1}(x)\sum_{k=0}^{n} w_{2k}^{2m+1}(x)$$

can be expressed as $(ax)^m(w_{2n+1}(x)+ax)\mathcal{S}_{2m}(x, w_{2n+1}(x))$, where $\mathcal{S}_{2m}(x,y)$ is a polynomial in two variables x and y with integer coefficients and of degree $2m$ in y.

5 The case $b = -1$

In this section, we discuss the case when $b = -1$. Recall that, in our notation, $V_n^{(a,-1)}(x) := \overline{W}_n(x)$. That is, the polynomial sequence $\{\overline{W}_n(x)\}_{n\geq 0}$ satisfies the recurrence relation

$$\overline{W}_n(x) = (ax)\overline{W}_{n-1}(x) - \overline{W}_{n-2}(x) \ \text{ for } n \geq 2,$$

with initial values $\overline{W}_0(x) = 0$ and $\overline{W}_1(x) = 1$. Let $v_n^{(a,-1)}(x) := \overline{w}_n(x)$ by analogy.

The following identities are easy to prove by the Binet formula for $\overline{W}(x)$ and $\overline{w}(x)$.

Lemma 5.1. *For any positive integers n and m, we have*

$$\sum_{k=1}^{n}\overline{W}_{2km}(x) = \frac{\overline{w}_{(2n+1)m}(x) - \overline{w}_m(x)}{(a^2x^2-4)\overline{W}_m(x)}, \quad \sum_{k=1}^{n}\overline{W}_{(2k-1)m}(x) = \frac{\overline{w}_{2nm}(x) - 2}{(a^2x^2-4)\overline{W}_m(x)},$$

and

$$\sum_{k=1}^{n}\overline{w}_{2km}(x) = \frac{\overline{W}_{(2n+1)m}(x)}{\overline{W}_m(x)} - 1, \quad \sum_{k=1}^{n}\overline{w}_{(2k-1)m}(x) = \frac{\overline{W}_{2nm}(x)}{\overline{W}_m(x)}.$$

From (6) and Lemma 5.1, we get

$$\sum_{k=1}^{n}\overline{w}_{2k}^{2m+1}(x) = \sum_{j=0}^{m}\binom{2m+1}{m-j}\sum_{k=1}^{n}\overline{w}_{(2j+1)2k}(x) = \sum_{j=0}^{m}\binom{2m+1}{m-j}\frac{\overline{W}_{(2n+1)(2j+1)}(x)}{\overline{W}_{2j+1}(x)} - 4^m. \tag{26}$$

Lemma 5.2. *For two non-negative integers n and q, we have*

$$\begin{aligned}\overline{W}_{(2q+1)n}(x) &= \sum_{\ell=0}^{q} \frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell}(a^2x^2-4)^{\ell}\overline{W}_n^{2\ell+1}(x)\\ &= \overline{W}_n(x)\sum_{\ell=0}^{q}(-1)^{q+\ell}\binom{q+\ell}{q-\ell}\overline{w}_n^{2\ell}(x),\end{aligned}\tag{27}$$

and

$$\begin{aligned}\overline{w}_{(2q+1)n}(x) &= \sum_{\ell=0}^{q}(-1)^{q+\ell}\frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell}\overline{w}_n^{2\ell+1}(x)\\ &= \overline{w}_n(x)\sum_{\ell=0}^{q}\binom{q+\ell}{q-\ell}(a^2x^2-4)^{\ell}\overline{W}_n^{2\ell}(x).\end{aligned}$$

Proof. Let $y=\overline{\alpha}^n(x), z=\overline{\beta}^n(x)$ and note that $yz=1$. We compute

$$\begin{aligned}\frac{\overline{W}_{(2q+1)n}(x)}{\overline{W}_n(x)} &= (y^{2q}+z^{2q})+(y^{2q-2}+z^{2q-2})+\cdots+(y^2+z^2)+1\\ &= \sum_{\ell=0}^{q}\frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell}(y-z)^{2\ell} \quad \text{by identity (13)}\\ &= \sum_{\ell=0}^{q}(-1)^{q+\ell}\binom{q+\ell}{q-\ell}(y+z)^{2\ell} \quad \text{by identity (14).}\end{aligned}$$

So the first assertion follows. Now,

$$\begin{aligned}&\frac{\overline{w}_{(2q+1)n}(x)}{\overline{w}_n(x)} = y^{2q}-y^{2q-1}z+y^{2q-2}z^2-\cdots-yz^{2q-1}+z^{2q}\\ &= (y^{2q}+z^{2q})-(y^{2q-2}+z^{2q-2})+\cdots+(-1)^{q-1}(y^2+z^2)+(-1)^q\\ &= \sum_{\ell=0}^{q}(-1)^{q+\ell}\frac{2q+1}{2\ell+1}\binom{q+\ell}{q-\ell}(y+z)^{2\ell} \quad \text{by identity (12)}\\ &= \sum_{\ell=0}^{q}\binom{q+\ell}{q-\ell}(y-z)^{2\ell} \quad \text{by identity (15).}\end{aligned}$$

Our second assertion follows and the proof completes. □

Taking $n=1$ in Lemma 5.2, it immediately yields

Corollary 5.1. *For any non-negative integer j, we have*

$$\overline{W}_{2j+1}(x) = \sum_{\ell=0}^{j}\binom{j+\ell}{j-\ell}\frac{2j+1}{2\ell+1}(a^2x^2-4)^{\ell},$$

and

$$\overline{w}_{2j+1}(x) = ax\sum_{\ell=0}^{j}\binom{j+\ell}{j-\ell}(a^2x^2-4)^{\ell}.$$

On one hand, we have

$$\begin{aligned}\frac{\overline{W}_{2qn}(x)}{\overline{W}_n(x)} &= \frac{y^{2q}-z^{2q}}{y-z} = (y^{2q-1}+y^{-2q+1})+(y^{2q-3}+y^{-2q+3})+\cdots+(y^1+y^{-1})\\ &= \sum_{\ell=1}^{q}(-1)^{q+\ell}\binom{q+\ell-1}{2\ell-1}(y+y^{-1})^{2\ell-1} \quad \text{by identity (18).}\end{aligned}$$

Thus, it implies that

$$\overline{W}_{2qn}(x) = \overline{W}_n(x) \sum_{\ell=1}^{q} (-1)^{q+\ell} \binom{q+\ell-1}{2\ell-1} \overline{w}_n^{2\ell-1}(x).$$

On the other hand,

$$\begin{aligned}\frac{(\sqrt{a^2x^2-4})\overline{W}_{2qn}(x)}{\overline{w}_n(x)} &= \frac{y^{2q}-z^{2q}}{y+z} \\ &= y^{2q-1} - y^{2q-2}z + y^{2q-3}z^2 - \cdots + yz^{2q-2} - z^{2q-1} \\ &= \sum_{\ell=1}^{q} \binom{q+\ell-1}{2\ell-1} (y-y^{-1})^{2\ell-1} \quad \text{by identity (19).}\end{aligned}$$

We have the second expression for $\overline{W}_{2qn}(x)$:

$$\overline{W}_{2qn}(x) = \overline{w}_n(x) \sum_{\ell=1}^{q} \binom{q+\ell-1}{2\ell-1} (a^2x^2-4)^{\ell-1} \overline{W}_n^{2\ell-1}(x).$$

So far, we conclude that $\overline{W}_m(x) \mid \overline{W}_n(x)$ if $m \mid n$ and $\overline{W}_m(x)\overline{w}_m(x) \mid \overline{W}_{nm}(x)$ if n is even.

Corollary 5.2. *Let the n-th $(a,-1)$-type Lucas number be $\overline{W}_n := \overline{W}_n(1)$. For any two odd primes p, q and a positive integer n, we have*

$$\overline{W}_{pn} \equiv \left(\frac{a^2-4}{p}\right) \overline{W}_n \pmod{p},$$

and

$$\overline{W}_{pq} \equiv \overline{W}_p \overline{W}_q \pmod{pq}.$$

A positive integer n is called a balancing number [1] if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some positive integer r. Behera and Panda [1] proved that if n is a balancing number, then $8n^2+1$ is a perfect square, and the positive square root of $8n^2+1$ is called a Lucas-Balancing number. We denote the n-th Balancing number by B_n and the n-the Lucas-Balancing number by C_n. (We admit that $B_1 = 1$ for convenience.) Notice that both $\{B_n\}_{n\geq 1}$ and $\{C_n\}_{n\geq 1}$ satisfies the recurrence relation

$$R_n = 6R_{n-1} - R_{n-2} \quad \text{for } n \geq 2,$$

with initial values $B_0 = 0, B_1 = 1$ and $C_0 = 1, C_1 = 3$. Also notice that $B_n(1) = B_n$ and $C_n(1) = C_n$ [10].

Hence, Corollary 5.2 implies that

$$B_p \equiv \left(\frac{32}{p}\right) \equiv \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} \pmod{p},$$

for any odd prime p.

Theorem 5.1. *We have Ozeki–Prodinger-like identities for the* $(a,1)$*-type Lucas polynomial sequences:*

(i)

$$\begin{aligned}
&\sum_{k=1}^{n}\overline{W}_{2k}^{2m+1}(x)\\
&=\frac{1}{(a^2x^2-4)^{m+1}}\sum_{\ell=0}^{m}(-1)^{m+\ell}\overline{w}_{2n+1}^{2\ell+1}(x)\sum_{j=\ell}^{m}\frac{1}{\overline{W}_{2j+1}(x)}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}\\
&\quad-\frac{1}{(a^2x^2-4)^{m+1}}\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\frac{\overline{w}_{2j+1}(x)}{\overline{W}_{2j+1}(x)}\\
&=\overline{w}_{2n+1}(x)\sum_{\ell=0}^{m}\frac{1}{(a^2x^2-4)^{m-\ell+1}}\overline{W}_{2n+1}^{2\ell}(x)\sum_{j=\ell}^{m}\frac{(-1)^{m-j}}{\overline{W}_{2j+1}(x)}\binom{2m+1}{m-j}\binom{j+\ell}{j-\ell}\\
&\quad-\frac{1}{(a^2x^2-4)^{m+1}}\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\frac{\overline{w}_{2j+1}(x)}{\overline{W}_{2j+1}(x)}.
\end{aligned}$$

(ii)

$$\begin{aligned}
&\sum_{k=1}^{n}\overline{w}_{2k}^{2m+1}(x)\\
&=\sum_{\ell=0}^{m}(a^2x^2-4)^{\ell}\overline{W}_{2n+1}^{2\ell+1}(x)\sum_{j=\ell}^{m}\frac{1}{\overline{W}_{2j+1}(x)}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}-4^m\\
&=\overline{W}_{2n+1}(x)\sum_{\ell=0}^{m}(-1)^{\ell}\overline{w}_{2n+1}^{2\ell}(x)\sum_{j=\ell}^{m}\frac{(-1)^{j}}{\overline{W}_{2j+1}(x)}\binom{2m+1}{m-j}\binom{j+\ell}{j-\ell}-4^m.
\end{aligned}$$

Proof. We should only prove **(ii)**,

$$\begin{aligned}
\sum_{k=1}^{n}\overline{w}_{2k}^{2m+1}(x)&=\sum_{j=0}^{m}\binom{2m+1}{m-j}\frac{\overline{W}_{(2n+1)(2j+1)}(x)}{\overline{W}_{2j+1}(x)}-4^m \quad \text{by identity (26),}\\
&=\sum_{j=0}^{m}\frac{1}{\overline{W}_{2j+1}(x)}\binom{2m+1}{m-j}\sum_{\ell=0}^{j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}(a^2x^2-4)^{\ell}\overline{W}_{2n+1}^{2\ell+1}(x)-4^m,
\end{aligned}$$

or

$$\sum_{k=1}^{n}\overline{w}_{2k}^{2m+1}(x)=\sum_{j=0}^{m}\binom{2m+1}{m-j}\frac{\overline{W}_{2n+1}(x)}{\overline{W}_{2j+1}(x)}\sum_{\ell=0}^{j}(-1)^{j+\ell}\binom{j+\ell}{j-\ell}\overline{w}_{2n+1}^{2\ell}(x)-4^m,$$

by identity (27). Then we obtain **(ii)** by just switching the order of summation. $\square$

There is an interesting implication when y is substituted for $\overline{\alpha}(x)/\overline{\beta}(x)$ in the Proposition 2.4. Then we obtain

$$(ax)^{2m+1}\mid(a^2x^2-4)^{m+1}\overline{W}_1(x)\overline{W}_3(x)\cdots\overline{W}_{2m+1}(x)\overline{C}_m(x), \tag{28}$$

where

$$\overline{C}_m(x)=\frac{1}{(a^2x^2-4)^{m+1}}\sum_{j=0}^{m}(-1)^{m-j}\binom{2m+1}{m-j}\frac{\overline{w}_{2j+1}(x)}{\overline{W}_{2j+1}(x)}.$$

It is time to prove our main theorem in this section.

Proof of Theorem 1.5. Let

$$\overline{W}(n,m;x) := (a^2x^2-4)^{m+1}\overline{W}_1(x)\overline{W}_3(x)\cdots\overline{W}_{2m+1}(x)\sum_{k=1}^{n}\overline{W}_{2k}^{2m+1}(x)$$

and

$$\begin{aligned}\overline{M}_{2m+1}(x,y) :=& \frac{1}{(a^2x^2-4)^{m+1}}\sum_{\ell=0}^{m}(-1)^{m+\ell}y^{2\ell+1}\\ &\times\sum_{j=\ell}^{m}\frac{1}{\overline{W}_{2j+1}(x)}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}-\overline{C}_m(x).\end{aligned}$$

We see that $\overline{M}_{2m+1}(x,ax)=0$ by Lemma 5.2. In addition, by Proposition 2.2,

$$\begin{aligned}&\frac{\partial}{\partial y}\overline{M}_{2m+1}(x,y)\Big|_{y=ax}\\ =&\frac{1}{(a^2x^2-4)^{m+1}}\sum_{j=0}^{m}\frac{(-1)^{m-j}}{\overline{W}_{2j+1}(x)}\binom{2m+1}{m-j}\sum_{\ell=0}^{j}(-1)^{j+\ell}(2j+1)\binom{j+\ell}{j-\ell}(ax)^{2\ell}\\ =&\frac{1}{(a^2x^2-4)^{m+1}}\sum_{j=0}^{m}(-1)^{m-j}(2j+1)\binom{2m+1}{m-j}\\ =&\,0.\end{aligned}$$

All together we conclude that

$$(\overline{w}_{2n+1}(x)-ax)^2 \ \mid\ \overline{W}(n,m;x).$$

To see that $\widetilde{M}_{2m-1}(x,y)\in\mathbb{Z}[x,y]$, we just notice that (25) and the fact that (28). □

Corollary 5.3. *For any positive integers n and m the sum*

$$2^{5m+3}B_1B_3\cdots B_{2m+1}\sum_{k=1}^{n}B_{2k}^{2m+1}$$

can be divisible by $(C_{2n+1}-3)^2$.

Theorem 5.2. *For any positive integers n and m, the sum*

$$\overline{W}_1(x)\overline{W}_3(x)\cdots\overline{W}_{2m+1}(x)\sum_{k=1}^{n}\overline{w}_{2k}^{2m+1}(x),$$

can be expressed as $(\overline{W}_{2n+1}(x)-1)\widetilde{N}_{2m}(x,\overline{W}_{2n+1}(x))$, where $\widetilde{N}_{2m}(x,y)$ is a polynomial in two variables x and y with integer coefficients and of degree $2m$ in y.

Proof. In view of **(ii)** in Theorem 5.1, we let

$$\begin{aligned}&\overline{N}_{2m+1}(x,y)\\=&\sum_{\ell=0}^{m}(a^2x^2-4)^{\ell}y^{2\ell+1}\sum_{j=\ell}^{m}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}\frac{1}{\overline{W}_{2j+1}(x)}-4^m.\end{aligned}$$

Then, by Corollary 5.1,

$$\begin{aligned}&\overline{N}_{2m+1}(x,1)\\=&\sum_{\ell=0}^{m}(a^2x^2-4)^{\ell}\sum_{j=\ell}^{m}\binom{2m+1}{m-j}\frac{2j+1}{2\ell+1}\binom{j+\ell}{j-\ell}\frac{1}{\overline{W}_{2j+1}(x)}-4^m\\=&\sum_{j=0}^{m}\binom{2m+1}{m-j}\frac{1}{\overline{W}_{2j+1}(x)}\sum_{\ell=0}^{j}\binom{j+\ell}{j-\ell}\frac{2j+1}{2\ell+1}(a^2x^2-4)^{\ell}-4^m\\=&\sum_{j=0}^{m}\binom{2m+1}{m-j}-4^m\\=&\,0.\end{aligned}$$

From this, we conclude that the Melham's sum

$$\overline{W}_1(x)\overline{W}_3(x)\cdots\overline{W}_{2m+1}(x)\sum_{k=1}^{n}\overline{w}_{2k}^{2m+1}(x)$$

can be divisible by $(\overline{W}_{2n+1}(x)-1)$. To see that $\widetilde{N}_{2m}(x,y)$ is an integer polynomial, we just notice (25). □

Corollary 5.4. *For any positive integers n and m the sum*

$$2B_1B_3\cdots B_{2m+1}\sum_{k=1}^{n}C_{2k}^{2m+1}$$

can be divisible by $B_{2n+1}-1$.

6 Conclusion

In summary, to study the divisibility of Melham's sum for (a,b)-type Lucas polynomial sequences, we only need to pay attention to the specialized cases $b=1$ and $b=-1$ (Theorem 1.3). We derive Ozeki-Prodinger-like identities for the $(a,1)$-type and $(a,-1)$-type Lucas polynomials (Theorem 4.1, 5.1), and extend Jennings' result (Lemma 4.2, 5.2). Finally, we prove some divisibility properties of Melham's sums for $(a,1)$-type and $(a,-1)$-type Lucas polynomial sequences (Theorem 1.4, 1.5, 4.2 and 5.2), which extend the scope of Melham's original conjectures, and have some interesting implications.

We raise an open question in studying Melham's sum for some Lucas polynomial sequences. The polynomials $\widetilde{H}_{2m-1}(x,y),\widetilde{S}_{2m}(x,y),\widetilde{M}_{2m-1}(x,y)$ and $\widetilde{N}_{2m}(x,y)$ can be viewed as polynomials in y of a suitable degree with integer coefficients. Are they irreducible polynomials in y for all $m\geq 1$? We hope to attract the attention of interested readers.

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