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A generalization of arithmetic derivative to p-adic fields and number fields

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Abstract: The arithmetic derivative is a function from the natural numbers to itself that sends all prime numbers to 1 and satisfies the Leibniz rule. The arithmetic partial derivative with respect to a prime p is the p-th component of the arithmetic derivative. In this paper, we generalize the arithmetic partial derivative to p-adic fields (the local case) and the arithmetic derivative to number fields (the global case). We study the dynamical system of the p-adic valuation of the iterations of the arithmetic partial derivatives. We also prove that for every integer $n \geq 0$, there are infinitely many elements with exactly n anti-partial derivatives. In the end, we study the p-adic continuity of arithmetic derivatives.

Keywords: Arithmetic derivative, Arithmetic partial derivative, Arithmetic subderivative, p-adic fields, Number fields, p-adic continuity.
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1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. The arithmetic derivative is a function $D : \mathbb{N} \to \mathbb{N}$ that satisfies the following two properties: $D(p) = 1$ for all primes p, and the Leibniz rule, $D(xy) =$

$D(x)y + xD(y)$ for all $x, y \in \mathbb{N}$. One of the questions on the 1950 Putnam competition [3] asked the contestants to predict the limit of the sequence $63, D(63), D^2(63), \ldots$. Many sources cite this as the origin of the arithmetic derivative. However we were able to find a paper by Shelly [14] published in 1911 which introduced this topic as well as some of the basic properties and generalizations of this function.

One can ask a more general question. If we fix $x \in \mathbb{N}$, what is the limit of the sequence $x, D(x), D^2(x), \ldots$. This is not easy to predict in general. Ufnarovski and Åhlander made the following conjecture.

Conjecture 1.1. [15, Conjecture 2] *For every $x \in \mathbb{N}$, exactly one of the following could happen: either $D^i(x) = 0$ or p^p for some prime p for sufficiently large i, or $\lim_{i \to +\infty} D^i(x) = +\infty$.*

We note that Shelly [14] alluded to this conjecture and Barbeau [1] made a similar conjecture. One corollary of this conjecture is that if the sequence $x, D(x), D^2(x), \ldots$ is eventually periodic, then the period is 1. That is $D^k(x) = p^p$ for some prime p when $k \gg 0$. Given $y > 1$, it is not hard to show [15, Corollary 3] that there are finitely many (possibly 0) x such that $D(x) = y$. We call x an anti-derivative of y. Ufnarovski and Åhlander made the following conjecture.

Conjecture 1.2. [15, Conjecture 8] *For every integer $n \geq 0$ there are infinitely many $x > 0$ such that x has exactly n anti-derivatives.*

Let ν_p be the p-adic valuation. One can show that $D(0) = 0$ and for $x > 0$, D has the following explicit formula

$$D(x) = x \sum_p \frac{\nu_p(x)}{p}.$$

This is a finite sum as there are only finitely many p such that $\nu_p(x) \neq 0$. It is natural to generalize D to $\mathbb{Q}$ as ν_p is well-defined over $\mathbb{Q}$. We will use D to denote the arithmetic derivative defined on $\mathbb{Q}$ in the introduction section. This generalization allows positive integers to have more anti-derivatives than they have in $\mathbb{N}$. For example, 2 does not have an anti-derivative in $\mathbb{N}$ but $D(-21/16) = 2$. The only anti-derivatives of 1 in $\mathbb{N}$ are the prime numbers but $D(-5/4) = 1$. Another direction to generalize D is, instead of differentiating with respect to all prime numbers, we only differentiate with respect to a set of primes. More specifically, let $T \subset \mathbb{P}$ be a nonempty set of rational primes. For $0 \neq x \in \mathbb{Q}$, we define

$$D_{\mathbb{Q},T}(x) = x \sum_{p \in T} \frac{\nu_p(x)}{p}.$$

This is called the arithmetic subderivative over $\mathbb{Q}$ with respect to T, first introduced by Haukkanen, Merikoski, and Tossavainen [5]. If $T = \mathbb{P}$, then $D_{\mathbb{Q},T} = D$. If $T = \{p\}$ contains a single prime number, then $D_{\mathbb{Q},T} = D_{\mathbb{Q},p}$ is called the arithmetic partial derivative with respect to p, first introduced by Kovič [9].

The authors of this paper have proved [2, Theorem 9] that the following sequence of integers

$$\nu_p(x),\ \nu_p(D_{\mathbb{Q},p}(x)),\ \nu_p(D^2_{\mathbb{Q},p}(x)),\ \ldots$$

is eventually periodic of period $\leq p$. An immediate corollary of this result is a positive answer to a conjecture similar to Conjecture 1.1 in the case of arithmetic partial derivative. We have to replace p^p in Conjecture 1.1 by bp^p where $\nu_p(b) = 0$ since $D_{\mathbb{Q},p}(bp^p) = bp^p$. In the same paper, we also proved a criterion to determine when an integer has integral anti-partial derivatives, and as application, we gave a positive answer to a conjecture similar to Conjecture 1.2 in the case of arithmetic partial derivative.

A natural next step is to generalize the arithmetic derivative to number fields and their rings of integers. The Leibniz rule can be used to generalize D to all unique factorization domains (UFD) R. In every equivalence class $\{x \text{ irreducible in } R \mid x = ux', u \in R^\times\}$, we choose an element x_0 and define $D_R(x_0) = 1$ (similar to $D(p) = 1$). For all units $u \in R^\times$, we define $D_R(u) = 0$ (similar to $D(\pm 1) = 0$). By the unique factorization property and the Leibniz rule, we can extend the definition of D to the entire ring R as well as its field of fraction Frac(R). Let $\mathcal{P}$ be a set of chosen irreducible elements as described above, one from each equivalence classes. For every $x \in \text{Frac}(R)$, if $x = up_1 \cdots p_k q_1^{-1} \cdots q_\ell^{-1}$ with $u \in R^\times$ and $p_i, q_j \in \mathcal{P}$ (p_i, q_j are not necessarily pairwise distinct) then

$$D_R(x) = x\Big(\sum_{i=1}^{k} \frac{1}{p_i} - \sum_{j=1}^{\ell} \frac{1}{q_j}\Big).$$

There are two major obstacles with this generalization. First, for every number field K, it is well known that $\mathcal{O}_K$ is not necessarily a UFD. It has been proved that this idea will fail for non-UFD [4]. Second, this definition of $D_R(x)$ depends on the choice of irreducible elements set $\mathcal{P}$ as well as the ring. There is no canonical way to choose x_0 within each equivalence classes. Also, for an irreducible element $x \in \mathcal{P} \subset R$, we have $D_R(x) = 1$. But if we consider $x \in \text{Frac}(R)$ and define the arithmetic derivative over Frac(R), then we will get $D_{\text{Frac}(R)}(x) = 0$ since all nonzero elements of Frac(R) are invertible. In other words, suppose $x \in R_1 \subset R_2$, we do not necessarily have $D_{R_1}(x) = D_{R_2}(x)$. This phenomenon makes it hard to generalize D to all number fields in a consistent way using this definition.

To get around the first obstacle, Mistri and Pandey [10] defined the arithmetic derivative of an ideal in the ring of integers $\mathcal{O}_K$ of a number field K. This generalization uses the fact that every fractional ideal of K can be uniquely factorized into a product of prime ideals in $\mathcal{O}_K$. Suppose $I = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_k$ is an ideal of $\mathcal{O}_K$ where $\mathfrak{p}_i$ are primes ideals of $\mathcal{O}_K$ with $\mathfrak{p}_i \mid p_i$ (again $\mathfrak{p}_i$ and p_i are not necessarily pairwise distinct). Then the arithmetic derivative of I is an ideal of $\mathcal{O}_K$ defined by

$$D_K(I) = \Big(p_1 p_2 \cdots p_k \sum_{i=1}^{k} \frac{1}{p_i}\Big).$$

This means that the arithmetic derivative of every ideal of $\mathcal{O}_K$ is a principal ideal in $\mathcal{O}_K$ generated by an integer. From the definition, it is easy to see that $D_{\mathbb{Z}}(n) = (D(n))$ where $D_{\mathbb{Z}}(n)$ is the arithmetic derivative of the ideal (n) and $D(n)$ is the usual arithmetic derivative of an integer. This property is certainly welcomed as part of the generalization but the second obstacle mentioned above still exists. For example, let $K = \mathbb{Q}(i)$ and we have $2\mathcal{O}_K = (1+i)(1-i)$, hence $D_K(2\mathcal{O}_K) = 4\mathcal{O}_K$. On the other hand, $D_{\mathbb{Z}}(2\mathbb{Z}) = \mathbb{Z}$. This means that if $x \in K_1 \subset K_2$, we do not necessarily have $D_{K_1}(x\mathcal{O}_{K_1}) \subset D_{K_2}(x\mathcal{O}_{K_2})$.

In this paper, we propose a new way to define the arithmetic derivative (resp. the arithmetic subderivative) D_K (resp. $D_{K,T}$) on every finite Galois extension $K/\mathbb{Q}$ in a consistent way in the following sense. First $D_K(x) = D(x)$ for all $x \in \mathbb{Q}$, so D_K is a true extension of D from $\mathbb{Q}$ to K. Second, if K_1 and K_2 are two finite Galois extensions, then for every $x \in K_1 \cap K_2$, we have $D_{K_1}(x) = D_{K_2}(x)$. This means that the definition of arithmetic derivative of x does not depend on the choice of the Galois extension. Because the arithmetic derivative satisfies $D_K(x)/x \in \mathbb{Q}$, we can even generalize it to every number field $L/\mathbb{Q}$ (not necessarily Galois) by taking a restriction $D_L(x) := D_K(x) = x \cdot (D_K(x)/x) \in L$ where K is a finite Galois extension containing x. Please refer to Section 3 for detailed definition.

At the local level, suppose K is a finite extension of the p-adic rational numbers $\mathbb{Q}_p$. Let $\nu_\mathfrak{p}$ be the unique valuation on K that extends the p-adic valuation ν_p on $\mathbb{Q}$. It only makes sense to study the arithmetic partial derivative $D_{K,\mathfrak{p}}$ over K. As part of the study of the behavior of the sequence $x, D_{K,\mathfrak{p}}(x), D^2_{K,\mathfrak{p}}(x), \ldots$, we give a complete description of the behavior of the following so-called $\nu_\mathfrak{p}$ sequence of x

$$\nu_\mathfrak{p}(x),\ \nu_\mathfrak{p}(D_{K,\mathfrak{p}}(x)),\ \nu_\mathfrak{p}(D^2_{K,\mathfrak{p}}(x)),\ \ldots.$$

Theorem 1.3. *Let K be a finite extension over $\mathbb{Q}_p$ and $\mathfrak{p}$ be the unique prime ideal of $\mathcal{O}_K$. For every $x \in K$, we have the following three properties.*

1. *If $\nu_\mathfrak{p}(\nu_\mathfrak{p}(x)) \geq 0$ or $\nu_\mathfrak{p}(x) \in \{0, +\infty\}$, then the $\nu_\mathfrak{p}$ sequence of x is eventually periodic of period $\leq p$.*
2. *If $\nu_\mathfrak{p}(\nu_\mathfrak{p}(x)) < 0$, then the $\nu_\mathfrak{p}$ sequence of x converges to $-\infty$.*
3. *The $\nu_\mathfrak{p}$ sequence of x is eventually $+\infty$ if and only if*

$$\nu_\mathfrak{p}(x) \in \{0, 1, \ldots, p-1, +\infty\}.$$

See Lemma 2.2, Proposition 2.4, and Theorem 2.8 for a proof of Theorem 1.3. Using the same idea as in our previous paper [2], we are also able to give a positive answer to a similar conjecture to Conjecture 1.2 in the p-adic fields case as well.

Theorem 1.4 (Theorem 2.14)**.** *Let K be a finite extension over $\mathbb{Q}_p$. For each positive integer n, there are infinitely many $x_0 \in K$ such that $D_{K,\mathfrak{p}}(x_0)$ has exactly n anti-partial derivatives in K.*

One difficulty of studying the iteration of arithmetic derivatives is that the arithmetic derivative is neither additive nor a group homomorphism. But if one considers the so-called logarithmic derivative $\mathrm{ld}(x) := D(x)/x$, it is not hard to see that $\mathrm{ld} : \mathbb{Q}^\times \to \mathbb{Q}$ is a group homomorphism from the multiplicative group to the additive group, just like the usual logarithmic function. As we generalize D to D_K, we also study the generalization of ld to ld_K. In particular, we have shown that $\mathrm{ld}_K(K^\times)$ are also isomorphic as subgroups of $\mathbb{Q}$ for any finite Galois extension K; see Theorem 4.2. We also give a concrete description of the exact image of $\mathrm{ld}_K(K^\times)$ when K is a quadratic extension.

It is not surprising that the arithmetic derivative function D is not continuous over $\mathbb{Q}$ because given two rational numbers that are close by (in the sense of the Archimedean metric), their prime factorizations can be drastically different. In fact, Haukkanen, Merikoski and Tossavainen [6] have shown that for every $x \in \mathbb{Q}$, the arithmetic subderivative $D_{\mathbb{Q},T}$ (and in particular the arithmetic derivative) can obtain arbitrary large values in any small neighborhood of x. Therefore $D_{\mathbb{Q},T}$ is clearly not continuous with respect to the standard Archimedean topology of $\mathbb{Q}$. But what about the p-adic topology? In another paper, Haukkanen, Merikoski and Tossavainen [7] have proved that the arithmetic partial derivative $D_{\mathbb{Q},p}$ is always continuous. They have also shown in some cases, the arithmetic subderivative $D_{\mathbb{Q},T}$ can be continuous at some points but discontinuous at other points. Major cases have been left open. For example, it is unknown whether $D_{\mathbb{Q},T}$ is continuous or not at nonzero points when T is an infinite set. As we generalize arithmetic partial derivatives to p-adic local fields and arithmetic subderivative to number fields, it makes sense to study whether the generalizations are $\mathfrak{p}$-adically continuous or not. We state our results in two theorems, one for the arithmetic partial derivative case and one for the arithmetic subderivative case.

Theorem 1.5. *Suppose K is a number field. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$. Then the arithmetic partial derivative $D_{K,\mathfrak{p}}$ is $\mathfrak{p}$-adically continuous at every point in K. Moreover $D_{K,\mathfrak{p}}$ is strictly differentiable and twice strictly differentiable (with respect to the ultrametric $|\cdot|_{\nu_\mathfrak{p}}$) at every nonzero point in K but $D_{K,\mathfrak{p}}$ is not strictly differentiable (with respect to the ultrametric $|\cdot|_{\nu_\mathfrak{p}}$) at* 0.

See Theorems 5.2, 5.3, and 5.4 for a proof of Theorem 1.5. The same result is true for arithmetic partial derivative over p-adic fields.

Theorem 1.6. *Suppose K is a number field. Let $\mathfrak{p}$ be a prime ideal and T be a nonempty subset of prime ideals of $\mathcal{O}_K$.*

1. *The arithmetic subderivative $D_{K,T}$ is $\mathfrak{p}$-adically continuous but not strictly differentiable (with respect to the ultrametric $|\cdot|_{\nu_\mathfrak{p}}$) at* 0.
2. *If $T \neq \{\mathfrak{p}\}$, then the arithmetic subderivative $D_{K,T}$ is $\mathfrak{p}$-adically discontinuous at every nonzero point in K.*

See Theorems 5.6, 5.8, 5.9, and 5.12 for a proof of Theorem 1.6. By letting $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$ in Theorem 1.6, we are able to give answers to all the open questions in [7, Section 7].

In general, it is unclear to us how to piece together the information of arithmetic partial derivatives to understand the arithmetic derivatives. New prime factors may arise in the dynamical system $D^i(x)$ following each successive differentiation and predicting new prime factors of $D(x)$ relies on the ability of predicting prime factors of $a+b$ when knowing the prime factors of a and b. There is a widespread intuition that the abc conjecture should be related to arithmetic derivatives of some sort. Pasten has formalized this idea in [11].

2 p-adic fields

2.1 Definition

Fix a rational prime p. Let $\mathbb{Q}_p$ be the field of p-adic rational numbers and ν_p the p-adic valuation. We denote the p-adic absolute value by $|\cdot|_{\nu_p}$. Recall that the arithmetic partial derivative (with respect to p) $D_{\mathbb{Q},p} : \mathbb{Q} \to \mathbb{Q}$ is defined by

$$D_{\mathbb{Q},p}(x) := \begin{cases} x\nu_p(x)/p, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

One can extend $D_{\mathbb{Q},p}$ to $D_{\mathbb{Q}_p,p}$ with the same formula because ν_p is well-defined on $\mathbb{Q}_p$. We can further extend $D_{\mathbb{Q}_p,p}$ to p-adic fields because ν_p can be uniquely extended to a discrete valuation over p-adic fields. Let K be a finite extension of $\mathbb{Q}_p$ of degree $n = [K : \mathbb{Q}_p]$. Let $\mathcal{O}_K$ be the ring of integers, which is a discrete valuation ring with maximal ideal $\mathfrak{p}$ and residue field $\mathcal{O}_K/\mathfrak{p}$. Let $f = f(K|\mathbb{Q}_p) = [\mathcal{O}_K/\mathfrak{p} : \mathbb{F}_p]$ be the inertia degree and $e = e(K|\mathbb{Q}_p)$ the ramification index, that is, the unique integer such that $p\mathcal{O}_K = \mathfrak{p}^e$. We have $n = ef$. It is well known [13, Chapter 2 Proposition 3] that K is again complete with respect to the $\mathfrak{p}$-adic topology. There exists a unique discrete valuation $\nu_{\mathfrak{p}} : K \to \mathbb{Q} \cup \{+\infty\}$ that extends ν_p defined by

$$\nu_{\mathfrak{p}}(x) := \frac{1}{n}\nu_p(N_{K/\mathbb{Q}_p}(x)),$$

where $N_{K/\mathbb{Q}_p} : K \to \mathbb{Q}_p$ is the norm. We know that $\nu_{\mathfrak{p}}(K) = \mathbb{Z}/e$. For every $x \in K$, we set $k = k(x) := \nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x))$, so $k \geq -\nu_p(e)$. The discrete valuation $\nu_{\mathfrak{p}}$ defines a unique absolute value on K, which will be denoted by $|\cdot|_{\nu_{\mathfrak{p}}}$, that extends the p-adic absolute value on $\mathbb{Q}_p$:

$$|x|_{\nu_{\mathfrak{p}}} = \sqrt[n]{\left|N_{K/\mathbb{Q}_p}(x)\right|_{\nu_p}}.$$

We can extend $D_{\mathbb{Q}_p,p}$ to $D_{K,\mathfrak{p}} : K \to K$ as follows:

$$D_{K,\mathfrak{p}}(x) := \begin{cases} x\nu_{\mathfrak{p}}(x)/p, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

One can check that $D_{K,\mathfrak{p}}$ satisfies the Leibniz rule. It is evident that $D_{K,\mathfrak{p}}(x) = D_{\mathbb{Q}_p,p}(x)$ for all $x \in \mathbb{Q}_p$. Note that the definition of $D_{K,\mathfrak{p}}$ is independent of the choice of uniformizers of $\mathcal{O}_K$.

Let K and K' be two finite extensions over $\mathbb{Q}_p$ such that $x \in K \cap K' =: K''$. Let $\nu_{\mathfrak{p}}$, $\nu_{\mathfrak{p}'}$, $\nu_{\mathfrak{p}''}$ be the unique discrete valuations that extend ν_p to K, K', and K'' respectively. Clearly $\nu_{\mathfrak{p}}|_{K''} = \nu_{\mathfrak{p}'}|_{K''} = \nu_{\mathfrak{p}''}$. Therefore we have $D_{K,\mathfrak{p}}(x) = x\nu_{\mathfrak{p}}(x)/p = x\nu_{\mathfrak{p}''}(x)/p = x\nu_{\mathfrak{p}'}(x)/p = D_{K',\mathfrak{p}'}(x) \in K \cap K'$. This implies that the definition of arithmetic partial derivative of x is independent of the choice of finite extensions where x lies.

Remark 2.1. Let q be another prime different from p. The q-adic valuation ν_q defined on $\mathbb{Q}$ does not extend to $\mathbb{Q}_p$ or finite extensions of $\mathbb{Q}_p$. Therefore, unlike the case of $\mathbb{Q}$ where we have one arithmetic partial derivative for each prime number, there is only one well-defined arithmetic partial derivative for $\mathbb{Q}_p$ and for finite extensions of $\mathbb{Q}_p$.

2.2 Periodicity of $\nu_{\mathfrak{p}}$ sequence

Let $K/\mathbb{Q}_p$ be a finite extension and let $x \in K$. Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}_K$ and $\nu_{\mathfrak{p}}$ the unique discrete valuation that extends ν_p. We call the following sequence

$$\nu_{\mathfrak{p}}(x),\ \nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)),\ \nu_{\mathfrak{p}}(D^2_{K,\mathfrak{p}}(x)),\ \ldots$$

the $\nu_{\mathfrak{p}}$ sequence of x. Note that the $\nu_{\mathfrak{p}}$ sequence of x is independent of the choice of K. If $\nu_{\mathfrak{p}}(D^j_{K,\mathfrak{p}}(x)) = +\infty$ for some integer $j \geq 0$, then $D^j_{K,\mathfrak{p}}(x) = 0$ and thus $D^i_{K,\mathfrak{p}}(x) = 0$ for all $i \geq j$. If $\nu_{\mathfrak{p}}(D^i_{K,\mathfrak{p}}(x)) < +\infty$ for all $i \geq 0$, then we call the sequence of increments of consecutive terms

$$\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) - \nu_{\mathfrak{p}}(x), \nu_{\mathfrak{p}}(D^2_{K,\mathfrak{p}}(x)) - \nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)), \nu_{\mathfrak{p}}(D^3_{K,\mathfrak{p}}(x)) - \nu_{\mathfrak{p}}(D^2_{K,\mathfrak{p}}(x)), \ldots$$

the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x. Suppose $\nu_{\mathfrak{p}}(x) = bp^k$ where $\nu_p(b) = 0$ and $k \geq -\nu_p(e)$. Then the increment is

$$\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) - \nu_{\mathfrak{p}}(x) = \nu_{\mathfrak{p}}(\frac{\nu_{\mathfrak{p}}(x)}{p}) = \nu_{\mathfrak{p}}(bp^{k-1}) = k - 1 = \nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x)) - 1. \tag{1}$$

Lemma 2.2. *The following two statements are equivalent:*

1. *The $\nu_{\mathfrak{p}}$ sequence of x is eventually $+\infty$.*
2. *$\nu_{\mathfrak{p}}(x) \in \{0, 1, 2, \ldots, p-1, +\infty\}$.*

Proof. Suppose $\nu_{\mathfrak{p}}(x) \in \{0, 1, 2, \ldots, p-1, +\infty\}$. If $\nu_{\mathfrak{p}}(x) = +\infty$, then $x = 0$, and $D_{K,\mathfrak{p}}(x) = 0$ for all $n \geq 0$. If $\nu_{\mathfrak{p}}(x) = 0$, then x is a unit in $\mathcal{O}_K$, and thus $D^n_{K,\mathfrak{p}}(x) = 0$ for all $n \geq 1$. If $\nu_{\mathfrak{p}}(x) = j$ for some $1 \leq j \leq p-1$, then $\nu_{\mathfrak{p}}(D^i_{K,\mathfrak{p}}(x)) = j - i$ for $1 \leq i \leq j$. From $\nu_{\mathfrak{p}}(D^i_{K,\mathfrak{p}}(x)) = 0$ we get $D^i_{K,\mathfrak{p}}(x)$ is a unit in $\mathcal{O}_K$, and thus $D^n_{K,\mathfrak{p}}(x) = 0$ for all $n > j$.

Now we show that if $\nu_{\mathfrak{p}}(x) \notin \{0, 1, 2, \ldots, p-1, +\infty\}$, then the $\nu_{\mathfrak{p}}$ sequence of x is not eventually $+\infty$. It suffices to show that $\nu_{\mathfrak{p}}(D^i_{K,\mathfrak{p}}(x)) \neq 0$ for all $i \geq 0$. We consider three mutually disjoint cases.

Case 1. Suppose $\nu_{\mathfrak{p}}(x) \notin \mathbb{Z}$. Then $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) \notin \mathbb{Z}$ by (1). By induction, we get $\nu_{\mathfrak{p}}(D^i_{K,\mathfrak{p}}(x)) \notin \mathbb{Z}$ since $\nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(D^{i-1}_{K,\mathfrak{p}}(x))) - 1 \in \mathbb{Z}$. In particular, $\nu_{\mathfrak{p}}(D^i_{K,\mathfrak{p}}(x)) \neq 0$.

Case 2. Suppose $\nu_{\mathfrak{p}}(x) \geq p$ is an integer. If $p \nmid \nu_{\mathfrak{p}}(x)$, then $\nu_{\mathfrak{p}}(x) > p$ and $k = 0$, and so $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) = \nu_{\mathfrak{p}}(x) - 1 \geq p$. If $p \mid \nu_{\mathfrak{p}}(x)$, then $k \geq 1$, and thus $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) \geq \nu_{\mathfrak{p}}(x) \geq p$ by (1). Therefore $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) \geq p > 0$. By induction, we get $\nu_{\mathfrak{p}}(D^i_{K,\mathfrak{p}}(x)) \neq 0$.

Case 3. Suppose $\nu_{\mathfrak{p}}(x) = bp^k < 0$ is an integer. Since $|bp^k| \geq p^k > k - 1$, we get $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) = bp^k + (k-1) < 0$. By induction, we get $\nu_{\mathfrak{p}}(D^i_{K,\mathfrak{p}}(x)) \neq 0$.

Combining all three cases, we have proved that if $\nu_{\mathfrak{p}}(x) \notin \{0, 1, 2, \ldots, p-1, +\infty\}$, then the $\nu_{\mathfrak{p}}$ sequence of x is not eventually $+\infty$. □

Remark 2.3. Ufnarovski and Åhlander conjecture [15, Conjecture 8] that there exists an infinite sequence a_n of different natural numbers such that $a_1 = 1$ and $D_{\mathbb{Q}}(a_n) = a_{n-1}$ for $n \geq 2$. Here $D_{\mathbb{Q}}$ is the arithmetic derivative (not arithmetic partial derivative) defined on $\mathbb{Q}$. The same question can be asked for $D_{K,\mathfrak{p}}$. Suppose there exists an infinite sequence $a_n \in K$ such that $a_1 = 1$ and $D_{K,\mathfrak{p}}(a_n) = a_{n-1}$ for $n \geq 2$. Let $N = p+1$ and we know that the $\nu_{\mathfrak{p}}$ sequence of a_N is eventually $+\infty$ because $\nu_{\mathfrak{p}}(D^N_{K,\mathfrak{p}}(a_N)) = \nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(a_1)) = \nu_{\mathfrak{p}}(0) = +\infty$. By the proof of Lemma 2.2, we know that $\nu_{\mathfrak{p}}(a_2) = 1, \nu_{\mathfrak{p}}(a_3) = 2, \ldots, \nu_{\mathfrak{p}}(a_{N-1}) = p-1$, and there does not exist a_N such that $D_{K,\mathfrak{p}}(a_N) = a_{N-1}$. Hence the conjecture is false over K for arithmetic partial derivative. On a related note, if we let $a_1 \in K \backslash \mathcal{O}_K^\times$ for some finite extension $K/\mathbb{Q}_p$, then it is possible to find an infinite sequence $a_n \in K$ such that $D_{K,\mathfrak{p}}(a_n) = a_{n-1}$ for all $n \geq 2$. For example, let $K = \mathbb{Q}$, $a_1 = p^{p^2}$, and for all $m \geq 1$, let $a_{2m} = p^{p^2+1}/(p^2+1)^m$ and $a_{2m+1} = p^{p^2}/(p^2+1)^m$. It is easy to check that $D_{\mathbb{Q},p}(a_{2m+1}) = a_{2m}$ and $D_{\mathbb{Q},p}(a_{2m}) = a_{2m-1}$.

The next proposition tells us if $\nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x)) < 0$, then the inc_p sequence of x is constant and negative. As a result of that, the $\nu_{\mathfrak{p}}$ sequence of x converges to $-\infty$.

Proposition 2.4. *Let $x \in K$ be a nonzero element such that $\nu_{\mathfrak{p}}(x) = bp^k$ with $\nu_p(b) = 0$ and $k < 0$. Then the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x is a constant sequence with negative terms*

$$(k-1, k-1, k-1, \ldots).$$

As a result, the $\nu_{\mathfrak{p}}$ sequence of x converges to $-\infty$.

Proof. Equation (1) implies that the first term of the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x is indeed $k-1$. Since

$$\nu_{\mathfrak{p}}(x) + (k-1) = bp^k + (k-1) = p^k(b + (k-1)p^{-k})$$

where $\nu_p(b + (k-1)p^{-k}) = 0$, we can write $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) = b'p^k$ where $b' := b + (k-1)p^{-k}$ with $\nu_p(b') = 0$. Since $\nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x))) = \nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x))$, we see that the second term of the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x is again $k-1$. In the meantime, we can write $\nu_{\mathfrak{p}}(D^2_{K,\mathfrak{p}}(x)) = b''p^k$ for some $b'' := b' + (k-1)p^{-k}$ where $\nu_p(b'') = 0$. By induction, we see that every term of the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x is equal to $k-1$. Therefore $\nu_{\mathfrak{p}}(D^n_{K,\mathfrak{p}}(x)) = \nu_{\mathfrak{p}}(x) + n(k-1) \to -\infty$ as $n \to \infty$. ☐

If the $\nu_{\mathfrak{p}}$ sequence of x is eventually $+\infty$, then it is periodic of period 1. For the rest of this subsection, we assume that the $\nu_{\mathfrak{p}}$ sequence of x is not eventually $+\infty$ and $\nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x)) > 0$. We will show that under these conditions, the $\nu_{\mathfrak{p}}$ sequence of x is eventually periodic of period $\leq p$. The next proposition gives a recipe of the initial terms of the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x if $\nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x)) > 0$.

Proposition 2.5. *Let $x \in K$ be a nonzero element such that $\nu_{\mathfrak{p}}(x) = bp^k$ with $\nu_p(b) = 0$ and $k > 0$. Denote $k' := (k-1 \bmod p) + 1 \leq p$. The first k' terms of the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x are*

$$(k-1, \underbrace{-1, -1, \ldots, -1}_{(k-1 \bmod p)\ copies}).$$

Proof. The first term of the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x is indeed $k-1$ by (1). We have

$$\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) = bp^k + (k-1).$$

If $k' = 1$, then there is nothing further to prove. If $k' = 2$, we have $k \equiv 2 \pmod p$ and thus $p \nmid (bp^k + (k-1))$. By (1) again, we get the second term of the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x is

$$\nu_{\mathfrak{p}}(D^2_{K,\mathfrak{p}}(x)) - \nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) = -1$$

and $\nu_{\mathfrak{p}}(D^2_{K,\mathfrak{p}}(x)) = bp^k + (k-2)$. The proof is complete by induction on k'. □

Corollary 2.6. *Let $x \in K$ be a nonzero element such that $\nu_{\mathfrak{p}}(x) = bp^k$ with $\nu_p(b) = 0$ and $1 \leq k \leq p$. Then the $\nu_{\mathfrak{p}}$ sequence and the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x are periodic of period k.*

Proof. If $1 \leq k \leq p$, then $k' = (k - 1 \bmod p) + 1 = k - 1 + 1 = k$. The first $k+1$ terms of the $\nu_{\mathfrak{p}}$ sequence are

$$(bp^k, bp^k + (k-1), bp^k + (k-2), \ldots, bp^k + 1, bp^k).$$

It is now clear that the $\nu_{\mathfrak{p}}$ sequence and the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x are periodic of period k. □

We will see later that the periodicity predicted by Corollary 2.6 will eventually happen as part of the $\nu_{\mathfrak{p}}$ sequence of x for all nonzero $x \in K$ as long as $\nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x)) \geq 0$ and the $\nu_{\mathfrak{p}}$ sequence of x is not eventually $+\infty$.

Definition 2.7. For any integer $k \geq 1$, we call the following sequence

$$\mathcal{S}_{k,p} := (k-1, \underbrace{-1, -1, \ldots, -1}_{(k-1 \bmod p) \text{ copies}})$$

the *k-segment* (with respect to p).

We define a sequence of integers $\kappa_0, \kappa_1, \kappa_2, \ldots$ recursively from $\nu_{\mathfrak{p}}(x)$ that will allow us to predict the period of the $\nu_{\mathfrak{p}}$ sequence of x. Let $\kappa_0 := \nu_{\mathfrak{p}}(x) \bmod p$ and $\kappa_1 := \nu_p(\lfloor \kappa_0 \rfloor_p)$. Here $\lfloor x \rfloor_p := x - (x \bmod p)$. For $i \geq 2$, we define

$$\kappa_i := \begin{cases} \nu_p(\lfloor \kappa_{i-1} - 1 \rfloor_p), & \text{if } \kappa_{i-1} < +\infty; \\ +\infty, & \text{if } \kappa_{i-1} = +\infty. \end{cases} \tag{2}$$

It is clear that if $1 \leq \kappa_i \leq p$, then $\kappa_{i+1} = +\infty$; if $p + 1 \leq \kappa_i < +\infty$, then $\kappa_{i+1} < \log_p(\kappa_i)$. If the $\nu_{\mathfrak{p}}$ sequence of x is not eventually $+\infty$, then there exists a unique positive integer $N = N(x)$ such that $1 \leq \kappa_N \leq p$, and $\kappa_i = +\infty$ for all $i > N$.

Theorem 2.8. *Let $x \in K$ be a nonzero element such that $\nu_{\mathfrak{p}}(x) = bp^k$ with $\nu_p(b) = 0$ and $k \geq 0$. If the $\nu_{\mathfrak{p}}$ sequence of x is not eventually $+\infty$, then the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x is of the form*

$$(\underbrace{-1, -1, \ldots, -1}_{\kappa_0 \text{ copies}}, \mathcal{S}_{\kappa_1,p}, \mathcal{S}_{\kappa_2,p}, \mathcal{S}_{\kappa_3,p}, \ldots, \mathcal{S}_{\kappa_N,p}, \mathcal{S}_{\kappa_N,p}, \mathcal{S}_{\kappa_N,p}, \ldots).$$

As a result, the $\nu_{\mathfrak{p}}$ sequence and the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x are eventually periodic of period κ_N.

Proof. For $0 \leq i \leq \kappa_0$, we have $\nu_{\mathfrak{p}}(D^i_{K,\mathfrak{p}}(x)) = b - i = \nu_{\mathfrak{p}}(x) - i$. Hence the first κ_0 terms of the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x are

$$(\underbrace{-1, -1, \ldots, -1}_{\kappa_0 \text{ copies}}).$$

We can write $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}^{\kappa_0}(x)) = b_0 p^{\kappa_1}$ with $\kappa_1 \geq 1$. By Proposition 2.5, we know that the next $\kappa_1' := (\kappa_1 - 1 \bmod p) + 1$ term of the $\mathrm{inc}_{\mathfrak{p}}$ sequence is the κ_1-segment

$$\mathcal{S}_{\kappa_1,p} = (\kappa_1 - 1, \underbrace{-1, -1, \ldots, -1}_{(\kappa_1 - 1 \bmod p) \text{ copies}}).$$

Furthermore, we get $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}^{\kappa_0+i}(x)) = bp^{\kappa_1} + (\kappa_1 - i)$ for $0 \leq i \leq \kappa_1'$. As $\kappa_1 - \kappa_1' = \lfloor \kappa_1 - 1 \rfloor_p$ and $\kappa_2 = \nu_p(\lfloor \kappa_1 - 1 \rfloor_p)$, we can write $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}^{\kappa_0+\kappa_1'+1}(x)) = b_1 p^{\kappa_2}$. If $\kappa_2 \geq 1$, by Proposition 2.5 again, we know that the next $\kappa_2' := (\kappa_2 - 1 \bmod p) + 1$ term of the $\mathrm{inc}_{\mathfrak{p}}$ sequence is the κ_2-segment. Let $N = N(x)$ be the unique positive integer such that $1 \leq \kappa_N \leq p$. By induction, we know that the initial terms of the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x is of the form

$$(\underbrace{-1, -1, \ldots, -1}_{\kappa_0 \text{ copies}}, \mathcal{S}_{\kappa_1,p}, \mathcal{S}_{\kappa_2,p}, \mathcal{S}_{\kappa_3,p}, \ldots, \mathcal{S}_{\kappa_N,p}).$$

Corollary 2.6 implies that if $b_{N-1}p^{\kappa_N}$ is a term in the $\nu_{\mathfrak{p}}$ sequence of x, then $\mathcal{S}_{k_N,p}$ will appear repeatedly in the $\mathrm{inc}_{\mathfrak{p}}$ sequence of x. This ends of the proof of the theorem. □

2.3 Anti-partial derivatives

We fix a finite extension $K/\mathbb{Q}_p$ in this subsection. Note that not all elements in K have an anti-partial derivative. For example, suppose $x \in K$ is an anti-partial derivative of $p^{p-1} \in K$, then $D_{K,\mathfrak{p}}^{p+1}(x) = 0$ and thus the $\nu_{\mathfrak{p}}$ sequence of x is eventually $+\infty$. By Lemma 2.2, $\nu_{\mathfrak{p}}(x) \in \{0, 1, 2, \ldots, p-1, +\infty\}$, but that is not possible as $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) = p - 1$. Therefore p^{p-1} does not have anti-partial derivative in K. Given an element $y \in K$, if y has an anti-partial derivative in K, we want to know how many there are. We start with $y = 0$. Let $x \in K$ such that

$$D_{K,\mathfrak{p}}(x) = \frac{x\nu_{\mathfrak{p}}(x)}{p} = 0.$$

Then $x\nu_{\mathfrak{p}}(x) = 0$ which implies that $x = 0$ or $\nu_{\mathfrak{p}}(x) = 0$. Hence the anti-partial derivative of 0 in K is

$$\{x \in K \, : \, \nu_{\mathfrak{p}}(x) = 0\} \cup \{0\}.$$

Lemma 2.9. *For every $0 \neq y \in K$, if there exists $x \in K$ such that $D_{K,\mathfrak{p}}(x) = y$, then $x \in \mathbb{Q}_p(y)$.*

Proof. Since $D_{K,\mathfrak{p}}(x) = x\nu_{\mathfrak{p}}(x)/p = y$ and $\nu_{\mathfrak{p}}(x)/p \in \mathbb{Q}$, we know that $x \in \mathbb{Q}_p(y)$. □

Let $x_1, x_2 \in K$ with $D_{K,\mathfrak{p}}(x_1) = D_{K,\mathfrak{p}}(x_2)$. If $\nu_{\mathfrak{p}}(x_1) = 0$, then $D_{K,\mathfrak{p}}(x_1) = 0 = D_{K,\mathfrak{p}}(x_2)$. Thus $\nu_{\mathfrak{p}}(x_2) = 0$. Hence $\nu_{\mathfrak{p}}(x_1) = 0$ if and only if $\nu_{\mathfrak{p}}(x_2) = 0$.

Suppose $\nu_{\mathfrak{p}}(x_1), \nu_{\mathfrak{p}}(x_2) \neq 0$. Let $\nu_{\mathfrak{p}}(x_1) = b_1 p^{k_1}$ and $\nu_{\mathfrak{p}}(x_2) = b_2 p^{k_2}$ where $\nu_p(b_1 b_2) = 0$. We get

$$b_1 p^{k_1} - b_2 p^{k_2} = k_2 - k_1. \tag{3}$$

Suppose $k_1 = k_2$, then (3) implies that $\nu_{\mathfrak{p}}(x_1) = \nu_{\mathfrak{p}}(x_2)$. Hence

$$x_1 = \frac{D_{K,\mathfrak{p}}(x_1)p}{\nu_{\mathfrak{p}}(x_1)} = \frac{D_{K,\mathfrak{p}}(x_2)p}{\nu_{\mathfrak{p}}(x_2)} = x_2.$$

This means that $x_1 = x_2$ if and only if $k_1 = k_2$.

If $k_1 \neq k_2$, without loss of generality, we assume $k_1 < k_2$. Suppose $k_1 < 0$, then (3) implies that

$$b_1 - b_2 p^{k_2-k_1} = p^{-k_1}(k_2 - k_1).$$

This is a contradiction because $\nu_{\mathfrak{p}}(b_1 - b_2 p^{k_2-k_1}) = 0$ and $\nu_{\mathfrak{p}}(p^{-k_1}(k_2-k_1)) \geq -k_1 > 0$. Hence if $k_1 < 0$, then $D_{K,\mathfrak{p}}(x_1)$ has exactly one anti-partial derivative.

Suppose $k_1 > 0$. There is an element $x_0 \in K$ in the set of all anti-partial derivatives of $D_{K,\mathfrak{p}}(x_1)$ with the smallest possible k_0. We call x_0 the *primitive* anti-partial derivative of $D_{K,\mathfrak{p}}(x_1)$. Equation (3) implies that

$$b_0 p^{k_0} - b p^{k_1} = k_1 - k_0, \tag{4}$$

As x_0 is primitive, we have $k_0 \leq k_1$ and (4) implies that $p^{k_0}(b_0 - bp^{k_1-k_0}) = k_1 - k_0$. Let $k_1 - k_0 = p^{k_0}c$ for some $c \in \mathbb{Z}_{\geq 0}$. Then $b_0 - bp^{p^{k_0}c} = c$. So $b = \dfrac{b_0 - c}{p^{p^{k_0}c}}$ and $\nu_p(b_0 - c) = p^{k_0}c$ since $\nu_p(b) = 0$. Let

$$C(x_0) := \Big\{ c \in \mathbb{Z}_{\geq 0} \,:\, \nu_p(b_0 - c) = p^{k_0}c \Big\}.$$

It is easy to see that $C(x_0)$ is finite because as $c \gg 0$, $\nu_p(b_0 - c) < p^{k_0}c$.

Theorem 2.10. *With the above notations, suppose x_0 is the primitive anti-partial derivative of $D_{K,\mathfrak{p}}(x_0)$. Let $\nu_{\mathfrak{p}}(x_0) = b_0 p^{k_0}$ with $\nu_p(b_0) = 0$ and $k_0 > 0$. There is a one-to-one correspondence between $C(x_0)$ and the set of all anti-partial derivatives of $D_{K,\mathfrak{p}}(x_0)$. Furthermore, suppose we fix a uniformizer $\pi \in \mathfrak{p} \subset \mathcal{O}_K$ and let e be the ramification index of $K/\mathbb{Q}_p$, we can write $x_0 = \alpha_0 \pi^{eb_0 p^{k_0}}$ and $p = \alpha_p \pi^e$ with $\alpha_0, \alpha_p \in \mathcal{O}_K^{\times}$. If $x = \alpha\pi^{ebp^k}$ is an anti-partial derivative of $D_{K,\mathfrak{p}}(x_0)$ such that $\nu_p(b) = 0$ and $\alpha \in \mathcal{O}_K^{\times}$, then there exists a unique $c \in C(x_0)$ such that*

$$k = p^{k_0}c + k_0 \in \mathbb{Z}_{\geq 0}, \quad b = \frac{b_0 - c}{p^{k-k_0}} = \frac{b_0 - c}{p^{p^{k_0}c}}, \quad \alpha = \frac{\alpha_0 b_0}{b}\alpha_p^{k_0-k} \in \mathcal{O}_K^{\times}.$$

Proof. We show that every anti-partial derivative x of $D_{K,\mathfrak{p}}(x_0)$ is associated with a unique $c \in C(x_0)$. If $x = x_0$, then we associate x with $c = 0$. Suppose $x \neq x_0$. Let $\nu_{\mathfrak{p}}(x) = bp^k$. Since x_0 is the primitive anti-partial derivative and $\nu_{\mathfrak{p}}(x_0) \neq 0$, we know that $b \neq 0$ and $k > k_0$. Then $p^{k_0}(b_0 - bp^{k-k_0}) = k - k_0$ and thus $\nu_p(k - k_0) = k_0$. Let $k - k_0 = p^{k_0}c$ where $c > 0$ and $\nu_p(c) = 0$. By plugging $k - k_0 = p^{k_0}c$ into $p^{k_0}(b_0 - bp^{k-k_0}) = k - k_0$, we get $b_0 - bp^{k-k_0} = c$. Since $\nu_p(b) = 0$, we know that $\nu_p(b_0 - c) = p^{k_0}c$.

Then we show that for each $c \in C(x_0)$, we can define a unique $x = x(c)$ such that $D_{K,\mathfrak{p}}(x) = D_{K,\mathfrak{p}}(x_0)$. Since $\nu_p(b_0 - c) = p^{k_0}c$, there exists $b \in \mathbb{Q}$ with $\nu_p(b) = 0$ such that $b_0 - c = bp^{p^{k_0}c}$. Set $k := p^{k_0}c + k_0$. We can compute

$$\begin{aligned} bp^k + k - 1 &= \frac{b_0 - c}{p^{k-k_0}}p^k + k - 1 = (b_0 - c)p^{k_0} + p^{k_0}c + k_0 - 1 \\ &= (b_0 - c)p^{k_0} + p^{k_0}c + k_0 - 1 = b_0 p^{k_0} + k_0 - 1. \end{aligned}$$

Set $x := \alpha\pi^{ebp^k}$ where $\alpha = \alpha_0 b_0 \alpha_p^{k_0-k}/b$. We have

$$\begin{aligned} D_{K,\mathfrak{p}}(x) &= \frac{x\nu_{\mathfrak{p}}(x)}{p} = \frac{\alpha\pi^{ebp^k}ebp^k}{p} = \alpha be\pi^{ebp^k}p^{k-1} = \alpha be\alpha_p^{k-1}\pi^{e(bp^k+k-1)} \\ &= \alpha_0 b_0 e\alpha_p^{k_0-1}\pi^{e(b_0p^{k_0}+k_0-1)} = \frac{\alpha_0 b_0 e}{p}\pi^{eb_0p^{k_0}}p^{k_0} = \frac{x_0\nu_{\mathfrak{p}}(x_0)}{p} = D_{K,\mathfrak{p}}(x_0). \end{aligned}$$

□

Corollary 2.11. *For any nonzero $y \in K$, the set $\{x \in K : D_{K,\mathfrak{p}}(x) = y\}$ is finite (possibly empty).*

For the rest of this subsection, we will prove Conjecture 1.2 for partial derivatives over any finite extension $K/\mathbb{Q}_p$. We will show that for each positive integer n, there exists infinitely many $x \in \mathbb{Q}_p$ such that $D_{\mathbb{Q}_p,p}(x)$ has exactly n anti-partial derivatives in $\mathbb{Q}_p$. By Lemma 2.9, we know that all anti-partial derivatives of $D_{\mathbb{Q}_p,p}(x)$ must be in $\mathbb{Q}_p$ and thus $D_{\mathbb{Q}_p,p}(x)$ has exactly n anti-partial derivatives in any finite extension $K/\mathbb{Q}_p$. The first lemma gives us a way to construct $k_0 \in \mathbb{Z}_{>0}$ such that if $\nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x_0)) = k_0$, then x_0 is the primitive anti-partial derivative of $D_{K,\mathfrak{p}}(x_0)$.

Lemma 2.12. *For every integer $m \geq 2$, let $k_0 = k_0(m) := p + p^2 + \cdots + p^m$. For every $x_0 \in \mathbb{Q}_p$, if $\nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x_0)) = k_0$, then x_0 is the primitive anti-partial derivative of $D_{\mathbb{Q}_p,p}(x_0)$.*

Proof. Suppose x_0 is not the primitive anti-partial derivative of $D_{\mathbb{Q}_p,p}(x_0)$. Let $x \neq x_0$ be another anti-partial derivative of $D_{\mathbb{Q}_p,p}(x_0)$ with $\nu_{\mathfrak{p}}(x) = bp^k$ such that $k < k_0$. If $k < 0$, we know that $D_{\mathbb{Q}_p,p}(x_0)$ has exactly one anti-partial derivative. Hence $k \geq 0$. Since $D_{\mathbb{Q}_p,p}(x) = D_{\mathbb{Q}_p,p}(x_0)$, we get $bp^k - b_0p^{k_0} = k_0 - k$. This means that $\nu_p(k_0 - k) = k$. It suffices to show that no $0 \leq k < k_0$ satisfies this relation. It is clear that $k \neq 0$ because $\nu_p(k_0) = 1$, and $k \neq 1$ because $\nu_p(k_0 - 1) = 0$. Suppose $k > 1$. If $\nu_p(k_0 - k') = k$ for some $k' > 0$, then $k' \geq p + \cdots + p^{k-1} > k$. Therefore there does not exist an anti-partial derivative x with $k < k_0$. This means that x_0 is the primitive anti-partial derivative of $D_{\mathbb{Q}_p,p}(x_0)$. □

The next lemma allows us to construct b_0 for every $k_0 > 0$ such that there are exactly $n - 1$ different possible values of $c \in \mathbb{Z}_{>0}$ such that $\nu_p(b_0 - c) = p^{k_0}c$. This means that the set $C(x_0)$ has exactly n elements (with 0 included).

Lemma 2.13. *Fix a positive integer k. Let $c_1 = 0$, and for $i \geq 2$, let $c_i := p^{p^k c_{i-1}} + c_{i-1}$. Suppose*

$$C_n := \{c \in \mathbb{Z}_{>0} \ : \ \nu_p(c_{n+1} - c) = p^k c\}.$$

Then $C_n = \{c_2, \ldots, c_n\}$.

Proof. We first note that for any $1 \leq i < j$,

$$c_j - c_i = \sum_{m=i}^{j-1} (c_{m+1} - c_m) = \sum_{m=i}^{j-1} p^{p^k c_m}$$

and so $\nu_p(c_j - c_i) = p^k c_i$. This shows that $c_m \in C_n$ if and only if $m \in \{2, 3, \ldots, n\}$.

Next, we show that no other integers are in C_n. If $c \in C_n$ where $c > c_{n+1}$, then $c - c_{n+1} = \alpha p^{p^k c}$, where $\alpha > 0$. By definition of c_{n+1}, $c - c_{n+1} = c - (c_n + p^{p^k c_n})$. Thus

$$c - c_n = \alpha\, p^{p^k c} + p^{p^k c_n} = p^{p^k c_n}\left(\alpha\, p^{p^k (c - c_n)} + 1\right).$$

This is a contradiction, since the expression on the right hand side is clearly larger than $c - c_n$. This shows that if $c \in C_n$, then $c \leq c_{n+1}$.

Suppose $c \in C_n$ where $c_m < c < c_{m+1}$ for some $2 \le m \le n$. We have $\nu_p(c_{n+1} - c_{m+1}) = p^k c_{m+1}$ when $m < n$. Since $\nu_p(c_{n+1} - c) = p^k c$, we have

$$\nu_p(c_{m+1} - c) = \nu_p\Big((c_{n+1} - c) - (c_{n+1} - c_{m+1})\Big) = p^k c.$$

Therefore $c_{m+1} - c = \gamma p^{p^k c}$ for some $\gamma > 0$. By definition, $c_{m+1} = p^{p^k c_m} + c_m$, and so we would have

$$p^{p^k c_m} + c_m = \gamma p^{p^k c} + c,$$

which is a contradiction, since the left side is clearly less than the right. This shows that if $c \in C_n$ and $c \le c_{n+1}$, then $c = c_m$ for some $2 \le m \le n$. This concludes the proof of the lemma. □

Theorem 2.14. *For each positive integer n, there are infinitely many $x_0 \in K$ such that $D_{K,\mathfrak{p}}(x_0)$ has exactly n anti-partial derivatives in K.*

Proof. By Lemma 2.9, it suffices to assume that $K = \mathbb{Q}_p$ and $\mathfrak{p} = (p)$. For every integer $m \ge 2$, let $k_0 = k_0(m)$ be defined as in Lemma 2.12, and let $b_0 = c_{n+1}$ be defined as in Lemma 2.13 for $k = k_0$. Set $x_0 := p^{b_0 p^{k_0}}$. Lemma 2.12 implies that x_0 is the primitive anti-partial derivative of $D_{\mathbb{Q}_p,p}(x_0)$. Lemma 2.13 implies that $D_{\mathbb{Q}_p,p}(x_0)$ has exactly n anti-partial derivatives. Therefore, for each positive integer n, there exists infinitely many $x_0 \in \mathbb{Q}_p$ such that $D_{\mathbb{Q}_p,p}(x_0)$ has exactly n anti-partial derivatives with x_0 being its primitive anti-partial derivative. □

3 Number fields

In this section, we will generalize arithmetic derivative and arithmetic partial derivative to number fields. Recall that the explicit formula of the arithmetic derivative defined on $\mathbb{Q}$:

$$D_{\mathbb{Q}}(x) = x \sum_{p \mid x} \frac{\nu_p(x)}{p}.$$

Let $K/\mathbb{Q}$ be a number field of finite degree. One could mimic the above formula and define the arithmetic derivative on K by the formula:

$$D_K(x) = x \sum_{\mathfrak{p} \mid x} \frac{\nu_{\mathfrak{p}}(x)}{p},$$

where $\mathfrak{p}$ are prime ideals in $\mathcal{O}_K$. The sum is finite as there are finitely many $\mathfrak{p}$ such that $\nu_{\mathfrak{p}}(x) \neq 0$. This formula presents a challenge. Let $p \in \mathbb{Q}$ be a rational prime. Then

$$D_K(p) = p \sum_{\mathfrak{p} \mid p} \frac{\nu_{\mathfrak{p}}(p)}{p} = \sum_{\mathfrak{p} \mid p} \nu_{\mathfrak{p}}(p) = g(p, K) \cdot 1 = g(p, K),$$

where $g(p, K)$ is the number of prime ideals in $\mathcal{O}_K$ that divide p. When $g(p, K) \neq 1$, $D_K(p) \neq D_{\mathbb{Q}}(p)$ so the above formula of D_K does not give a true extension of $D_{\mathbb{Q}}$. In order for $D_K(x) = D_{\mathbb{Q}}(x)$ for all $x \in \mathbb{Q}$, we will need to divide $g(p, K)$. Furthermore, let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two prime ideals in $\mathcal{O}_K$ that divide p and let L/K be a finite extension. We know that in general $g(\mathfrak{p}_1, L) \neq g(\mathfrak{p}_2, L)$ unless L/K is finite Galois. So we will start the generalization of $D_{\mathbb{Q}}$ to finite Galois extensions. Then we can further generalize $D_{\mathbb{Q}}$ to number fields by taking restriction.

3.1 Finite Galois extensions

Let K be a finite Galois extension of $\mathbb{Q}$ of degree n. Let $\mathcal{O}_K$ be the ring of integers and $\mathfrak{p}$ a nonzero prime ideal of $\mathcal{O}_K$ such that $p \in \mathfrak{p}$. There is a discrete valuation $\nu_{\mathfrak{p}}$ on K that extends the p-adic valuation ν_p on $\mathbb{Q}$. This induces a norm $|\cdot|_{\nu_{\mathfrak{p}}} = [\mathcal{O}_K : \mathfrak{p}]^{-\nu_{\mathfrak{p}}(\cdot)}$ on K. Let $K_{\nu_{\mathfrak{p}}}$ be the completion of K with respect to the $\mathfrak{p}$-adic topology and thus $K_{\nu_{\mathfrak{p}}}$ is a finite extension of $\mathbb{Q}_p$ such that $\mathfrak{p} \cap \mathbb{Q} = (p)$ (denoted by $\mathfrak{p} \mid p$). Let $e(K_{\nu_{\mathfrak{p}}}|\mathbb{Q}_p)$ be the ramification index and $f(K_{\nu_{\mathfrak{p}}}|\mathbb{Q}_p) := [\mathcal{O}_K/\mathfrak{p} : \mathbb{F}_p]$ be the inertia degree of the extension $K_{\nu_{\mathfrak{p}}}/\mathbb{Q}_p$. One has the following decomposition:

$$p\mathcal{O}_K = \prod_{\mathfrak{p}|p} \mathfrak{p}^{e(K_{\nu_{\mathfrak{p}}}|\mathbb{Q}_p)}.$$

It is well known that for every fixed prime number p, we have the formula

$$n = \sum_{\mathfrak{p}|p} e(K_{\nu_{\mathfrak{p}}}|\mathbb{Q}_p) f(K_{\nu_{\mathfrak{p}}}|\mathbb{Q}_p). \tag{5}$$

The Galois group $G(K/\mathbb{Q})$ acts transitively on the set of prime ideals $\{\mathfrak{p} \subset \mathcal{O}_K : \mathfrak{p} \mid p\}$ for every fixed prime $p \in \mathbb{Q}$ [13, Chapter 1, Section 7, Proposition 19]. This implies that for every nonzero prime ideal $\mathfrak{p} \mid p$, the ramification index $e(K_{\nu_{\mathfrak{p}}}|\mathbb{Q}_p)$ and the inertia degree $f(K_{\nu_{\mathfrak{p}}}|\mathbb{Q}_p)$ depend only on p. If we denote them by $e(p, K)$ and $f(p, K)$ respectively, then formula (5) becomes

$$n = e(p, K) f(p, K) g(p, K), \tag{6}$$

where $g(p, K)$ (again only depends on p) is the number of distinct prime ideals $\mathfrak{p}$ such that $\mathfrak{p} \mid p$. Now we can extend the arithmetic derivative $D_{\mathbb{Q}}$ to K. For every nonzero $x \in K$, we define

$$D_K(x) := x \sum_{\mathfrak{p}|x} \frac{\nu_{\mathfrak{p}}(x)}{pg(p, K)}.$$

One can check that D_K satisfies the Leibniz rule:

$$\begin{aligned} D_K(xy) &= xy \sum_{\mathfrak{p}|xy} \frac{\nu_{\mathfrak{p}}(xy)}{pg(p,K)} = xy \sum_{\mathfrak{p}|xy} \frac{\nu_{\mathfrak{p}}(x) + \nu_{\mathfrak{p}}(y)}{pg(p,K)} \\ &= \Big(\sum_{\mathfrak{p}|xy} \frac{x\nu_{\mathfrak{p}}(x)}{pg(p,K)} \Big) y + x \Big(\sum_{\mathfrak{p}|xy} \frac{y\nu_{\mathfrak{p}}(y)}{pg(p,K)} \Big) \\ &= \Big(\sum_{\mathfrak{p}|x} \frac{x\nu_{\mathfrak{p}}(x)}{pg(p,K)} \Big) y + x \Big(\sum_{\mathfrak{p}|y} \frac{y\nu_{\mathfrak{p}}(y)}{pg(p,K)} \Big) \\ &= D_K(x)y + xD_K(y). \end{aligned}$$

It is easy to check that $D_K(0) = 0$. To check that $D_K : K \to K$ extends $D_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{Q}$, recall that for every prime p, we have $\nu_{\mathfrak{p}}(x) = \nu_p(x)$ for every $x \in \mathbb{Q}$. And so for every nonzero $x \in \mathbb{Q}$, we get

$$D_K(x) = x \sum_{\mathfrak{p}|x} \frac{\nu_{\mathfrak{p}}(x)}{pg(p,K)} = x \sum_{p|x} \Big(\frac{g(p,K) \cdot \nu_p(x)}{pg(p,K)} \Big) = x \sum_{p|x} \frac{\nu_p(x)}{p} = D_{\mathbb{Q}}(x).$$

3.2 Number fields

Let $K/\mathbb{Q}$ be a number field and let L/K be an extension such that $L/\mathbb{Q}$ is finite Galois (e.g., one can take L to be a Galois closure of $K/\mathbb{Q}$). For every $x \in K$, one can define $D_K(x) = D_L(x)$. But we want to make sure that $D_L(x) = D_K(x)$ for all $x \in K$ so the definition of D_K does not depend on the choice of Galois extensions.

Lemma 3.1. *Suppose $K/\mathbb{Q}$ and $L/\mathbb{Q}$ are finite Galois extensions. We have $D_K(x) = D_L(x)$ for every $x \in K \cap L$.*

Proof. We first assume that $K \subset L$. Since $L/\mathbb{Q}$ is Galois, we know that L/K is also Galois. For every rational prime p and nonzero prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of $\mathcal{O}_K$ with $\mathfrak{p}_1 \mid p$ and $\mathfrak{p}_2 \mid p$, we get $g(\mathfrak{p}_1, L) = g(\mathfrak{p}_2, L)$. Let $\mathfrak{p}$ and $\mathfrak{P}$ be two prime ideals in $\mathcal{O}_K$ and O_L respectively such that $\mathfrak{P} \mid \mathfrak{p} \mid p$. For every nonzero $x \in K$, we have

$$\begin{aligned} D_L(x) &= x \sum_{\mathfrak{P}|x} \frac{\nu_{\mathfrak{P}}(x)}{pg(p, L)} = x \sum_{\mathfrak{p}|x} \sum_{\mathfrak{P}|\mathfrak{p}} \frac{\nu_{\mathfrak{p}}(x)}{pg(p, L)} \\ &= x \sum_{\mathfrak{p}|x} \frac{g(\mathfrak{p}, L)\nu_{\mathfrak{p}}(x)}{pg(\mathfrak{p}, L)g(p, K)} = x \sum_{\mathfrak{p}|x} \frac{\nu_{\mathfrak{p}}(x)}{pg(p, K)} = D_K(x). \end{aligned}$$

This shows that $D_K(x) = D_L(x)$ for all $x \in K$ if $K \subset L$.

Now suppose $K/\mathbb{Q}$ and $L/\mathbb{Q}$ are two arbitrary finite Galois extensions. Since $K \cap L$ is also a finite Galois extension of $\mathbb{Q}$, for every $x \in K \cap L$, we have $D_K(x) = D_{K\cap L}(x)$ by the previous paragraph. Using the same argument, we get $D_L(x) = D_{K\cap L}(x)$ for every $x \in K \cap L$, and therefore $D_K(x) = D_L(x)$ for every $x \in K \cap L$. □

Suppose $K/\mathbb{Q}$ is a number field (not necessarily Galois). For every $x \in K$, we can define $D_K(x) := D_{K^{\mathrm{Gal}}}(x)$ where K^{Gal} is a Galois closure of $K/\mathbb{Q}$. When $x \neq 0$, it is clear that $D_K(x)/x \in \mathbb{Q}$ and thus $D_K(x) \in K$. We have a well-defined arithmetic derivative $D_K : K \to K$ when K is a number field.

3.3 Arithmetic subderivative

Let S be a (finite or infinite) subset of the prime numbers $\mathbb{P}$. One can define the so-called arithmetic subderivative $D_{\mathbb{Q},S} : \mathbb{Q} \to \mathbb{Q}$ by

$$D_{\mathbb{Q},S}(x) = \sum_{p\in S} x\nu_p(x)/p.$$

It is easy to see that $D_{\mathbb{Q},S} = \sum_{p\in S} D_p$ and $D_{\mathbb{Q}} = \sum_{p\in\mathbb{P}} D_p$. One can extend $D_{\mathbb{Q},S}$ to all finite Galois extensions $K/\mathbb{Q}$. Let T be a set of prime ideals of $\mathcal{O}_K$. For every nonzero $x \in K$, we define

$$D_{K,T}(x) := x \sum_{\mathfrak{p}\in T, \mathfrak{p}|p} \frac{\nu_{\mathfrak{p}}(x)}{pg(p, K)}.$$

If $T = \{\mathfrak{p}\}$ contains only one prime ideal, then we call $D_{K,T} = D_{K,\mathfrak{p}}$ the arithmetic partial derivative with respect to $\mathfrak{p}$. By taking $K = \mathbb{Q}$ and $\mathfrak{p} = \{p\}$, we can see $D_{K,\mathfrak{p}}$ is the generalization of arithmetic partial derivative with respect to p. Suppose L/K is a finite Galois extension. Let

$$T_{L/K} = \{\mathfrak{P} : \mathfrak{P} \text{ prime ideal of } \mathcal{O}_L, \exists\, \mathfrak{p} \in T \text{ such that } \mathfrak{P} \mid \mathfrak{p}\}.$$

For every nonzero $x \in K$, we have

$$\begin{aligned} D_{L,T_{L/K}}(x) &= \sum_{\mathfrak{P} \in T_{L/K}, \mathfrak{P}|p} \frac{x\nu_{\mathfrak{P}}(x)}{pg(p,L)} = \sum_{\mathfrak{p} \in T} \sum_{\mathfrak{P} \in T_{L/K}, \mathfrak{P}|\mathfrak{p}} \frac{x\nu_{\mathfrak{P}}(x)}{pg(p,L)} \\ &= \sum_{\mathfrak{p} \in T} g(\mathfrak{p}, L) \frac{x\nu_{\mathfrak{p}}(x)}{pg(p,K)g(\mathfrak{p},L)} = \sum_{\mathfrak{p} \in T} \frac{x\nu_{\mathfrak{p}}(x)}{pg(p,K)} = D_{K,T}(x). \end{aligned}$$

In this case, $D_{L,T_{L/K}}$ extends $D_{K,T}$.

If $K/\mathbb{Q}$ is a number field (not necessarily Galois), we can define $D_{K,T}$ via a larger Galois extension. Let L/K be a finite extension such that $L/\mathbb{Q}$ is Galois. Let $T_{L/K}$ be defined as above. We can define $D_{K,T}(x) := D_{L,T_{L/K}}(x)$ for all $x \in K$. Again this definition does not depend on the choice of Galois extensions. Let L_1/K and L_2/K be finite extensions such that $L_1/\mathbb{Q}$ and $L_2/\mathbb{Q}$ are Galois. Let $L_3 := L_1 \cap L_2$ and $T' := T_{L_3/K}$. We note that $T_{L_1/K} = T'_{L_1/L_3}$ and $T_{L_2/K} = T'_{L_2/L_3}$. Therefore for every $x \in K \subset L_3$, we have

$$D_{L_1,T_{L_1/K}}(x) = D_{L_1,T'_{L_1/L_3}}(x) = D_{L_3,T'}(x) = D_{L_2,T'_{L_2/L_3}}(x) = D_{L_2,T_{L_2/K}}(x).$$

Remark 3.2. Let $K/\mathbb{Q}$ be a finite Galois extension. Just like in the local case, one can ask whether Theorems 1.3 and 1.4 are true for $D_{K,\mathfrak{p}}$. Note that in the global case $D_{K,\mathfrak{p}}(x) = \frac{x\nu_{\mathfrak{p}}(x)}{pg(p,K)}$, whereas in the local case $g(p,K) = 1$. If $\nu_p(g(p,K)) = 0$, then $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) = \nu_{\mathfrak{p}}(x) + \nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x)) - 1$, which is the same as Equation (1). In this case, Theorems 1.3 and 1.4 are still true and can be proved in a similar fashion. If $\nu_p(g(p,K)) = a > 0$, then $\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)) = \nu_{\mathfrak{p}}(x) + \nu_{\mathfrak{p}}(\nu_{\mathfrak{p}}(x)) - 1 - a$. In this case, the behavior of the $\nu_{\mathfrak{p}}$ sequence of x warrants further study.

4 Arithmetic logarithmic derivative

4.1 Local case

The logarithmic partial derivative (with respect to p) $\mathrm{ld}_{\mathbb{Q},p} : \mathbb{Q}^\times \to \mathbb{Q}$ is a homomorphism defined by the formula

$$\mathrm{ld}_{\mathbb{Q},p}(x) = D_{\mathbb{Q},p}(x)/x$$

because

$$\mathrm{ld}_{\mathbb{Q},p}(xy) = \frac{D_{\mathbb{Q},p}(xy)}{xy} = \frac{D_{\mathbb{Q},p}(x)y + xD_{\mathbb{Q},p}(y)}{xy} = \mathrm{ld}_{\mathbb{Q},p}(x) + \mathrm{ld}_{\mathbb{Q},p}(y).$$

The image of $\mathrm{ld}_{\mathbb{Q},p}$ is

$$\mathrm{ld}_{\mathbb{Q},p}(\mathbb{Q}^\times) = \{m/p : m \in \mathbb{Z}\} = \langle 1/p \rangle \cong \mathbb{Z}$$

and thus $\mathrm{ld}_{\mathbb{Q},p}$ is not onto. Suppose $\mathrm{ld}_{\mathbb{Q},p}(x) = 0$, then $D_{\mathbb{Q},p}(x) = 0$ and thus $\nu_p(x) = 0$. Therefore

$$\mathrm{Ker}(\mathrm{ld}_{\mathbb{Q},p}) = \{x \in \mathbb{Q}^\times : \nu_p(x) = 0\}.$$

One can extend $\mathrm{ld}_{\mathbb{Q},p}$ to $\mathbb{Q}_p^\times$ by the formula $\mathrm{ld}_{\mathbb{Q}_p,p}(x) := D_{\mathbb{Q}_p,p}(x)/x \in \mathbb{Q}$. Using the same argument, we get

$$\mathrm{ld}_{\mathbb{Q}_p,p}(\mathbb{Q}_p^\times) = \{m/p : m \in \mathbb{Z}\}, \qquad \mathrm{Ker}(\mathrm{ld}_{\mathbb{Q}_p,p}) = \{x \in \mathbb{Q}_p^\times : \nu_p(x) = 0\}.$$

Let $K/\mathbb{Q}_p$ be a finite extension. We can define $\mathrm{ld}_{K,\mathfrak{p}} : K^\times \to \mathbb{Q}$ as

$$\mathrm{ld}_{K,\mathfrak{p}}(x) := \frac{D_{K,\mathfrak{p}}(x)}{x} = \frac{\nu_{\mathfrak{p}}(x)}{p}.$$

It is easy to see the kernel of $\mathrm{ld}_{K,\mathfrak{p}}$ is

$$\mathrm{Ker}(\mathrm{ld}_{K,\mathfrak{p}}) = \{x \in K^\times : \nu_{\mathfrak{p}}(x) = 0\}.$$

The description of the image of $\mathrm{ld}_{K,\mathfrak{p}}$ depends on whether p divides the ramification index e. Let $e = p_1^{r_1} p_2^{r_2} \cdots p_j^{r_j}$ be the unique factorization of the ramification index into prime powers. If $p \notin \{p_1, p_2, \ldots, p_j\}$, then

$$\mathrm{ld}_{K,\mathfrak{p}}(K^\times) = \{m/pe : m \in \mathbb{Z}\} = \langle 1/p, 1/p_1^{r_1}, \ldots, 1/p_j^{r_j}\rangle \cong \mathbb{Z}.$$

If $p \in \{p_1, p_2, \ldots, p_j\}$ and assume $p = p_1$, then

$$\mathrm{ld}_{K,\mathfrak{p}}(K^\times) = \{m/pe : m \in \mathbb{Z}\} = \langle 1/p_1^{r_1+1}, 1/p_2^{r_2}, \ldots, 1/p_j^{r_j}\rangle \cong \mathbb{Z}.$$

4.2 Global case

If $K/\mathbb{Q}$ is a finite Galois extension, one can define the arithmetic logarithmic derivative $\mathrm{ld}_K : K^\times \to \mathbb{Q}$ as

$$\mathrm{ld}_K(x) = \frac{D_K(x)}{x} = \sum_{\mathfrak{p}|x} \frac{\nu_{\mathfrak{p}}(x)}{pg(p,K)} \in \mathbb{Q}.$$

It is easy to show that ld_K is a group homomorphism. When $K = \mathbb{Q}$, we get that $\mathrm{ld}_{\mathbb{Q}}(x) = \sum_{p|x} \frac{\nu_p(x)}{p}$. Hence $\mathrm{ld}_{\mathbb{Q}}(\mathbb{Q}^\times) = \langle \frac{1}{p} : p \in \mathbb{P}\rangle$. For every finite Galois extension $K/\mathbb{Q}$, one can show that $\mathrm{ld}_K(K^\times)$ are isomorphic as subgroups of $\mathbb{Q}$. Before we prove this result, we need to recall a concept called p-height in the classification of subgroups of $\mathbb{Q}$. Let G be an (additive) subgroup of $\mathbb{Q}$ and $g \in G$. The p-height of g in G is k if $p^k x = g$ is solvable in G and $p^{k+1}x = g$ is not. If $p^k x = a$ has a solution for every k, then we say that the p-height of a in G is infinite. Let $H_{p_i,G}(g)$ be the p_i-height of g in G. Set $H_G(g) := (H_{2,G}(g), H_{3,G}(g), H_{5,G}(g), \ldots)$. It turned out that $H_G(1)$ is an invariant of the subgroup G in the following sense.

Theorem 4.1. *[8, Theorem 4] Let G_1 and G_2 be two subgroups of $\mathbb{Q}$. Then $G_1 \cong G_2$ if and only if $H_{G_1}(1)$ and $H_{G_2}(1)$ only differ in finitely many indices, and in the case $H_{p_i,G_1}(1) \neq H_{p_i,G_2}(1)$, both of them are finite.*

Theorem 4.2. *Let $K/\mathbb{Q}$ be a finite Galois extension. Then $\mathrm{ld}_K(K^\times) \cong \langle \frac{1}{p} : p \in \mathbb{P}\rangle < \mathbb{Q}$.*

Proof. Let $G := \langle \frac{1}{p} : p \in \mathbb{P} \rangle < \mathbb{Q}$. It is easy to see that

$$H_G = (1, 1, 1, \ldots).$$

Let $[K : \mathbb{Q}] = n$ and $\overline{\nu}_{\mathfrak{p}}(x) := \nu_{\mathfrak{p}}(x) e(p, K)$ be the normalized discrete valuation. For every $x \in K^\times$, we have

$$\mathrm{ld}_K(x) = \sum_{\mathfrak{p}|x} \frac{\nu_{\mathfrak{p}}(x)}{pg(p,K)} = \sum_{\mathfrak{p}|x} \frac{\overline{\nu}_{\mathfrak{p}}(x)}{pg(p,K)e(p,K)} = \frac{1}{n} \sum_{\mathfrak{p}|x} \frac{\overline{\nu}_{\mathfrak{p}}(x) f(p,K)}{p}.$$

Therefore

$$\begin{aligned}
\mathrm{ld}_K(K^\times) &= \Big\{ \frac{1}{n} \sum_{\mathfrak{p}|x} \frac{\overline{\nu}_{\mathfrak{p}}(x) f(p,K)}{p} \mid x \in K^\times \Big\} \\
&= \Big\langle \frac{f(p,K)}{np} \mid p \in \mathbb{P} \Big\rangle \\
&= \Big\langle \frac{1}{p^{1+\nu_p(n)-\nu_p(f(p,K))}} \mid p \in \mathbb{P} \Big\rangle.
\end{aligned}$$

For every $p \in \mathbb{P}$, we denote $m(p) := 1 + \nu_p(n) - \nu_p(f(p,K))$. It is easy to see that

$$H_{\mathrm{ld}_K(K^\times)} = (m(2), m(3), m(5), \ldots).$$

As $f(p,K) \mid n$, we know that $1 \leq m(p) < +\infty$. When $p > n$, we have $\nu_p(n) = \nu_p(f(p,K)) = 0$. This implies that $m(p) = 1$ for all except for finitely many primes. Hence H_G and $H_{\mathrm{ld}_K(K^\times)}$ only differ in finitely many indices, and in the case $H_{p_i,G}(1) \neq H_{p_i,\mathrm{ld}_K(K^\times)}$, both of them are finite. Hence $\mathrm{ld}_K(K^\times) \cong G$ by Theorem 4.1. □

To determine the exact image of ld_K in general is not easy. We give an example.

Example 4.3. Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic extension, where D is a square free integer. We rewrite the formula of ld_K using the normalized discrete valuation $\overline{\nu}_{\mathfrak{p}} = \nu_{\mathfrak{p}} \cdot e(p,K)$

$$\mathrm{ld}_K(x) = \sum_{\mathfrak{p}|x} \frac{\nu_{\mathfrak{p}}(x)}{pg(p,K)} = \sum_{\mathfrak{p}|x} \frac{\overline{\nu}_{\mathfrak{p}}(x)}{pg(p,K)e(p,K)} = \frac{1}{2} \sum_{\mathfrak{p}|x} \frac{\overline{\nu}_{\mathfrak{p}}(x) f(p,K)}{p}.$$

It remains to determine when 2 is inert in K, that is, $f(2,K) = 2$. Let Δ_K be the discriminant of K, that is, $\Delta_K = D$ if $D \equiv 1 \pmod 4$ and $\Delta_K = 4D$ if $D \equiv 2, 3 \pmod 4$. Hence $\Delta_K \equiv 0, 1, 4, 5 \pmod 8$. We know that $\mathcal{O}_K = \mathbb{Z}[\frac{\Delta_K + \sqrt{\Delta_K}}{2}]$. The minimal polynomial of $\frac{\Delta_K + \sqrt{\Delta_K}}{2}$ is

$$(X - \frac{\Delta_K + \sqrt{\Delta_K}}{2})(X - \frac{\Delta_K - \sqrt{\Delta_K}}{2}) = X^2 - \Delta_K X + \frac{\Delta_K^2 - \Delta_K}{4}.$$

We discuss the cases based on the value of $\Delta_K \bmod 8$.

1. If $\Delta_K \equiv 0 \pmod 8$, then $\Delta_K^2 - \Delta_K \equiv 8 \pmod 8$. Hence $\frac{\Delta_K^2 - \Delta_K}{4} \equiv 0 \pmod 2$. Therefore
$$X^2 - \Delta_K X + \frac{\Delta_K^2 - \Delta_K}{4} \equiv X^2 \pmod 2,$$
and $(2) = (2, \frac{\Delta_K + \sqrt{\Delta_K}}{2})^2$ is ramified in this case, that is, $e(2,K) = 2$.

2. If $\Delta_K \equiv 1 \pmod 8$, then $\Delta_K^2 - \Delta_K \equiv 1-1 \equiv 0 \pmod 8$. Hence $\frac{\Delta_K^2-\Delta_K}{4} \equiv 0 \pmod 2$. Therefore

$$X^2 - \Delta_K X + \frac{\Delta_K^2 - \Delta_K}{4} \equiv X^2 + X \equiv X(X+1) \pmod 2,$$

and $(2) = (2, \frac{\Delta_K+\sqrt{\Delta_K}}{2})(2, \frac{\Delta_K+\sqrt{\Delta_K}}{2}+1)$ is totally split in this case, that is, $g(2,K) = 2$.

3. If $\Delta_K \equiv 4 \pmod 8$, then $\Delta_K^2 - \Delta_K \equiv 0-4 \equiv 4 \pmod 8$. Hence $\frac{\Delta_K^2-\Delta_K}{4} \equiv 1 \pmod 2$. Therefore

$$X^2 - \Delta_K X + \frac{\Delta_K^2 - \Delta_K}{4} \equiv X^2 + 1 \equiv (X+1)^2 \pmod 2,$$

and $(2) = (2, \frac{\Delta_K+\sqrt{\Delta_K}}{2}+1)^2$ is ramified in this case, that is, $e(2,K) = 2$.

4. If $\Delta_K \equiv 5 \pmod 8$, then $\Delta_K^2 - \Delta_K \equiv 1-5 \equiv 4 \pmod 8$. Hence $\frac{\Delta_K^2-\Delta_K}{4} \equiv 1 \pmod 2$. Therefore

$$X^2 - \Delta_K X + \frac{\Delta_K^2 - \Delta_K}{4} \equiv X^2 + X + 1 \pmod 2,$$

which is irreducible. In this case, 2 is inert, that is, $f(2,K) = 2$.

If $\Delta_K \equiv 5 \pmod 8$, then $\Delta_K \equiv 1 \pmod 4$. In this case, $\Delta_K = D$ and thus $D \equiv 5 \pmod 8$. Therefore

$$\operatorname{ld}_K(K^\times) = \begin{cases} \langle 1/2, 1/3, 1/5, \ldots, \rangle, & \text{if } D \equiv 5 \pmod 8; \\ \langle 1/4, 1/3, 1/5, \ldots, \rangle, & \text{otherwise.} \end{cases}$$

5 p-adic continuity and discontinuity

In this section, we study when arithmetic partial derivatives and arithmetic subderivatives are p-adically continuous and discontinuous. When they are continuous, we will also study if they are strictly differentiable. We first recall some definitions.

Let K be a field and $\nu : K \to \mathbb{R} \cup \{+\infty\}$ be a discrete valuation. For all $x, y \in K$, we have $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$. An important property of ν that we will use repeatedly in this subsection is that if $\nu(x) \neq \nu(y)$, then $\nu(x+y) = \min\{\nu(x), \nu(y)\}$. If c is a real number number between 0 and 1, then the discrete valuation ν induces an absolute value on K as follows:

$$|x|_\nu := \begin{cases} c^{\nu(x)}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

We then have the formula $|x+y|_\nu \leq \max\{|x|_\nu, |y|_\nu\}$ and thus $|\cdot|$ is an ultrametric absolute value. The subset $\mathcal{O}_K = \{x \in K : \nu(x) \geq 0\}$ is a ring with the unique maximal ideal $\mathfrak{p} = \{x \in K : \nu(x) > 0\}$. Let $f : K \to K$ be a function. We say that f is $\mathfrak{p}$-adically continuous at a point $x \in K$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $|y-x|_\nu < \delta$, we have $|f(y) - f(x)|_\nu < \epsilon$. Equivalently, to show that f is $\mathfrak{p}$-adically continuous at x, it is enough to show that for every sequence x_i,

$$\lim_{i\to+\infty}\nu(x-x_i)=+\infty \quad \text{implies} \quad \lim_{i\to+\infty}\nu(f(x)-f(x_i))=+\infty.$$

On the contrary, to show that f is $\mathfrak{p}$-adically discontinuous at x, it is enough to find one sequence x_i such that

$$\lim_{i\to+\infty}\nu(x-x_i)=+\infty \quad \text{and} \quad \lim_{i\to+\infty}\nu(f(x)-f(x_i))\neq+\infty.$$

Recall that f is differentiable at a point x if the difference quotients $(f(y)-f(x))/(y-x)$ have a limit as $y\to x$ ($y\neq x$) in the domain of f. When the absolute value of the domain is ultrametric, we study the so-called strict differentiability. For more details on p-adic analysis, we refer the reader to [12].

Definition 5.1. Let K be a field equipped with an ultrametric absolute value $|\cdot|_\nu$. We say that $f:K\to K$ is *strictly differentiable* at a point $x\in K$ (with respect to $|\cdot|_\nu$) if the difference quotients

$$\Phi f(u,v)=\frac{f(u)-f(v)}{u-v}$$

have a limit as $(u,v)\to(x,x)$ while u and v remaining distinct. Similarly, we say that f is *twice strictly differentiable* at a point x if

$$\Phi_2 f(u,v,w)=\frac{\Phi f(u,w)-\Phi f(v,w)}{u-v}$$

tends to a limit as $(u,v,w)\to(x,x,x)$ while u, v, and w remaining pairwise distinct.

5.1 Partial derivative

Let $K/\mathbb{Q}$ be a finite Galois extension of degree n. Let $p\in\mathbb{Q}$ be a rational prime and $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$ such that $\mathfrak{p}\mid p$. The discrete valuation $\nu_\mathfrak{p}$ that extends ν_p defines an ultrametric absolute value on K by

$$|x|_{\nu_\mathfrak{p}}=\sqrt[n]{|N_{K_{\nu_\mathfrak{p}}/\mathbb{Q}_p}(x)|_{\nu_p}}.$$

Theorem 5.2. *Let K be a number field and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$. The arithmetic partial derivative $D_{K,\mathfrak{p}}$ is $\mathfrak{p}$-adically continuous on K.*

Proof. Suppose $K/\mathbb{Q}$ is Galois. We first show that $D_{K,\mathfrak{p}}$ is continuous at nonzero $x\in K$. Let x_i be a sequence that converges to x $\mathfrak{p}$-adically. Since $x\neq0$, we can rename the sequence as x_ix without loss of generality. As $i\to+\infty$, we know that

$$\nu_\mathfrak{p}(x-x_ix)=\nu_\mathfrak{p}(x)+\nu_\mathfrak{p}(1-x_i)\to+\infty.$$

This implies that $\nu_\mathfrak{p}(1-x_i)\to+\infty$ as $i\to+\infty$. As a result, we also know that $\nu_\mathfrak{p}(x_i)=0$ when $i\gg0$ because if $\nu_\mathfrak{p}(x_i)\neq0$, then $\nu_\mathfrak{p}(1-x_i)=\min\{\nu_\mathfrak{p}(1),\nu_\mathfrak{p}(x_i)\}=0$. Therefore $D_{K,\mathfrak{p}}(x_i)=0$ when $i\gg0$. To show that $D_{K,\mathfrak{p}}(x_i)$ converges to $D_{K,\mathfrak{p}}(x)$ $\mathfrak{p}$-adically, it is enough to observe that

$$\begin{aligned}\nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)-D_{K,\mathfrak{p}}(x_ix)) &= \nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)-D_{K,\mathfrak{p}}(x)x_i-D_{K,\mathfrak{p}}(x_i)x)\\ &= \nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x)(1-x_i))\\ &= \nu_{\mathfrak{p}}(D_{K,\mathfrak{p}}(x))+\nu_{\mathfrak{p}}(1-x_i)\to+\infty\end{aligned}$$

as $i\to+\infty$. The case $x=0$ will be covered in Theorem 5.6.

Suppose $K/\mathbb{Q}$ is a number field, not necessarily Galois. Let L/K be a finite extension such that $L/\mathbb{Q}$ is Galois. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$ such that $\mathfrak{P}\mid\mathfrak{p}$. By the previous paragraph, we know that $D_{L,\mathfrak{P}}$ is $\mathfrak{P}$-adically continuous on L (and thus on K). Let $x_i\in K$ be a sequence that converges to $x\in K$ $\mathfrak{p}$-adically. Since $\nu_{\mathfrak{p}}(y)=\nu_{\mathfrak{P}}(y)$ for all $y\in K$, we know that x_i converges to x $\mathfrak{P}$-adically. As $D_{L,\mathfrak{P}}$ is $\mathfrak{P}$-adically continuous on L, we know that $D_{L,\mathfrak{P}}(x_i)$ converges to $D_{L,\mathfrak{P}}(x)$ $\mathfrak{P}$-adically, and thus $\mathfrak{p}$-adically. This shows that $D_{L,\mathfrak{P}}$ is $\mathfrak{p}$-adically continuous on K. Let $T=\{\mathfrak{p}\}$. We know that by definition $D_{K,\mathfrak{p}}(x)=D_{L,T_{L/K}}(x)=\sum_{\mathfrak{P}\mid\mathfrak{p}}D_{L,\mathfrak{P}}$. This implies that $D_{K,\mathfrak{p}}$ is continuous on K. □

Since $D_{K,\mathfrak{p}}$ is $\mathfrak{p}$-adically continuous on K, the next question is whether $D_{K,\mathfrak{p}}$ is strictly differentiable on K with respect to the ultrametric $|\cdot|_{\nu_{\mathfrak{p}}}$.

Theorem 5.3. *Let K be a number field and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$. The arithmetic partial derivative $D_{K,\mathfrak{p}}$ is strictly differentiable and twice strictly differentiable (with respect to the ultrametric $|\cdot|_{\nu_{\mathfrak{p}}}$) at every nonzero $x\in K$.*

Proof. Let L/K be a finite extension such that $L/\mathbb{Q}$ is Galois. Let $T=\{\mathfrak{p}\}$. We have $D_{K,\mathfrak{p}}(x)=D_{L,T_{L/K}}(x)=\sum_{\mathfrak{P}\mid\mathfrak{p}}D_{L,\mathfrak{P}}$.

We first show that $D_{K,\mathfrak{p}}$ is strictly differentiable at $x\neq0$. Suppose a sequence (u_i,v_i) converges to (x,x) $\mathfrak{p}$-adically while u_i and v_i remaining distinct. This implies that (u_i,v_i) converges to (x,x) $\mathfrak{P}$-adically. When $i\gg0$, we have $\nu_{\mathfrak{P}}(u_i)=\nu_{\mathfrak{P}}(v_i)=\nu_{\mathfrak{P}}(x)$. We can compute

$$\begin{aligned}\Phi D_{K,\mathfrak{p}}(u_i,v_i)&=\frac{D_{K,\mathfrak{p}}(u_i)-D_{K,\mathfrak{p}}(v_i)}{u_i-v_i}=\frac{\sum_{\mathfrak{P}\mid\mathfrak{p}}D_{L,\mathfrak{P}}(u_i)-\sum_{\mathfrak{P}\mid\mathfrak{p}}D_{L,\mathfrak{P}}(v_i)}{u_i-v_i}\\ &=\frac{\sum_{\mathfrak{P}\mid\mathfrak{p}}\frac{u_i\nu_{\mathfrak{P}}(x)}{pg(p,L)}-\sum_{\mathfrak{P}\mid\mathfrak{p}}\frac{v_i\nu_{\mathfrak{P}}(x)}{pg(p,L)}}{u_i-v_i}=\sum_{\mathfrak{P}\mid\mathfrak{p}}\frac{\nu_{\mathfrak{P}}(x)}{pg(p,L)}=\frac{D_{K,\mathfrak{p}}(x)}{x}.\end{aligned}$$

Therefore the limit of $\Phi D_{K,\mathfrak{p}}(u_i,v_i)$ is equal to $D_{K,\mathfrak{p}}(x)/x$ as $i\to+\infty$. This shows that $D_{K,\mathfrak{p}}$ is strictly differentiable at any nonzero $x\in K$, and the derivative of $D_{K,\mathfrak{p}}$ is a constant function, defined by

$$(D_{K,\mathfrak{p}})'(x)=D_{K,\mathfrak{p}}(x)/x=\mathrm{ld}_{K,\mathfrak{p}}(x).$$

We then show that $D_{K,\mathfrak{p}}$ is twice strictly differentiable at nonzero points. Suppose a sequence (u_i,v_i,w_i) converges to (x,x,x) $\mathfrak{p}$-adically while u_i, v_i, and w_i remaining pairwise distinct. Then for all $i\gg0$, we have

$$\Phi_2D_{K,\mathfrak{p}}(u_i,v_i,w_i)=\frac{\Phi D_{K,\mathfrak{p}}(u_i,w_i)-\Phi D_{K,\mathfrak{p}}(v_i,w_i)}{u_i-v_i}=\frac{0}{u_i-v_i}=0.$$

Hence $D_{K,\mathfrak{p}}$ is twice strictly differentiable at nonzero points and the second derivative is the constant zero function. □

Theorem 5.4. *Let K be a number field and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$. The arithmetic partial derivative $D_{K,\mathfrak{p}}$ is not strictly differentiable (with respect to the ultrametric $|\cdot|_{\nu_\mathfrak{p}}$) at 0.*

Proof. This theorem is a direct corollary of a more generalized Theorem 5.8. □

Remark 5.5. Theorems 5.2, 5.3, and 5.4 hold in the local case of finite extensions over $\mathbb{Q}_p$.

5.2 Subderivative

Theorem 5.6. *Let $K/\mathbb{Q}$ be a number field and $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$. Let T be a nonempty set of prime ideals in $\mathcal{O}_K$. The arithmetic subderivative $D_{K,T}$ is $\mathfrak{p}$-adically continuous at $x = 0$.*

Proof. Let L/K be a finite extension such that $L/\mathbb{Q}$ is Galois. Suppose $x_i \in K$ is a sequence that converges to x $\mathfrak{p}$-adically in K. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$ such that $\mathfrak{P} \mid \mathfrak{p}$. Then x_i converges to x $\mathfrak{P}$-adically in L. Hence

$$\lim_{i \to +\infty} \nu_\mathfrak{P}(x - x_i) = \lim_{i \to +\infty} \nu_\mathfrak{P}(x_i) = +\infty.$$

We have

$$\begin{aligned}
\nu_\mathfrak{p}(D_{K,T}(x_i)) &= \nu_\mathfrak{P}(D_{L,T_{L/K}}(x_i)) \\
&= \nu_\mathfrak{P}\Big(x_i \sum_{\mathfrak{Q} \in T_{L/K}, \mathfrak{Q} \mid q} \frac{\nu_\mathfrak{Q}(x_i)}{qg(q,L)}\Big) \\
&= \nu_\mathfrak{P}(x_i) + \nu_\mathfrak{P}\Big(\sum_{\mathfrak{Q} \in T_{L/K}, \mathfrak{Q} \mid q} \frac{\nu_\mathfrak{Q}(x_i)}{qg(q,L)}\Big) \\
&= \nu_\mathfrak{P}(x_i) + \nu_\mathfrak{P}\Big(\frac{1}{[L:\mathbb{Q}]} \sum_{\mathfrak{Q} \in T_{L/K}, \mathfrak{Q} \mid q} \frac{\nu_\mathfrak{Q}(x_i) e(q,L) f(q,L)}{q}\Big) \\
&\geq \nu_\mathfrak{P}(x_i) - \nu_\mathfrak{P}([L:\mathbb{Q}])) - \nu_\mathfrak{P}\big(\prod_{\mathfrak{Q} \in T_{L/K}, \mathfrak{Q} \mid q} q\big).
\end{aligned}$$

As $\lim_{i \to +\infty} \nu_\mathfrak{P}(x_i) = +\infty$, we have

$$\lim_{i \to +\infty} \nu_\mathfrak{p}(D_{K,T}(x) - D_{K,T}(x_i)) = +\infty.$$ □

Corollary 5.7. *Let T be a nonempty set of (rational) prime numbers. The arithmetic subderivative $D_{\mathbb{Q},T}$ is p-adically continuous at $x = 0$.*

Theorem 5.8. *Let $K/\mathbb{Q}$ be a number field and $\mathfrak{p}$ a prime ideal of $\mathcal{O}_K$. Let T be a nonempty set of prime ideals in $\mathcal{O}_K$. The arithmetic subderivative $D_{K,T} : K \to K$ is not strictly differentiable (with respect to the ultrametric $|\cdot|_{\nu_\mathfrak{p}}$) at 0.*

Proof. Let L/K be a finite extension such that $L/\mathbb{Q}$ is Galois. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_L$ such that $\mathfrak{P} \mid \mathfrak{p} \mid p$.

We prove this theorem in two cases. First, we assume that there exists a prime ideal $\mathfrak{p}' \in T$ such that $\mathfrak{p}' \mid p$. Let m_p be the number of prime ideals in $T_{L/K}$ that divide p. For positive integer

$i \geq 1$, define $u_i = p^{i+1}, v_i = p^i$. It is clear that $u_i \neq v_i$ and (u_i, v_i) converges to $(0,0)$ $\mathfrak{p}$-adically. We can compute the difference quotient

$$\begin{aligned}\Phi D_{K,T}(u_i, v_i) &= \frac{D_{K,T}(u_i) - D_{K,T}(v_i)}{u_i - v_i} = \frac{D_{L,T_{L/K}}(u_i) - D_{L,T_{L/K}}(v_i)}{u_i - v_i} \\ &= \frac{\frac{(i+1)p^{i+1}m_p}{pg(p,L)} - \frac{ip^i m_p}{pg(p,L)}}{p^{i+1} - p^i} = \frac{m_p}{g(p,L)} \frac{(i+1)p - i}{p^2 - p}.\end{aligned}$$

The $\mathfrak{p}$-adic valuation of $\Phi D_{K,T}(u_i, v_i)$ is greater than or equal to $\nu_{\mathfrak{p}}(m_p) - \nu_{\mathfrak{p}}(g(p,L))$ if $p \mid i$ and is equal to $\nu_{\mathfrak{p}}(m_p) - \nu_{\mathfrak{p}}(g(p,L)) - 1$ if $p \mid i$. Hence $\Phi D_{K,T}(u_i, v_i)$ does not have a limit as the sequence $(u_i, v_i) \to (0,0)$.

Second, we assume that there does not exist a prime ideal $\mathfrak{p}' \in T$ such that $\mathfrak{p}' \mid p$. Let $\mathfrak{q} \in T$ be such that $\mathfrak{q} \nmid p$ and $\mathfrak{Q} \in T_{L/K}$ such that $\mathfrak{Q} \mid \mathfrak{q} \mid q$. Let m_q be the number of prime ideals in $T_{L/K}$ that divide q. For positive integer $i \geq 1$, define $u_i = (pq)^{i+1}, v_i = (pq)^i$. It is clear that $u_i \neq v_i$ and (u_i, v_i) converges to $(0,0)$ $\mathfrak{p}$-adically. We can compute the difference quotient

$$\begin{aligned}\Phi D_{K,T}(u_i, v_i) &= \frac{D_{K,T}(u_i) - D_{K,T}(v_i)}{u_i - v_i} = \frac{D_{L,T_{L/K}}(u_i) - D_{L,T_{L/K}}(v_i)}{u_i - v_i} \\ &= \frac{\frac{(i+1)(pq)^{i+1}m_q}{qg(q,K)} - \frac{i(pq)^i m_q}{qg(q,K)}}{(pq)^{i+1} - (pq)^i} = \frac{m_q}{g(q,K)} \frac{(i+1)pq - i}{pq^2 - q}.\end{aligned}$$

The $\mathfrak{p}$-adic valuation of $\Phi D_{K,T}(u_i, v_i)$ is greater than or equal to $\nu_{\mathfrak{p}}(m_q) - \nu_{\mathfrak{p}}(g(q,K)) + 1$ if $p \mid i$ and is equal to $\nu_{\mathfrak{p}}(m_q) - \nu_{\mathfrak{p}}(g(q,K))$ if $p \mid i$. Hence $\Phi D_{K,T}(u_i, v_i)$ does not have a limit as the sequence $(x_i, y_i) \to (0,0)$. □

Theorem 5.9. *Let $K/\mathbb{Q}$ be a number field of degree n. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ with $\mathfrak{p} \mid p$. Let $\{\mathfrak{p}\} \neq T$ be a nonempty set of prime ideals in $\mathcal{O}_K$ such that there exists a prime ideal in T that does not divide p. Then the arithmetic subderivative $D_{K,T} : K \to K$ is $\mathfrak{p}$-adically discontinuous at every nonzero $x \in K$.*

Proof. We first assume $K/\mathbb{Q}$ is Galois. For each prime $q \in \mathbb{P}$, let r_q be the number of prime ideals $\mathfrak{q} \in T$ such that $\mathfrak{q} \mid q$. Let $\mathbb{P}_T := \{q \in \mathbb{P} \mid r_q \neq 0, q \neq p\}$ and we know $0 \leq \nu_p(g(q,K)) \leq \nu_p(n)$ for all $q \in \mathbb{P}_T$. Let $q_0 \in \mathbb{P}_T$ be a prime such that $\nu_p(g(q_0,K)) = \min\{\nu_p(g(q,K)) \mid q \in \mathbb{P}_T\}$. Let $M := \max\{\nu_p(j) : 1 \leq j \leq n\} + 1$. For each integer $i \geq 1$, the Dirichlet's theorem on arithmetic progression implies there are infinitely many primes in the arithmetic progression $q_0^{p^M}, q_0^{p^M} + p^i, q_0^{p^M} + 2p^i, \ldots$. Set $n_0 := 0$. For each $i \geq 1$, let $n_i > n_{i-1}$ be a positive integer such that $q_i := q_0^{p^M} + n_i p^i$ is a prime, that is, one prime from each arithmetic progression. Hence we know that $p, q_0, q_1, q_2, \ldots$ is a list of pairwise distinct prime numbers. Let $x_i := q_0^{p^M} x / q_i \in K$. One can show that

$$\lim_{i \to +\infty} \nu_{\mathfrak{p}}(x - x_i) = \lim_{i \to +\infty} \nu_{\mathfrak{p}}\Big(\frac{x n_i p^i}{q_i}\Big) = \lim_{i \to +\infty} \nu_{\mathfrak{p}}(x n_i p^i) = +\infty.$$

This means that the sequence x_i converges to x $\mathfrak{p}$-adically. We now show that $D_{K,T}(x_i)$ does not converge to $D_{K,T}(x)$ $\mathfrak{p}$-adically. We have

$$
\begin{aligned}
D_{K,T}(x) - D_{K,T}(x_i) &= D_{K,T}(x) - \Big(\frac{q_0^{p^M}}{q_i} D_{K,T}(x) + x D_{K,T}\big(\frac{q_0^{p^M}}{q_i}\big)\Big) \\
&= \frac{n_i p^i}{q_i} D_{K,T}(x) - x \frac{D_{K,T}(q_0^{p^M}) q_i - q_0^{p^M} D_{K,T}(q_i)}{q_i^2} \\
&= \frac{n_i p^i}{q_i} D_{K,T}(x) - \frac{x r_{q_0} p^M q_0^{p^M - 1}}{g(q_0, K) q_i} + \frac{x q_0^{p^M} D_{K,T}(q_i)}{q_i^2}.
\end{aligned}
$$

We analyze the $\mathfrak{p}$-adic valuation of each of three summands separately. For the first summand, we have

$$\lim_{i \to +\infty} \nu_{\mathfrak{p}}\Big(\frac{n_i p^i}{q_i} D_{K,T}(x)\Big) = \lim_{i \to +\infty} \nu_{\mathfrak{p}}(p^i) = +\infty.$$

For the second summand, as $i \gg 0$, we have

$$\nu_{\mathfrak{p}}\Big(\frac{x r_{q_0} p^M q_0^{p^M - 1}}{g(q_0, K) q_i}\Big) = \nu_{\mathfrak{p}}\Big(\frac{x r_{q_0} p^M}{g(q_0, K)}\Big) = \nu_{\mathfrak{p}}\Big(\frac{x r_{q_0}}{g(q_0, K)}\Big) + M.$$

For the third summand, if $q_i \notin \mathbb{P}_T$, then $D_{K,T}(q_i) = 0$ so it has no contribution to the $\mathfrak{p}$-adic valuation. On the other hand, if $q_i \in \mathbb{P}_T$, then we have

$$\nu_{\mathfrak{p}}\Big(\frac{x q_0^{p^M} D_{K,T}(q_i)}{q_i^2}\Big) = \nu_{\mathfrak{p}}\Big(\frac{x q_0^{p^M} r_{q_i}}{g(q_i, K) q_i^2}\Big) = \nu_{\mathfrak{p}}\Big(\frac{x r_{q_i}}{g(q_i, K)}\Big).$$

Since $1 \leq r_{q_i} \leq n$, we know that $M > \nu_p(r_{q_i})$ by definition. We also know that $\nu_p(g(q_0, K)) \leq \nu_p(g(q_i, K))$ for all $i \geq 1$. Hence

$$\nu_{\mathfrak{p}}\Big(\frac{x r_{q_0}}{g(q_0, K)}\Big) + M > \nu_{\mathfrak{p}}\Big(\frac{x r_{q_i}}{g(q_i, K)}\Big).$$

This implies that

$$\nu_{\mathfrak{p}}(D_{K,T}(x) - D_{K,T}(x_i)) = \begin{cases} \nu_{\mathfrak{p}}\Big(\frac{x r_{q_i}}{g(q_i, K)}\Big), & \text{if } q_i \in \mathbb{P}_T; \\ \nu_{\mathfrak{p}}\Big(\frac{x r_{q_0}}{g(q_0, K)}\Big) + M, & \text{if } q_i \notin \mathbb{P}_T. \end{cases}$$

This implies that

$$\lim_{i \to +\infty} \nu_{\mathfrak{p}}(D_{K,T}(x) - D_{K,T}(x_i)) \neq +\infty.$$

Now we assume that $K/\mathbb{Q}$ is not necessarily Galois. Let L/K be a finite extension such that $L/\mathbb{Q}$ is Galois, and $\mathfrak{P}$ a prime ideal of $\mathcal{O}_L$ such that $\mathfrak{P} \mid \mathfrak{p}$. Since T contains a prime ideal that does not divide p, we know that $T_{L/K}$ also contains a prime ideal that does not divide p. Let $x_i \in K$ be defined as above. Then we know that x_i converges to x $\mathfrak{p}$-adically in K, and thus $\mathfrak{P}$-adically in L since $\nu_{\mathfrak{p}}$ and $\nu_{\mathfrak{P}}$ agree on K. Since $L/\mathbb{Q}$ is Galois, we know that

$$\lim_{i \to +\infty} (\nu_{\mathfrak{P}}(D_{L,T_{L/K}}(x_i) - D_{L,T_{L/K}}(x))) \neq +\infty.$$

Hence

$$\lim_{i \to +\infty} (\nu_{\mathfrak{p}}(D_{K,T}(x_i) - D_{K,T}(x)) = \lim_{i \to +\infty} (\nu_{\mathfrak{P}}(D_{L,T_{L/K}}(x_i) - D_{L,T_{L/K}}(x))) \neq +\infty.$$

This shows that $D_{K,T}$ is discontinuous at x. □

Corollary 5.10. *Let $\{p\} \neq T$ be a nonempty set of prime numbers. The arithmetic subderivative $D_{\mathbb{Q},T}$ is p-adically discontinuous at any nonzero $x \in \mathbb{Q}$.*

Proof. Apply Theorem 5.9 by taking $K = \mathbb{Q}$ and $\mathfrak{p} = (p)$. □

Remark 5.11. Corollaries 5.7 and 5.10 together give answers to all open questions about p-adic continuity and discontinuity of arithmetic subderivative over $\mathbb{Q}$ listed in [7, Section 7].

The only case that is left for consideration is when all prime ideals in T sit above the same p. This case will be fully answered by the next theorem when we assume T is finite.

Theorem 5.12. *Let $K/\mathbb{Q}$ be a number field of degree n. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ with $\mathfrak{p} \mid p$. Let $\{\mathfrak{p}\} \neq T$ be a nonempty finite set of prime ideals in $\mathcal{O}_K$. Then the arithmetic subderivative $D_{K,T} : K \to K$ is $\mathfrak{p}$-adically discontinuous at any nonzero $x \in K$.*

Proof. We first assume $K/\mathbb{Q}$ is Galois. Let $T \setminus \{\mathfrak{p}\} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. By the Chinese remainder theorem, for each $i \geq 1$, there exists $x_i \in K$ such that $\nu_{\mathfrak{p}}(1-x_i) = i$, $\nu_{\mathfrak{p}_1}(x_i) = 1$, and $\nu_{\mathfrak{p}_j}(x_i) = 0$ for $2 \leq j \leq n$. This implies that $\nu_{\mathfrak{p}}(x_i) = 0$. Hence for all $i \geq 1$, we have

$$D_{K,T}(x_i) = \frac{x_i}{p_1 g(p_1, K)}.$$

The sequence $x_i x$ converges to x $\mathfrak{p}$-adically because as $i \to +\infty$, we have

$$\nu_{\mathfrak{p}}(x - x_i x) = \nu_{\mathfrak{p}}(1 - x_i) + \nu_{\mathfrak{p}}(x) \to +\infty.$$

On the other hand, $D_{K,T}(x_i x)$ does not converge to $D_{K,T}(x)$ $\mathfrak{p}$-adically because as $i \gg 0$, we have

$$\begin{aligned}
\nu_{\mathfrak{p}}(D_{K,T}(x) - D_{K,T}(x_i x)) &= \nu_{\mathfrak{p}}(D_{K,T}(x) - x_i D_{K,T}(x) - x D_{K,T}(x_i)) \\
&= \nu_{\mathfrak{p}}\Big(D_{K,T}(x)(1 - x_i) - \frac{x x_i}{p_1 g(p_1, K)}\Big) \\
&= \nu_{\mathfrak{p}}(x) - \nu_{\mathfrak{p}}(p_1) - \nu_{\mathfrak{p}}(g(p_1, K)).
\end{aligned}$$

Hence

$$\lim_{i \to +\infty} \nu_{\mathfrak{p}}(D_{K,T}(x) - D_{K,T}(x_i x)) \neq +\infty,$$

and $D_{K,T}$ is discontinuous at x.

If $K/\mathbb{Q}$ is not necessarily Galois, then one can prove that $D_{K,T}$ is discontinuous at x using the same strategy as in Theorem 5.9. □

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