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Infinite multisets: Basic properties and cardinality

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Abstract: This research work presents the topic of infinite multisets, their basic properties and cardinality from a somewhat different perspective. In this work, a new property of multisets, 'm-cardinality', is defined using multiset functions. M-cardinality unifies and generalizes the definitions of cardinality, injection, bijection, and surjection to apply to multisets. M-cardinality takes into account both the number of distinct elements in a multiset and the number of copies of each element (i.e., the multiplicity of the elements). Based on m-cardinality, 'm-cardinal numbers' are defined as a generalization of cardinal numbers in the context of multisets. Some properties of m-cardinal numbers associated with finite and infinite msets have been researched. Concrete examples of transfinite m-cardinal numbers are given, corresponding to infinite msets which are less than $\aleph_0$ (the cardinality of the countably infinite set). It has been established that between finite numbers and $\aleph_0$ there exist hierarchies of transfinite m-cardinals, corresponding to infinite msets. Furthermore, there are examples of infinite msets with negative multiplicity that have a cardinality less than zero. We prove that there is a decreasing sequence of transfinite m-cardinal numbers, corresponding to infinite msets with negative multiplicity, and in this sequence, there is not a smallest transfinite m-cardinal number.

Keywords: Multiset theory, M-cardinality, Infinite multiset, M-cardinal numbers.
2020 Mathematics Subject Classification: 03E10, 03E75.

1 Introduction

In mathematics, a multiset (shortened to mset) is a modification of the concept of a set. Unlike sets, in msets each element may occur more than once. The multiplicity of an element is defined as the number of times (number of copies) it occurs in the mset. For example, in the mset $\{\!\{a, a, b, c\}\!\}$, the element a has multiplicity 2, and the two elements b, c have multiplicity 1. Multiple occurrences of an element in an mset are treated without preference [7]. The same elements in the mset (in this case the elements a, a) are indistinguishable from one another.

This research work presents the topic of infinite msets, their basic properties and cardinality. A definition of msets is given in the first part of the work and the basic operations and relations with them are presented. By means of the introduced operations with msets, the possibility of getting msets with negative and even fraction multiplicity is illustrated. The present article proposes broadening the concept of cardinality so that this concept is applicable to finite and infinite msets. A new property of multisets, 'm-cardinality', is defined using multiset functions. Relations are introduced which allow comparing finite and infinite msets and defining their m-cardinality. Some of the basic properties of these relations are formulated and proved. In the second part of the work the m-cardinality of infinite msets is investigated, comparisons are made, and the m-cardinality of concrete msets is determined. The existence of still unresearched hierarchies of infinities (corresponding to infinite msets) with m-cardinality less than the cardinality of a countably infinite set ($\aleph_0$), has been established. Therefore, between finite numbers and $\aleph_0$ there is an unlimited number of different transfinite m-cardinals, corresponding to infinite msets.

2 Definitions and properties

All elements of the viewed msets are taken in a fixed set U which we will call a universe. An element of U that does not belong to a given mset will naturally have multiplicity of 0 in this mset.

An mset A can be formally defined as an ordered pair

$$A = \{\!\{A^*, m_A\}\!\},$$

where A^* is a given set. Here $m_A: A^* \to \mathbb{K}_{\neq 0}$ is a function giving each element its multiplicity from A^* to a class $\mathbb{K}_{\neq 0}$ of non-zero rational numbers and non-zero cardinal numbers ($m_A \neq 0$). For each $a \in A^*$, $m_A(a)$ is the characteristic value of a in A and indicates the number of occurrences of the element a in A. (Hereon, msets will be denoted with $\{\!\{\ \}\!\}$ brackets, and traditional sets will be denoted with { } brackets.) In fact, A^* is the set of all distinguishable (distinct) elements of an mset A and is called its support or root set:

$$A^* \coloneqq \{x \in U \mid m_A(x) \neq 0\}. \tag{1}$$

In the present work, the root set A^* is a countable set (finite or infinite). The mset will also be written in the following way:

$$A = \{\!\{a\}\!\}_{m_A} = \{\!\{a_1, a_2, a_3, \dots\}\!\}_{m_A(a_1), m_A(a_2), m_A(a_3), \dots} .$$

The root set will also be denoted in the following way:

$$A^* = \{\!\{a_1, a_2, a_3, \dots\}\!\}_{m_A(a_1), m_A(a_2), m_A(a_3), \dots}{}^* = \{a_1, a_2, a_3, \dots\}.$$

The cardinality of the root set A^* is denoted by $|A^*| = \left|\{\!\{a_1, a_2, a_3, \dots\}\!\}_{m_A(a_1), m_A(a_2), m_A(a_3), \dots}{}^*\right|$.

Let a appear m times in the mset A. It is denoted by $a \in^m A$. Let $\{\!\{x\}\!\}_t$ denote the mset that contains exactly t copies of x and nothing else.

We will call the sum of the multiplicities of all elements of the mset total multiplicity of the mset, and we will denote it in the following way:

$$\mu(A) = \mu\left(\{\!\{a_1, a_2, a_3, \dots\}\!\}_{m_A(a_1), m_A(a_2), m_A(a_3), \dots}\right) = \sum_{a \in A^*} m_A(a). \qquad (2)$$

Let us look at msets $A = \{\!\{A^*, m_A\}\!\}$ and $B = \{\!\{B^*, m_B\}\!\}$. We will define the following relations:

A is an msubset of B, denoted by $A \subseteq B$, if

$$m_A(x) \le m_B(x), \qquad \forall x \in A^*.$$

Two msets A and B are equal, denoted by $A = B$, if

$$m_A(x) = m_B(x), \qquad x \in A^* \vee x \in B^*.$$

The union C of two msets A and B is defined as:

$$C = A \cup B = \{\!\{A^* \cup B^*, \max(m_A, m_B)\}\!\}.$$

The intersection C between two msets A and B is defined as:

$$C = A \cap B = \{\!\{A^* \cap B^*, \min(m_A, m_B)\}\!\}.$$

The sum C of these two msets A and B is defined as:

$$C = A + B = \{\!\{A^* \cup B^*, m_A + m_B\}\!\}.$$

The difference C of these two msets A and B is defined as:

$$C = A - B = \{\!\{A^* \cup B^*, m_A - m_B\}\!\}.$$

Since the range of the characteristic function is the class of all of non-zero rational numbers and non-zero cardinal numbers, there is no limitation on negative values of the multiplicity in the last operation – the difference of the msets. It is interesting to note that the limitation on negative values of the multiplicity can be ignored without considerably affecting the other properties and operations. These generalizations of the msets with negative multiplicity are examined in detail in many research works [2, 3, 5].

In this way, we can examine msets with a negative multiplicity (sometimes called a signed mset or, hybrid/shadow mset [3, 9]), a fraction multiplicity and even a multiplicity which is a random real number (for example, real-valued msets, fuzzy sets [2]). The negative multiplicity is, in fact, the removing or deleting of an element from the mset.

For example, in the mset $A = \{\!\{a, b, c\}\!\}_{2,-3,-2}$ the element a appears twice, the element b is removed three times, and the element c is removed twice. If we look at the mset $B = \{\!\{a, b, c\}\!\}_{1,4,2}$, then

$$A + B = \{\!\{a, b, c\}\!\}_{3,1,0} = \{\!\{a, b\}\!\}_{3,1}.$$

Let us look at the mset $D = \{\!\{a, d\}\!\}_{1,\frac{1}{2}}$. Here the element d has a fraction multiplicity which is equal to $\frac{1}{2}$. We will have:

$$D + D = 2D = \{\!\{a, d\}\!\}_{2,1}.$$

The empty mset $\emptyset = \{\!\{\ \}\!\}$ is the unique mset with an empty support and has total multiplicity 0. The empty mset is an msubset of any possible mset: $\forall A\ (\emptyset \subseteq A)$.

Let $A = \{\!\{A^*, m_A\}\!\}$ and $B = \{\!\{B^*, m_B\}\!\}$ be two msets. Let us denote:

$$\bar{A} = -A = \{\!\{A^*, -m_A\}\!\} \text{ and } \bar{B} = -B = \{\!\{B^*, -m_B\}\!\}.$$

We will call msets $\bar{A}$ and $\bar{B}$ absolute complement msets.

Theorem 2.1. *It is easy to prove that De Morgan properties are valid:*

$$\overline{A \cup B} = \bar{A} \cap \bar{B}\,, \tag{3}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}\,. \tag{4}$$

Proof. Indeed,

$$\overline{A \cup B} = \overline{\{\!\{A^* \cup B^*, \max(m_A, m_B)\}\!\}} = \{\!\{\overline{A^* \cup B^*}, -\max(m_A, m_B)\}\!\}$$
$$= \{\!\{A^* \cap B^*, \min(-m_A, -m_B)\}\!\} = \bar{A} \cap \bar{B}$$

and

$$\overline{A \cap B} = \overline{\{\!\{A^* \cap B^*, \min(m_A, m_B)\}\!\}} = \{\!\{\overline{A^* \cap B^*}, -\min(m_A, m_B)\}\!\}$$
$$= \{\!\{A^* \cup B^*, \max(-m_A, -m_B)\}\!\} = \bar{A} \cup \bar{B}\,. \qquad \square$$

Let Ω denote the empty mset, which contains null multiplicities for all elements in the respective support [5]. We will have:

$$\Omega = \{\!\{\underbrace{x_1, x_2, \ldots, x_N}_{N \text{ elements}}\}\!\}_{\underbrace{0,0,\ldots,0}_{N \text{ times}}} = \{\!\{\ \}\!\}\,. \tag{5}$$

The absolute complement can be formally defined as follows:

$$\bar{A} = \Omega - A. \tag{6}$$

(In the definition of the absolute complement, the empty mset has been used instead of the universe mset [5].)

The Cartesian product of two msets A and B is defined as [6]:

$$A \times B = \{\!\{a, b, (a,b)\}\!\}_{m_A, m_B, k}\,, \qquad \text{where } k = m_A m_B.$$

The entry of the form $\{\!\{a, b, (a,b)\}\!\}_{m_A, m_B, k}$ denotes that a is repeated m_A times, b is repeated m_B times and the pair (a, b) is repeated k times. The counts of the members of the domain and codomain vary with respect to the counts of the a coordinate and b coordinate in $\{\!\{a, b, (a,b)\}\!\}_{m_A, m_B, k}$. We introduce the notation $m_1(a, b)$ and $m_2(a, b)$, where $m_1(a, b)$ denotes the count of the first coordinate in the ordered pair (a, b) and $m_2(a, b)$ denotes the count of the second coordinate in the ordered pair (a, b).

Example 2.1. If $A = \{\!\{x, y\}\!\}_{2,3}, B = \{\!\{z\}\!\}_4$ are two msets, then

$$A \times B = \{\!\{x, y\}\!\}_{2,3} \times \{\!\{z\}\!\}_4 = \{\!\{x, z, (x, z)\}\!\}_{2,4,8}, \{\!\{y, z, (y, z)\}\!\}_{3,4,12}\,.$$

The Cartesian product of three or more nonempty msets is defined by generalizing the definition of the Cartesian product of two msets.

An msubset R of $A \times A$ is said to be an mset relation on A if every member $\{\!\{a, b\}\!\}_{m_A, m_B}$ of R has a count, product of $m_1(a, b)$ and $m_2(a, b)$, [6]. We denote $\{\!\{a\}\!\}_{m_A}$ related to $\{\!\{b\}\!\}_{m_B}$ by $\{\!\{a\}\!\}_{m_A} R \{\!\{b\}\!\}_{m_B}$.

The Domain and Range of the mset relation R on A are defined as follows [6]:

$$\text{Dom } R = [\{\!\{ a \in^t A : \exists\, b \in^s A \text{ such that } \{\!\{a\}\!\}_t R \{\!\{b\}\!\}_s\}\!\} \text{ where } m_{DomR}(a) = \sup\{m_1(a,b) : a \in^t A],$$

$$\text{Ran } R = \{\!\{ b \in^s A : \exists\, a \in^t A \text{ such that } \{\!\{a\}\!\}_t R \{\!\{b\}\!\}_s\}\!\} \text{ where } m_{RanR}(b) = \sup\{m_2(a,b) : b \in^s A].$$

An mset relation f is called an mset function if for every element $\{\!\{a\}\!\}_r$ in Dom f, there is exactly one $\{\!\{b\}\!\}_l$ in Ran f such that $\{\!\{a,b\}\!\}_{r,l}$ is in f with the pair occurring only $m_1(a,b)$ times, [6]. For functions between arbitrary msets, it is essential that images of indistinguishable elements of the domain must be indistinguishable elements of the range but the images of the distinct elements of the domain need not be distinct elements of the range.

Most authors [1, 4, 6, 8] identify the cardinality of the mset with its total multiplicity. But the two notions have different significance and meaning. In the set theory, if two sets have the same cardinalities, then a bijection (one-to-one correspondence) can be established between them, and vice versa. A similar dependence would be expected in the mset theory, too, when examining two msets with the same total multiplicities. The following elementary example shows that it is not the case. The two msets $\{\!\{a\}\!\}_3$ and $\{\!\{a,b\}\!\}_{2,1}$ have the same total multiplicities but obviously, a bijection between the corresponding root sets $\{a\}$ and $\{a,b\}$ cannot be found (as it is in the set theory). This means that the notion of total multiplicity in the mset theory is not a generalization of the notion of cardinality in the set theory. Furthermore, if we examine infinite msets, then we will obtain infinite total multiplicities, but this will not allow us to compare these msets in a way analogous to comparing infinite sets. The infinite total multiplicity of the mset may be due to the infinite cardinality of the root sets, or to the infinite multiplicity of some of the elements of the mset, or both. In the case of infinite msets a new concept of cardinality is needed, similar to the concept of the cardinality of infinite sets. For this purpose, we will define the notions of m-injection (one-to-one), m-surjection (onto), m-bijection (one-to-one and onto) and m-cardinality which are applicable to finite and infinite msets, and are a generalization of the corresponding notions of the set theory. On the basis of these notions we will define the binary relations '$\prec, \asymp, \succ$', through which we will compare the m-cardinality of the finite and infinite msets.

Let $A = \{\!\{A^*, m_A\}\!\}$ and $B = \{\!\{B^*, m_B\}\!\}$ be two msets.

Definition I. The mset function $f: A \to B$ is an m-injection if and only if
(i) $f: A^* \to B^*$ is an injection and
(ii) $\mu(A) \le \mu(B)$, i.e., $\sum_{a \in A^*} m_A(a) \le \sum_{b \in B^*} m_B(b)$.

Definition II. The mset function $f: A \to B$ is an m-bijection if and only if
(i) $f: A^* \to B^*$ is a bijection and
(ii) $\mu(A) = \mu(B)$, i.e., $\sum_{a \in A^*} m_A(a) = \sum_{b \in B^*} m_B(b)$.

Definition III. The mset function $f: A \to B$ is an m-surjection if and only if
(i) $f: A^* \to B^*$ is a surjection and
(ii) $\mu(A) \geq \mu(B)$, i.e., $\sum_{a \in A^*} m_A\,(a) \geq \sum_{b \in B^*} m_B\,(b)$.

The modification of the definitions for injection, bijection, and surjection allows for the application of the concept of cardinality to msets in a similar way as in set theory.

For msets A and B, we define the binary relations:

Definition IV. $A \preccurlyeq B$ means that there is an m-injection $f: A \to B$ (or equivalently there exists an m-surjection $g: B \to A$). In this case we will say that the m-cardinality of A is less than or equal to the m-cardinality of B and we will denote it in the following way: $|A| \leq |B|$.

Definition V. $A \asymp B$ means that there is an m-bijection $f: A \to B$. In this case we will say that the m-cardinality of A is equal to the m-cardinality of B and will denote it in the following way: $|A| = |B|$.

Definition VI. $A \prec B$ means that $A \preccurlyeq B \wedge \sim A \asymp B$. In this case we will say that the m-cardinality of A is less than the m-cardinality of B and will denote it in the following way: $|A| < |B|$.

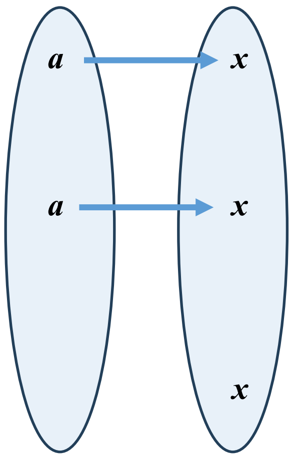
Therefore, the 'm-cardinality' is determined by the total multiplicity of the msets and by the cardinality of the root sets.

The Definitions I–III, which we propose, differ minimally from the definitions for injection, bijection and surjection ([1], p. 46, [6], p.765). For example, in the definitions for injection, bijection and surjection, the conditions (ii) $\sum_{a \in A^*} m_A(a) \gtreqless \sum_{b \in B^*} m_B(b)$ are replaced with the conditions $\forall x\big[x \in A^* \to m_1\,(x, f(x)\,) \gtreqless m_2\,(x, f(x)\,)\big]$, respectively (see ([1], p. 46, [6], p.765).

By introducing 'm-cardinality', we generalize the definitions of cardinality, injection, bijection, and surjection to apply to msets.

Figures 1.1–1.12 below present different cases that illustrate the similarities and differences between injection, surjection, bijection and m-injection, m-surjection, m-bijection respectively.

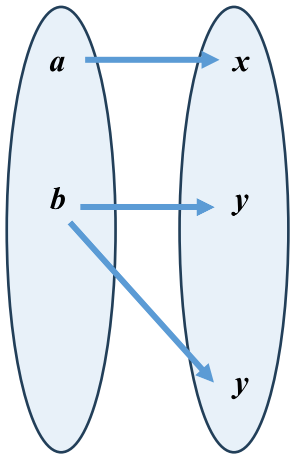
1.1. Injection and M-injection



$f: \{\!\{a\}\!\}_2 \to \{\!\{x\}\!\}_3$,
$f = \{\!\{a, x, (a, x)\}\!\}_{2,3,2}$.

$|\{\!\{a\}\!\}_2{}^*| = |\{\!\{x\}\!\}_3{}^*| = 1$;
$m_1(a, x) = 2 < m_2(a, x) = 3$;
$\mu(\{\!\{a\}\!\}_2) = 2 < \mu(\{\!\{x\}\!\}_3) = 3$.

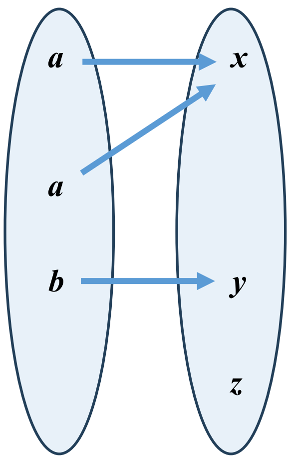
1.2. Injection and M-injection



$f: \{\!\{a, b\}\!\}_{1,1} \to \{\!\{x, y\}\!\}_{1,2}$,
$f = \{\!\{a, x, (a, x)\}\!\}_{1,1,1}, \{\!\{b, y, (b, y)\}\!\}_{1,2,1}$.

$\left|\{\!\{a, b\}\!\}_{1,1}{}^*\right| = \left|\{\!\{x, y\}\!\}_{1,2}{}^*\right| = 2$;
$m_1(a, x) = m_2(a, x) = 1$,
$m_1(b, y) = 1 < m_2(b, y) = 2$;
$\mu\left(\{\!\{a, b\}\!\}_{1,1}\right) = 2 < \mu\left(\{\!\{x, y\}\!\}_{1,2}\right) = 3$.

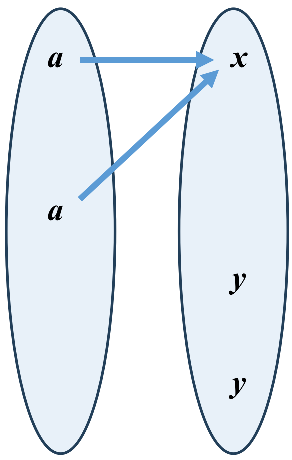
1.3. M-injection, but not injection



$f: \{\!\{a, b\}\!\}_{2,1} \to \{\!\{x, y, z\}\!\}_{1,1,1}$,
$f = \{\!\{a, x, (a, x)\}\!\}_{2,1,2}, \{\!\{b, y, (b, y)\}\!\}_{1,1,1}$.

$\left|\{\!\{a, b\}\!\}_{2,1}{}^*\right| = 2 < \left|\{\!\{x, y, z\}\!\}_{1,1,1}{}^*\right| = 3$;
$m_1(a, x) = 2 > m_2(a, x) = 1$,
$m_1(b, y) = m_2(b, y) = 1$;
$\mu\left(\{\!\{a, b\}\!\}_{2,1}\right) = \mu\left(\{\!\{x, y, z\}\!\}_{1,1,1}\right) = 3$.

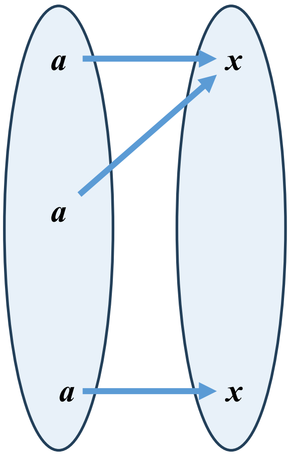
1.4. M-injection, but not injection



$f: \{\!\{a\}\!\}_2 \to \{\!\{x, y\}\!\}_{1,2}$,
$f = \{\!\{a, x, (a, x)\}\!\}_{2,1,2}$.

$|\{\!\{a\}\!\}_2{}^*| = 1 < \left|\{\!\{x, y\}\!\}_{1,2}{}^*\right| = 2$;
$m_1(a, x) = 2 > m_2(a, x) = 1$;
$\mu(\{\!\{a\}\!\}_2) = 2 < \mu\left(\{\!\{x, y\}\!\}_{1,2}\right) = 3$.

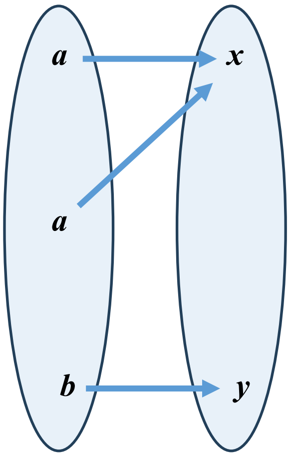
1.5. Surjection and M-surjection



$f\colon \{\!\{a\}\!\}_3 \rightarrow \{\!\{x\}\!\}_2$,
$f = \{\!\{a, x, (a, x)\}\!\}_{3,2,3}$.

$|\{\!\{a\}\!\}_3{}^*| = |\{\!\{x\}\!\}_2{}^*| = 1$;
$m_1(a, x) = 3 > m_2(a, x) = 2$;
$\mu(\{\!\{a\}\!\}_3) = 3 > \mu(\{\!\{x\}\!\}_2) = 2$.

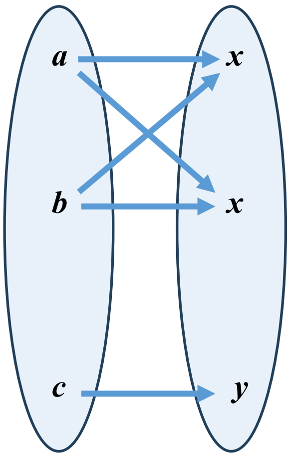
1.6. Surjection and M-surjection



$f\colon \{\!\{a, b\}\!\}_{2,1} \rightarrow \{\!\{x, y\}\!\}_{1,1}$,
$f = \{\!\{a, x, (a, x)\}\!\}_{2,1,2}, \{\!\{b, y, (b, y)\}\!\}_{1,1,1}$.

$\left|\{\!\{a, b\}\!\}_{2,1}{}^*\right| = \left|\{\!\{x, y\}\!\}_{1,1}{}^*\right| = 2$;
$m_1(a, x) = 2 > m_2(a, x) = 1$,
$m_1(b, y) = m_2(b, y) = 1$;
$\mu\left(\{\!\{a, b\}\!\}_{2,1}\right) = 3 > \mu\left(\{\!\{x, y\}\!\}_{1,1}\right) = 2$.

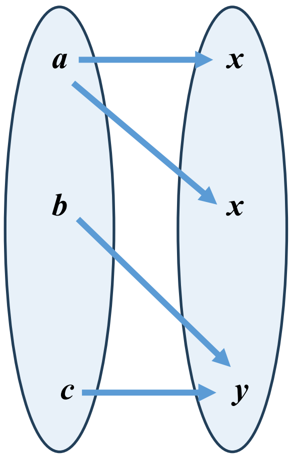
1.7. M-surjection but not surjection



$f\colon \{\!\{a, b, c\}\!\}_{1,1,1} \rightarrow \{\!\{x, y\}\!\}_{2,1}$,
$f = \{\!\{a, x, (a, x)\}\!\}_{1,2,1}, \{\!\{b, x, (b, x)\}\!\}_{1,2,1},$
$\{\!\{c, y, (c, y)\}\!\}_{1,1,1}$.

$\left|\{\!\{a, b, c\}\!\}_{1,1,1}{}^*\right| = 3 > \left|\{\!\{x, y\}\!\}_{2,1}{}^*\right| = 2$;
$m_1(a, x) = 1 < m_2(a, x) = 2$,
$m_1(b, x) = 1 < m_2(b, x) = 2$,
$m_1(c, y) = m_2(c, y) = 1$;
$\mu\left(\{\!\{a, b, c\}\!\}_{1,1,1}\right) = \mu\left(\{\!\{x, y\}\!\}_{2,1}\right) = 3.$

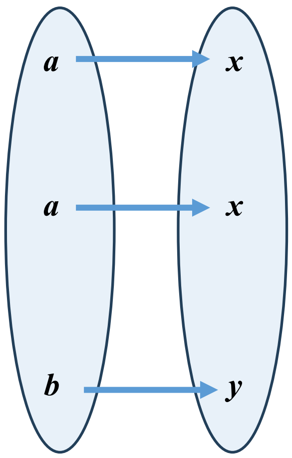
1.8. M-surjection but not surjection



$f\colon \{\!\{a, b, c\}\!\}_{1,1,1} \rightarrow \{\!\{x, y\}\!\}_{2,1}$,
$f = \{\!\{a, x, (a, x)\}\!\}_{1,2,1}, \{\!\{b, y, (b, y)\}\!\}_{1,1,1},$
$\{\!\{c, y, (c, y)\}\!\}_{1,1,1}$.

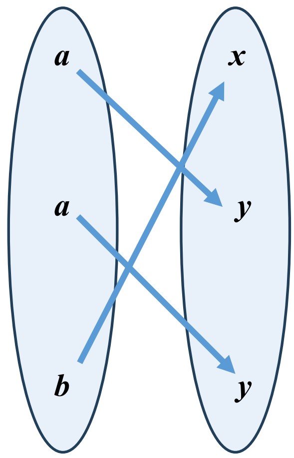
$\left|\{\!\{a, b, c\}\!\}_{1,1,1}{}^*\right| = 3 > \left|\{\!\{x, y\}\!\}_{2,1}{}^*\right| = 2$;
$m_1(a, x) = 1 < m_2(a, x) = 2$,
$m_1(b, y) = m_2(b, y) = 1$,
$m_1(c, y) = m_2(c, y) = 1$;
$\mu\left(\{\!\{a, b, c\}\!\}_{1,1,1}\right) = \mu\left(\{\!\{x, y\}\!\}_{2,1}\right) = 3.$

1.9. Bijection and m-bijection



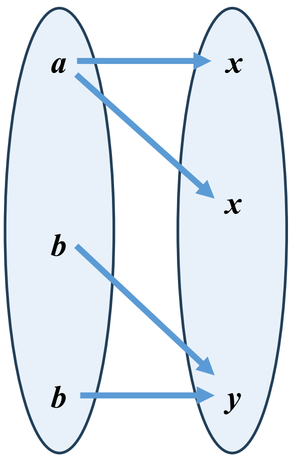
$f: \{\!\!\{a,b\}\!\!\}_{2,1} \to \{\!\!\{x,y\}\!\!\}_{2,1}$,
$f = \{\!\!\{a,x,(a,x)\}\!\!\}_{2,2,2}, \{\!\!\{b,y,(b,y)\}\!\!\}_{1,1,1}$.
$\left|\{\!\!\{a,b\}\!\!\}_{2,1}{}^{*}\right| = \left|\{\!\!\{x,y\}\!\!\}_{2,1}{}^{*}\right| = 2$;
$m_1(a,x) = m_2(a,x) = 2$,
$m_1(b,y) = m_2(b,y) = 1$;
$\mu(\{\!\!\{a,b\}\!\!\}_{2,1}) = \mu(\{\!\!\{x,y\}\!\!\}_{2,1}) = 3$.

1.10. Bijection and m-bijection



$f: \{\!\!\{a,b\}\!\!\}_{2,1} \to \{\!\!\{x,y\}\!\!\}_{1,2}$,
$f = \{\!\!\{a,y,(a,y)\}\!\!\}_{2,2,2}, \{\!\!\{b,x,(b,x)\}\!\!\}_{1,1,1}$.
$\left|\{\!\!\{a,b\}\!\!\}_{2,1}{}^{*}\right| = \left|\{\!\!\{x,y\}\!\!\}_{1,2}{}^{*}\right| = 2$;
$m_1(a,y) = m_2(a,y) = 2$,
$m_1(b,x) = m_2(b,x) = 1$;
$\mu(\{\!\!\{a,b\}\!\!\}_{2,1}) = \mu(\{\!\!\{x,y\}\!\!\}_{1,2}) = 3$.

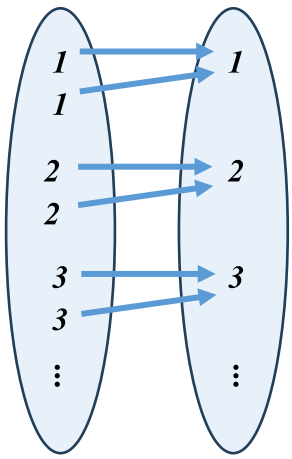
1.11. M-bijection, but not bijection



$f: \{\!\!\{a,b\}\!\!\}_{1,2} \to \{\!\!\{x,y\}\!\!\}_{2,1}$,
$f = \{\!\!\{a,x,(a,x)\}\!\!\}_{1,2,1}, \{\!\!\{b,y,(b,y)\}\!\!\}_{2,1,2}$.

$\left|\{\!\!\{a,b\}\!\!\}_{1,2}{}^{*}\right| = \left|\{\!\!\{x,y\}\!\!\}_{2,1}{}^{*}\right| = 2$;
$m_1(a,x) = 1 < m_2(a,x) = 2$,
$m_1(b,y) = 2 > m_2(b,y) = 1$;
$\mu(\{\!\!\{a,b\}\!\!\}_{1,2}) = \mu(\{\!\!\{x,y\}\!\!\}_{2,1}) = 3$.

1.12. M-bijection, but not bijection (only surjection)



$f: \{\!\!\{1,2,3,\ldots\}\!\!\}_{\underbrace{2,2,2,\ldots}_{\aleph_0 \text{ times}}} \to \{\!\!\{1,2,3,\ldots\}\!\!\}_{\underbrace{1,1,1,\ldots}_{\aleph_0 \text{ times}}} = \{\underbrace{1,2,3,\ldots}_{\aleph_0 \text{ elements}}\} = \mathbb{Z}^+$,
$f = \{\!\!\{1,1,(1,1)\}\!\!\}_{2,1,2}, \{\!\!\{2,2,(2,2)\}\!\!\}_{2,1,2}, \{\!\!\{3,3,(3,3)\}\!\!\}_{2,1,2}, \ldots$,
$\left|\{\!\!\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\!\!\}_{\underbrace{2,2,2,\ldots}_{\aleph_0}}{}^{*}\right| = \left|\{\!\!\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\!\!\}_{\underbrace{1,1,1,\ldots}_{\aleph_0}}{}^{*}\right| = \aleph_0$;
$m_1(1,1) = 2 > m_2(1,1) = 1$,
$m_1(2,2) = 2 > m_2(2,2) = 1$,
$m_1(3,3) = 2 > m_2(3,3) = 1, \ldots$;
$\mu\left(\{\!\!\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\!\!\}_{\underbrace{2,2,2,\ldots}_{\aleph_0}}\right) = \mu\left(\{\!\!\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\!\!\}_{\underbrace{1,1,1,\ldots}_{\aleph_0}}\right) = \aleph_0$.

Figure 1. Comparison between injection, surjection, bijection and m-injection, m-surjection, m-bijection, respectively.

Msets with equal total multiplicities do not always have the same m-cardinalities. For example, the msets $\{a,b,c\}$ and $⦃x,y⦄_{2,1}$ both contain three elements, but $\{a,b,c\} \succ ⦃x,y⦄_{2,1}$, because the mset $\{a,b,c\}$ contains three different elements, and the mset $⦃x,y⦄_{2,1}$ contains two different elements (i.e., the mset functions $f_{1,2}: ⦃a,b,c⦄_{1,1,1} \to ⦃x,y⦄_{2,1}$, $f_1 = ⦃a,x,(a,x)⦄_{1,2,1}$, $⦃b,x,(b,x)⦄_{1,2,1}$, $⦃c,y,(c,y)⦄_{1,1,1}$ and $f_2 = ⦃a,x,(a,x)⦄_{1,2,1}$, $⦃b,y,(b,y)⦄_{1,1,1}$, $⦃c,y,(c,y)⦄_{1,1,1}$ are m-surjective; see Figures 1.7 and 1.8).

Furthermore, it is not always possible to compare two msets for their m-cardinality. For example, for the two msets $⦃a,b⦄_{2,2}$ and $\{x,y,z\}$ none of the three relations is fulfilled, i.e., $⦃a,b⦄_{2,2} \nprec \{x,y,z\}$, $⦃a,b⦄_{2,2} \nsucc \{x,y,z\}$, $⦃a,b⦄_{2,2} \not\asymp \{x,y,z\}$. The reason for this is that there is neither an m-injection, nor an m-surjection between the two msets. Indeed, $\left|⦃a,b⦄_{2,2}{}^*\right| = 2$ $< |\{x,y,z\}| = 3$ and $\mu\left(⦃a,b⦄_{2,2}\right) = 4 > \mu(\{x,y,z\}) = 3$, i.e., Definitions I–VI are not fulfilled.

Let us consider the function

$$f: ⦃1,2,3,\ldots⦄_{\underbrace{2,2,2,\ldots}_{\aleph_0 \text{ times}}} \to ⦃1,2,3,\ldots⦄_{\underbrace{1,1,1,\ldots}_{\aleph_0 \text{ times}}} = \{\underbrace{1,2,3,\ldots}_{\aleph_0 \text{ elements}}\} = \mathbb{Z}^+,$$

$$f = ⦃1,1,(1,1)⦄_{2,1,2}, ⦃2,2,(2,2)⦄_{2,1,2}, ⦃3,3,(3,3)⦄_{2,1,2}, \ldots .$$

This function is an m-bijection, but not a bijection (only a surjection), see Figure 1.12. If we define the m-cardinality of msets using the relations of injection, bijection, and surjection, then we will arrive at the inequality:

$$\left|⦃1,2,3,\ldots⦄_{\underbrace{2,2,2,\ldots}_{\aleph_0}}\right| > \left|\{\underbrace{1,2,3,\ldots}_{\aleph_0}\}\right|.$$

However, this inequality is illogical. Considering that $⦃1,2,3,\ldots⦄_{\underbrace{2,2,2,\ldots}_{\aleph_0}} = \{\underbrace{1,2,3,\ldots}_{\aleph_0}\} + \{\underbrace{1,2,3,\ldots}_{\aleph_0}\}$ and $\aleph_0 + \aleph_0 = \aleph_0$, we should expect

$$\left|⦃1,2,3,\ldots⦄_{\underbrace{2,2,2,\ldots}_{\aleph_0}}\right| = \left|\{\underbrace{1,2,3,\ldots}_{\aleph_0}\}\right| = \aleph_0.$$

On the other hand, if we define the m-cardinality of msets using the relations of m-injection, m-bijection, and m-surjection, then we will arrive at the logical equality

$$\left|⦃1,2,3,\ldots⦄_{\underbrace{2,2,2,\ldots}_{\aleph_0}}\right| = \left|\{\underbrace{1,2,3,\ldots}_{\aleph_0}\}\right|.$$

This example shows that it is more logical to define the m-cardinality of msets through the relations of m-injection, m-bijection, and m-surjection, rather than through the relations of injection, bijection, and surjection.

If the mset A is a traditional set, then the notion of m-cardinality will actually coincide with the notion of the cardinality of a set. Indeed, if $m_A(a) = 1, \forall a \in A^*$, i.e., $A = ⦃A^*, m_A⦄ = A^*$, then we will have: $|A| = |A^*| = \mu(A)$.

Let there be two msets A and B, with corresponding root sets $A^*$and B^* and corresponding total multiplicities $\mu(A)$ and $\mu(B)$. The following properties follow from the above definitions.

Property I. *If* $\mu(A) = \mu(B)$ *and* $|A^*| < |B^*|$, *then* $A \prec B$, *i.e.,* $|A| < |B|$.

Proof. If $|A^*| < |B^*|$, then the function $f: A^* \to B^*$ which maps elements of set A^* to elements of set B^* is an injection, and vice versa. Equivalently, if $|A^*| < |B^*|$, then the function $g: B^* \to A^*$ is a surjection (the Partition Principle). Because $|A^*| \neq |B^*|$, the function $f: A^* \to B^*$ is not a bijection, hence $A \not\asymp B$. (Equivalently, the function $g: B^* \to A^*$ is not a bijection.) By condition, $\mu(A) = \mu(B)$. According to Definition I, the mset function $f: A \to B$ is an m-injection. (Furthermore, according to Definition III, the mset function $g: B \to A$ is an m-surjection.) According to Definition IV and Definition VI, we will have $A \prec B$, by which we proved Property I. ☐

Property II. *If* $|A^*| = |B^*|$ *and* $\mu(A) < \mu(B)$, *then* $A \prec B$, *i.e.,* $|A| < |B|$.

Proof. If $|A^*| = |B^*|$, then the function $f: A^* \to B^*$ which maps elements of set A^* to elements of set B^*, is a bijection (both injective and surjective), and vice versa. (Equivalently, if $|A^*| = |B^*|$, then the function $g: B^* \to A^*$ is a bijection.) But by condition $\mu(A) < \mu(B)$, hence $A \not\asymp B$. According to Definition I, the mset function $f: A \to B$ is an m-injection. (Furthermore, according to Definition III, the mset function $g: B \to A$ is an m-surjection.) According to Definition IV and Definition VI, we will have $A \prec B$ by which we proved Property II. ☐

Property III. $|\bar{A}| = -|A|$ *is fulfilled for msets* A *and* $\bar{A}$. *If* $\mu(A) = 0$, *then* $\bar{A} \asymp A$, *i.e.,* $|\bar{A}| = |A|$.

Proof. Since $\bar{A} = -A$, and $A^* = \bar{A}^*$, $|A^*| = |\bar{A}^*|$, so the function $f: A^* \to \bar{A}^*$ is a bijection (both injective and surjective). (Equivalently, the function $g: \bar{A}^* \to A^*$ is a bijection.) Furthermore, $\mu(\bar{A}) = -\mu(A)$, hence we will have: $|\bar{A}| = |-A| = -|A|$. If we assume that $\mu(A) = 0$, then $\mu(\bar{A}) = \mu(A)$, and, therefore, $\bar{A} \asymp A$. ☐

Property IV. *Let us assume that* $|A| < |B|$. *If* $\mu(A) = \mu(B)$, *then* $|\bar{A}| < |\bar{B}|$. *If* $|A^*| = |B^*|$, *then* $|\bar{A}| > |\bar{B}|$.

Proof. It is clear that $\bar{A}^* = A^*$, $\bar{B}^* = B^*$, $\mu(\bar{A}) = -\mu(A)$, $\mu(\bar{B}) = -\mu(B)$. Let us first suppose that $\mu(A) = \mu(B)$. We will have $\mu(\bar{A}) = \mu(\bar{B})$. Since $|A| < |B|$, so in this case $|A^*| < |B^*|$ (i.e., $|\bar{A}^*| < |\bar{B}^*|$.) Therefore, $|\bar{A}| < |\bar{B}|$.

Now let us suppose that $|A^*| = |B^*|$ (i.e., $|\bar{A}^*| = |\bar{B}^*|$). Since $|A| < |B|$, so in this case $\mu(A) < \mu(B)$, i.e., $\mu(\bar{A}) > \mu(\bar{B})$. Therefore, $|\bar{A}| > |\bar{B}|$. ☐

We will introduce a new concept, *'m-cardinal number'*. The m-cardinal number describes the m-cardinality of a mset. If the considered mset is a traditional set, then the notion of m-cardinal number coincides with the notion of the cardinal number of a set. It is clear that $\{\!\{a\}\!\}_2$, $\{a\}$ and $\{a, b\}$ have different m-cardinal numbers. It is easy to verify that according to Properties I and II, $\{a\} \prec \{\!\{a\}\!\}_2 \prec \{a, b\}$ is fulfilled. (The cardinal number of the set $\{a\}$ is 1, and that of the set $\{a, b\}$ is 2.) Therefore, the m-cardinal number of the mset $\{\!\{a\}\!\}_2$ is greater than 1, but less than 2 (see Properties I and II).

3 M-cardinality of infinite msets

We will go on to examine infinite msets. With these msets the cardinality of the root set and/or the multiplicity of the individual elements are infinite cardinal numbers. Accordingly, when we operate with these numbers, we will apply the cardinal arithmetic, assuming the Axiom of Choice.

Let us examine the two infinite msets A and $\mathbb{Z}^+$:

$$\begin{array}{ccccccc} \mathbb{Z}^+ & & 1 & 2 & 3 & \dots \\ \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ A & & 1 & 1 & 1 & 1 \end{array}$$

Here $\mathbb{Z}^+ = \{1,2,3,\dots\}$ is the set of the positive integers $\mathbb{Z}^+$ and

$$A = \underbrace{\{\!\!\{1\}\!\!\}_1 + \{\!\!\{1\}\!\!\}_1 + \{\!\!\{1\}\!\!\}_1 + \dots}_{\aleph_0} = \{\!\!\{\underbrace{1,1,1,\dots}_{\aleph_0}\}\!\!\} = \{\!\!\{\underbrace{1,1,1,\dots}_{\aleph_0}\}\!\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}} = \{\!\!\{1\}\!\!\}_{\aleph_0}.$$

It is clear that

$$\{\!\!\{1,2,3,\dots\}\!\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}} = \{1,2,3,\dots\} = \mathbb{Z}^+.$$

Theorem 3.1. *It holds that*

$$\{\!\!\{\underbrace{1,1,1,\dots}_{\aleph_0 \text{ times}}\}\!\!\} \prec \{\underbrace{1,2,3,\dots}_{\aleph_0 \text{ elements}}\}, \text{ i.e.,}$$

$$\{\!\!\{1\}\!\!\}_{\aleph_0} \prec \mathbb{Z}^+ \quad \text{or} \quad \left|\{\!\!\{1\}\!\!\}_{\aleph_0}\right| < |\mathbb{Z}^+| = \aleph_0. \tag{7}$$

Proof. We will have:

$$f: \{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\} \to \{\!\!\{1\}\!\!\}_{\aleph_0},$$

$$f = \{\!\!\{1,1,(1,1)\}\!\!\}_{1,\aleph_0,1}, \{\!\!\{2,1,(2,1)\}\!\!\}_{1,\aleph_0,1}, \{\!\!\{3,1,(3,1)\}\!\!\}_{1,\aleph_0,1}, \dots = \{\!\!\{k,1,(k,1)\}\!\!\}_{1,\aleph_0,1}, \qquad k \in \mathbb{Z}^+.$$

The proof follows from Property I. Indeed, $\mu\left(\{\!\!\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\!\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}}\right) = \mu\left(\{\!\!\{1\}\!\!\}_{\aleph_0}\right) = \aleph_0$. Furthermore, $\{\!\!\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\!\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}}^* = \{\underbrace{1,2,3,\dots}_{\aleph_0 \text{ elements}}\} = \mathbb{Z}^+$ and $\{\!\!\{1\}\!\!\}_{\aleph_0}^* = \{1\}$. It is clear that $\left|\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\right| = \aleph_0 > |\{1\}| = 1$. The mset function $f: \{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\} \to \{\!\!\{1\}\!\!\}_{\aleph_0}$ is an m-surjection. Therefore,

$$\{\!\!\{1\}\!\!\}_{\aleph_0} \prec \{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\} \quad \text{or} \quad \left|\{\!\!\{1\}\!\!\}_{\aleph_0}\right| < \left|\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\right|. \qquad \square$$

What we got is that the infinity $\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}$ is "greater" than the infinity $\{\!\!\{\underbrace{1,1,1,\dots}_{\aleph_0 \text{ times}}\}\!\!\}$. The reason is that the infinity $\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}$ is "richer" in elements, it contains an infinite number of different elements ($\aleph_0$ different elements); whereas the infinity $\{\!\!\{\underbrace{1,1,1,\dots}_{\aleph_0}\}\!\!\}$ is "poorer" in elements, it only contains one single element which repeats infinite times ($\aleph_0$ times). The m-cardinality of the mset $\{\!\!\{\underbrace{1,1,1,\dots}_{\aleph_0}\}\!\!\}$ is infinite, but less than the smallest transfinite cardinal number $\aleph_0$. Therefore, a transfinite cardinal m-number less than $\aleph_0$ should be associated with the mset $\{\!\!\{\underbrace{1,1,1,\dots}_{\aleph_0}\}\!\!\}$. Let us denote it with $\mathcal{M}_1^1 = \left|\{\!\{1\}\!\}_{\aleph_0}\right|$. We will have:

$$\mathcal{M}_1^1 < \aleph_0.$$

Analogously, it is proved that

$$\begin{gathered} \left|\{\!\{1\}\!\}_{\aleph_0}\right| < |X| = \aleph_0, \\ \left|\{\!\{1\}\!\}_{\aleph_1}\right| < |2^X| = \aleph_1, \\ \left|\{\!\{1\}\!\}_{\aleph_2}\right| < \left|2^{2^X}\right| = \aleph_2, \dots . \end{gathered} \tag{8}$$

Here X is an infinite set with cardinality $\aleph_0$. Here 2^X denotes the power set of X.

It is easily verified that $\mathcal{M}_1^1 > 1$. Indeed, $\left|\{\!\{1\}\!\}_{\aleph_0}{}^*\right| = \left|\{\!\{1\}\!\}_1{}^*\right| = |\{1\}| = 1$ and $\mu(\{\!\{1\}\!\}_{\aleph_0}) = \aleph_0 > \mu(\{\!\{1\}\!\}_1) = 1$. According to Property II, we will have: $\left|\{\!\{1\}\!\}_{\aleph_0}\right| > |\{\!\{1\}\!\}_1| = 1$, i.e., $\mathcal{M}_1^1 > 1$. (The mset function $f: \{\!\{1\}\!\}_{\aleph_0} \to \{\!\{1\}\!\}_1$, $f = \{\!\{1,1,(1,1)\}\!\}_{\aleph_0,1,\aleph_0}$ is an m-surjection.)

Now let us look at the following mset:

$$\{\!\{1\}\!\}_{\aleph_0} + \{\!\{1,2\}\!\}_{1,1} = \{\!\{1,2\}\!\}_{(\aleph_0+1),1} = \{\!\{1,2\}\!\}_{\aleph_0,1} .$$

Theorem 3.2. *It holds that*

$$\{\!\{1\}\!\}_{\aleph_0} \prec \{\!\{1,2\}\!\}_{\aleph_0,1} \prec \{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}. \tag{9}$$

Proof. Here

$$f: \{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\} \to \{\!\{1,2\}\!\}_{\aleph_0,1} , f = \{\!\{1,2,(1,2)\}\!\}_{1,1,1}, \{\!\{t,1,(t,1)\}\!\}_{1,\aleph_0,1}, \qquad t > 1, t \in \mathbb{Z}^+$$

and

$$g: \{\!\{1,2\}\!\}_{\aleph_0,1} \to \{\!\{1\}\!\}_{\aleph_0} , g = \{\!\{1,1,(1,1)\}\!\}_{\aleph_0,\aleph_0,\aleph_0}, \{\!\{2,1,(1,1)\}\!\}_{1,\aleph_0,1} .$$

The proof follows from Property I. Indeed, $\mu\left(\{\!\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}}\right) = \mu(\{\!\{1,2\}\!\}_{\aleph_0,1}) = \mu(\{\!\{1\}\!\}_{\aleph_0}) = \aleph_0$. Furthermore, $\{\!\{1,2,3,\dots\}\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}}{}^* = \{1,2,3,\dots\} = \mathbb{Z}^+$, $\{\!\{1,2\}\!\}_{\aleph_0,1}{}^* = \{1,2\}$, $\{\!\{1\}\!\}_{\aleph_0}{}^* = \{1\}$. It is clear that $\left|\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\right| = \aleph_0 > |\{1,2\}| = 2 > |\{1\}| = 1$. Therefore, the mset functions $f: \{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\} \to \{\!\{1,2\}\!\}_{\aleph_0,1}$ and $g: \{\!\{1,2\}\!\}_{\aleph_0,1} \to \{\!\{1\}\!\}_{\aleph_0}$ are m-surjective. □

Using Property I, it is proved analogously that

$$\{\!\{1\}\!\}_{\aleph_0} \prec \{\!\{1,2\}\!\}_{\aleph_0,1} \prec \{\!\{1,2,3\}\!\}_{\aleph_0,1,1} \prec \dots \prec \{\!\{1,2,3,\dots\}\!\}_{\aleph_0,\underbrace{1,1,\dots}_{\aleph_0}} \asymp \{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}. \tag{10}$$

Let us look at the transfinite m-cardinal numbers $\mathcal{M}_2^1 = \left|\{\!\{1,2\}\!\}_{\aleph_0,1}\right|$, $\mathcal{M}_3^1 = \left|\{\!\{1,2,3\}\!\}_{\aleph_0,1,1}\right|, \dots$ We will have:

$$1 < \mathcal{M}_1^1 < \mathcal{M}_2^1 < \mathcal{M}_3^1 < \dots < \aleph_0. \tag{11}$$

Taking into consideration that $\left|\{\!\{1,2\}\!\}_{1,1}\right| = |\{1,2\}| = 2$, $\left|\{\!\{1,2,3\}\!\}_{1,1,1}\right| = |\{1,2,3\}| = 3, \dots$, then formulae (9)–(11) can also be written in the following way:

$$\mathcal{M}_2^1 = \mathcal{M}_1^1 + 2 > \mathcal{M}_1^1,$$
$$\mathcal{M}_3^1 = \mathcal{M}_2^1 + 3 > \mathcal{M}_2^1, \qquad (12)$$
$$\mathcal{M}_4^1 = \mathcal{M}_3^1 + 4 > \mathcal{M}_3^1, \dots$$

We obtained the following interesting property of m-cardinal numbers: In the case under consideration, if we add a finite positive number to a transfinite m-cardinal number, we will get a new transfinite m-cardinal number that is greater than the initial one.

Theorem 3.3. *It holds that for an arbitrary finite positive integer s,*

$$\mathcal{M}_s^1 + \mathcal{M}_{s+1}^1 = \mathcal{M}_{s+1}^1 . \qquad (13)$$

Proof. Indeed, $\mathcal{M}_s^1 = |\{\!\{1,2,3,\dots,s\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s-1}}|$, $\mathcal{M}_{s+1}^1 = |\{\!\{1,2,3,\dots,s,s+1\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s}}|$ and $\mathcal{M}_s^1 + \mathcal{M}_{s+1}^1 = |\{\!\{1,2,3,\dots,s,s+1\}\!\}_{(\aleph_0+\aleph_0),\underbrace{2,2,\dots,2}_{s-1},1}|$. It is clear that

$$\left|\{\!\{1,2,3,\dots,s,s+1\}\!\}_{(\aleph_0+\aleph_0),\underbrace{2,2,\dots,2}_{s-1},1}^{\;*}\right| = \left|\{\!\{1,2,3,\dots,s,s+1\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s}}^{\;*}\right|$$
$$= |\{1,2,3,\dots,s,s+1\}| = s+1$$

and

$$\mu\left(\{\!\{1,2,3,\dots,s,s+1\}\!\}_{(\aleph_0+\aleph_0),\underbrace{2,2,\dots,2}_{s-1},1}\right) = \mu\left(\{\!\{1,2,3,\dots,s,s+1\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s}}\right) = \aleph_0.$$

Therefore,

$$\mathcal{M}_s^1 + \mathcal{M}_{s+1}^1 = \left|\{\!\{1,2,3,\dots,s,s+1\}\!\}_{(\aleph_0+\aleph_0),\underbrace{2,2,\dots,2}_{s-1},1}\right| = \left|\{\!\{1,2,3,\dots,s,s+1\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s}}\right| = \mathcal{M}_{s+1}^1.$$

This completes the proof. □

Let us denote: $\mathcal{M}_s^2 = \left|\{\!\{1\}\!\}_{\mathcal{M}_s^1}\right|, \mathcal{M}_s^3 = \left|\{\!\{1\}\!\}_{\mathcal{M}_s^2}\right|, \mathcal{M}_s^4 = \left|\{\!\{1\}\!\}_{\mathcal{M}_s^3}\right|, \dots$.

Theorem 3.4. *It holds that*

$$\aleph_0 > \mathcal{M}_s^1 > \mathcal{M}_s^2 > \mathcal{M}_s^3 > \mathcal{M}_s^4 > \cdots > 1. \qquad (14)$$

Proof. It is clear that $\mathcal{M}_s^1 = \left|\{\!\{1,2,3,\dots,s\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s-1}}\right|$. Using Property I, we can easily prove that $\aleph_0 > \mathcal{M}_s^1$. Indeed,

$$\left|\{\!\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}}^{\;*}\right| = \aleph_0 > \left|\{\!\{1,2,3,\dots,s\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s-1}}^{\;*}\right| = s$$

and

$$\mu\left(\{\!\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}}\right) = \mu\left(\{\!\{1,2,3,\dots,s\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s-1}}\right) = \aleph_0.$$

Moreover, we will have $\mathcal{M}_s^1 > \mathcal{M}_s^2 > 1$. Indeed,

$$\left|\{\!\{1,2,3,\dots,s\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s-1}}^{\;*}\right| = s \geq \left|\{\!\{1\}\!\}_{\mathcal{M}_s^1}^{\;*}\right| = \left|\{\!\{1\}\!\}_1^{\;*}\right| = 1$$

and

$$\mu\left(\{\!\{1,2,3,\dots,s\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{s-1}}\right) = \aleph_0 > \mu\left(\{\!\{1\}\!\}_{\mathcal{M}_s^1}\right) = \mathcal{M}_s^1 \geq \mu\left(\{\!\{1\}\!\}_{\mathcal{M}_1^1}\right) = \mathcal{M}_1^1 > \mu(\{\!\{1\}\!\}_1) = 1.$$

According to Property II, we will have $\mathcal{M}_s^2 > \mathcal{M}_s^3 > 1$. Indeed, $\left|\{\!\{1\}\!\}_{\mathcal{M}_s^1}{}^*\right| = \left|\{\!\{1\}\!\}_{\mathcal{M}_s^2}{}^*\right| = \left|\{\!\{1\}\!\}_1{}^*\right| = 1$ and $\mu\big(\{\!\{1\}\!\}_{\mathcal{M}_s^1}\big) = \mathcal{M}_s^1 > \mu\big(\{\!\{1\}\!\}_{\mathcal{M}_s^2}\big) = \mathcal{M}_s^2 > \mu(\{\!\{1\}\!\}_1) = 1$. This process can continue infinitely, and thus we will obtain the formula (14). □

Theorem 3.5. *For an arbitrary finite positive integer t, it holds that:*

$$1 < \mathcal{M}_1^t < \mathcal{M}_2^t < \mathcal{M}_3^t < \cdots < \aleph_0. \tag{15}$$

Proof. We will prove (15) using formula (11) and mathematical induction on t, starting with $t = 1$. According to formula (11), we will have $1 < \mathcal{M}_1^1 < \mathcal{M}_2^1 < \mathcal{M}_3^1 < \cdots < \aleph_0$. Let us assume that $1 < \mathcal{M}_1^p < \mathcal{M}_2^p < \mathcal{M}_3^p < \cdots < \aleph_0$, where p is an arbitrary finite positive integer. It is clear that $\mathcal{M}_1^{p+1} = \left|\{\!\{1\}\!\}_{\mathcal{M}_1^p}\right|, \mathcal{M}_2^{p+1} = \left|\{\!\{1\}\!\}_{\mathcal{M}_2^p}\right|, \mathcal{M}_3^{p+1} = \left|\{\!\{1\}\!\}_{\mathcal{M}_3^p}\right|$, ... We will have

$$\left|\{\!\{1\}\!\}_1{}^*\right| = \left|\{\!\{1\}\!\}_{\mathcal{M}_1^p}{}^*\right| = \left|\{\!\{1\}\!\}_{\mathcal{M}_2^p}{}^*\right| = \left|\{\!\{1\}\!\}_{\mathcal{M}_3^p}{}^*\right| = \cdots = \left|\{\!\{1\}\!\}_{\aleph_0}{}^*\right| = 1$$

and

$$\mu(\{\!\{1\}\!\}_1) = 1 < \mu\left(\{\!\{1\}\!\}_{\mathcal{M}_1^p}\right) = \mathcal{M}_1^p < \mu\left(\{\!\{1\}\!\}_{\mathcal{M}_2^p}\right) = \mathcal{M}_2^p < \mu\left(\{\!\{1\}\!\}_{\mathcal{M}_3^p}\right) = \mathcal{M}_3^p < \cdots < \mu\big(\{\!\{1\}\!\}_{\aleph_0}\big) = \aleph_0.$$

According to Property II, we will have $\{\!\{1\}\!\}_1 \prec \{\!\{1\}\!\}_{\mathcal{M}_1^p} \prec \{\!\{1\}\!\}_{\mathcal{M}_2^p} \prec \{\!\{1\}\!\}_{\mathcal{M}_3^p} \prec \cdots \prec \{\!\{1\}\!\}_{\aleph_0}$, i.e., $1 < \mathcal{M}_1^{p+1} < \mathcal{M}_2^{p+1} < \mathcal{M}_3^{p+1} < \cdots < \mathcal{M}_1^1 < \aleph_0$. □

Now let us denote:

$$\mathcal{M}_{s,k-1}^{t+1} = \left|\{\!\{1,2,3,\dots,k\}\!\}_{(\mathcal{M}_s^t),\underbrace{1,1,\dots,1}_{k-1}}\right|,$$

where t, s, k are arbitrary finite positive integers.

Example 3.1. $\mathcal{M}_{1,0}^2 = \mathcal{M}_1^2 = \left|\{\!\{1\}\!\}_{\mathcal{M}_1^1}\right|$, $\mathcal{M}_{1,0}^3 = \mathcal{M}_1^3 = \left|\{\!\{1\}\!\}_{\mathcal{M}_1^2}\right|$, ..., $\mathcal{M}_{2,0}^2 = \mathcal{M}_2^2 = \left|\{\!\{1\}\!\}_{\mathcal{M}_2^1}\right|$, ..., $\mathcal{M}_{1,1}^2 = \left|\{\!\{1,2\}\!\}_{\mathcal{M}_1^1,1}\right|$, ..., $\mathcal{M}_{3,2}^2 = \left|\{\!\{1,2,3\}\!\}_{\mathcal{M}_3^1,1,1}\right|$, ...

Theorem 3.6. *Using Properties I and II, we can prove the following relations (16)–(17):*

$$\aleph_0 > \mathcal{M}_k^1 > \mathcal{M}_{s,k-1}^2 > \mathcal{M}_{s,k-1}^3 > \mathcal{M}_{s,k-1}^4 > \cdots > k-1, \tag{16}$$

$$1 < \mathcal{M}_1^{t+1} \le \mathcal{M}_{1,k-1}^{t+1} < \mathcal{M}_{2,k-1}^{t+1} < \mathcal{M}_{3,k-1}^{t+1} < \cdots < \aleph_0. \tag{17}$$

(In formula (17), equality $\mathcal{M}_1^{t+1} = \mathcal{M}_{1,k-1}^{t+1}$ is obtained when $k = 1$.)

Proof. Indeed,

$$\left|\{\!\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}}{}^*\right| = \aleph_0 > \left|\{\!\{1,2,3,\dots,k\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{k-1}}{}^*\right| = \left|\{\!\{1,2,3,\dots,k\}\!\}_{(\mathcal{M}_s^1),\underbrace{1,1,\dots,1}_{k-1}}{}^*\right|$$
$$= \left|\{\!\{1,2,3,\dots,k\}\!\}_{(\mathcal{M}_s^2),\underbrace{1,1,\dots,1}_{k-1}}{}^*\right| = \cdots = |\{1,2,3,\dots,k\}| = k$$

and

$$\mu\left(\{\!\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{1,1,1,\dots}_{\aleph_0}}\right) = \aleph_0 = \mu\left(\{\!\{1,2,3,\dots,k\}\!\}_{\aleph_0,\underbrace{1,1,\dots,1}_{k-1}}\right) = \aleph_0 + k - 1$$
$$> \mu\left(\{\!\{1,2,3,\dots,k\}\!\}_{(\mathcal{M}_s^1),\underbrace{1,1,\dots,1}_{k-1}}\right) = \mathcal{M}_s^1 + k - 1 > \mu\left(\{\!\{1,2,3,\dots,k\}\!\}_{(\mathcal{M}_s^2),\underbrace{1,1,\dots,1}_{k-1}}\right)$$
$$= \mathcal{M}_s^2 + k - 1 > \cdots > k - 1.$$

(Here we used formula (14).) Therefore, formula (16) is fulfilled.
Furthermore,

$$\left|⦃1⦄_1{}^*\right| = \left|⦃1⦄_{(\mathcal{M}_1^t)}{}^*\right| = 1 \le \left|⦃1,2,3,\dots,k⦄_{(\mathcal{M}_1^t),\underbrace{1,1,\dots,1}_{k-1}}{}^*\right| = \left|⦃1,2,3,\dots,k⦄_{(\mathcal{M}_2^t),\underbrace{1,1,\dots,1}_{k-1}}{}^*\right| = \cdots$$

$$= |\{1,2,3,\dots,k\}| = k < \left|⦃\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}⦄_{\underbrace{1,1,1,\dots}_{\aleph_0}}{}^*\right| = \aleph_0$$

and

$$\mu(⦃1⦄_1) = 1 < \mu\left(⦃1⦄_{(\mathcal{M}_1^t)}\right) = \mathcal{M}_1^t \le \mu\left(⦃1,2,3,\dots,k⦄_{(\mathcal{M}_1^t),\underbrace{1,1,\dots,1}_{k-1}}\right) = \mathcal{M}_1^t + k - 1$$

$$< \mu\left(⦃1,2,3,\dots,k⦄_{(\mathcal{M}_2^t),\underbrace{1,1,\dots,1}_{k-1}}\right) = \mathcal{M}_2^t + k - 1 < \cdots < \mu\left(⦃\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}⦄_{\underbrace{1,1,1,\dots}_{\aleph_0}}\right) = \aleph_0$$

$$= \aleph_0 + k - 1.$$

(Here we used formula (15).) Therefore, formula (17) is fulfilled. □

Theorem 3.7. *The following relations hold:*

$$1 < \mathcal{M}_s^{t+1} < \mathcal{M}_{s,1}^{t+1} < \mathcal{M}_{s,2}^{t+1} < \mathcal{M}_{s,3}^{t+1} < \cdots < \aleph_0. \tag{18}$$

Proof. Indeed, Using Property I,

$$(⦃1⦄_1) = 1 < \mathcal{M}_1^t \le \mu\left(⦃1⦄_{\mathcal{M}_s^t}\right) = \mathcal{M}_s^t \le \mu\left(⦃1,2⦄_{(\mathcal{M}_s^t),1}\right) = \mathcal{M}_s^t + 1 \le \mu\left(⦃1,2,3⦄_{(\mathcal{M}_s^t),1,1}\right)$$

$$= \mathcal{M}_s^t + 2 \le \cdots \quad < \mu\left(⦃\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}⦄_{\underbrace{1,1,1,\dots}_{\aleph_0}}\right) = \aleph_0$$

and

$$\left|⦃1⦄_1{}^*\right| = \left|⦃1⦄_{\mathcal{M}_s^t}{}^*\right| = 1 < \left|⦃1,2⦄_{\mathcal{M}_s^t,1}{}^*\right| = 2 < \left|⦃1,2,3⦄_{\mathcal{M}_s^t,1,1}{}^*\right| = 3 < \cdots < \left|⦃\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}⦄_{\underbrace{1,1,1,\dots}_{\aleph_0}}{}^*\right| = \aleph_0.$$

This completes the proof. □

We obtained an infinite number of different sequences of decreasing transfinite m-cardinal numbers which are less than $\aleph_0$ (the cardinal number corresponding to the countably infinite set). Therefore, if we examine the m-cardinality of infinite msets, it will appear that between the finite numbers and $\aleph_0$ there are hierarchies of infinities which have not been investigated so far.

Theorem 3.8. *Using Properties I and II, we prove that:*

$$⦃\underbrace{1,2,3,4,\dots}_{\mathbb{Z}^+}⦄_{\aleph_0,\underbrace{1,1,1,\dots}_{\aleph_0}} \asymp ⦃\underbrace{1,2,3,4,\dots}_{\mathbb{Z}^+}⦄_{\aleph_0,\aleph_0,\underbrace{1,1,\dots}_{\aleph_0}} \asymp \cdots \asymp ⦃\underbrace{1,2,3,4,\dots}_{\mathbb{Z}^+}⦄_{\underbrace{\aleph_0,\aleph_0,\aleph_0,\aleph_0,\dots}_{\aleph_0}} \asymp \{\underbrace{1,2,3,4,\dots}_{\mathbb{Z}^+}\}. \tag{19}$$

Proof. Here we have used the fact that

$$\mu\left(⦃1,2,3,4\dots⦄_{\aleph_0,\underbrace{1,1,1,\dots}_{\aleph_0}}\right) = \mu\left(⦃1,2,3,4,\dots⦄_{\aleph_0,\aleph_0,\underbrace{1,1,\dots}_{\aleph_0}}\right) = \cdots = \mu\left(⦃1,2,3,4,\dots⦄_{\underbrace{\aleph_0,\aleph_0,\aleph_0,\aleph_0,\dots}_{\aleph_0}}\right)$$

$$= \mu\left(\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\right) = \aleph_0.$$ □

Theorem 3.9. *The following relations hold between msets with negative multiplicity*:

$$⦃1⦄_1 \succ ⦃1⦄_{-1} \succ ⦃1⦄_{-2} \succ ⦃1⦄_{-3} \succ \cdots \succ ⦃1⦄_{-\aleph_0}. \tag{20}$$

Proof. We will prove, for example, that $\{\!\{1\}\!\}_1 \succ \{\!\{1\}\!\}_{-1}$. The other relations are proved in an analogous way. The proof follows from Property II. Indeed, $\{\!\{1\}\!\}_{-1}{}^* = \{\!\{1\}\!\}_1{}^* = \{1\}$. It is clear that $|\{\!\{1\}\!\}_{-1}{}^*| = |\{\!\{1\}\!\}_1{}^*| = |\{1\}| = 1$. Furthermore, $\mu(\{\!\{1\}\!\}_1) = 1 > \mu(\{\!\{1\}\!\}_{-1}) = -1$. Therefore, $\{\!\{1\}\!\}_1 \succ \{\!\{1\}\!\}_{-1}$. Analogously, it is proved that $\{\!\{1\}\!\}_1 \succ \{\!\{1\}\!\}_{-1} \succ \{\!\{1\}\!\}_{-2} \succ \{\!\{1\}\!\}_{-3} \succ \cdots \succ \{\!\{1\}\!\}_{-\aleph_0}$. □

Theorem 3.10. *The following relations hold:*

$$\begin{gathered}
\{\!\{1\}\!\}_{-\aleph_0} \prec \{\!\{1\}\!\}_{\aleph_0}, \\
\{\!\{1,2\}\!\}_{-\aleph_0,-1} \prec \{\!\{1,2\}\!\}_{\aleph_0,1}, \\
\{\!\{1,2,3\}\!\}_{-\aleph_0,-1,-1} \prec \{\!\{1,2,3\}\!\}_{\aleph_0,1,1}, \\
\cdots \\
\{\!\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{-1,-1,-1,\ldots}_{\aleph_0}} \prec \{\!\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{1,1,1,\ldots}_{\aleph_0}} \asymp \{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}.
\end{gathered} \tag{21}$$

Proof. The proof follows from Property II. For example, to prove that $\{\!\{1\}\!\}_{-\aleph_0} \prec \{\!\{1\}\!\}_{\aleph_0}$, we take into consideration that $\left|\{\!\{1\}\!\}_{-\aleph_0}{}^*\right| = \left|\{\!\{1\}\!\}_{\aleph_0}{}^*\right| = 1$ and $\mu\left(\{\!\{1\}\!\}_{-\aleph_0}\right) = -\aleph_0 < \mu\left(\{\!\{1\}\!\}_{\aleph_0}\right) = \aleph_0$. □

Let us denote with $\bar{\mathcal{M}}_1^1, \bar{\mathcal{M}}_2^1, \bar{\mathcal{M}}_3^1, \ldots, \bar{\aleph}_0$ the transfinite m-cardinal numbers corresponding to the viewed msets: $\bar{\mathcal{M}}_1^1 = \left|\{\!\{1\}\!\}_{-\aleph_0}\right|, \bar{\mathcal{M}}_2^1 = \left|\{\!\{1,2\}\!\}_{(-\aleph_0),-1}\right|, \bar{\mathcal{M}}_3^1 = \left|\{\!\{1,2,3\}\!\}_{(-\aleph_0),-1,-1}\right|, \ldots,$ $\bar{\aleph}_0 = \left|\{\!\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{-1,-1,-1,\ldots}_{\aleph_0}}\right| = -\aleph_0$.

Theorem 3.11. *We will have:*

$$\{\!\{1\}\!\}_{-\aleph_0} \prec \{\!\{1,2\}\!\}_{(-\aleph_0),-1} \prec \{\!\{1,2,3\}\!\}_{(-\aleph_0),-1,-1} \prec \cdots \prec \{\!\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\!\}_{(-\aleph_0),\underbrace{-1,-1,\ldots}_{\aleph_0}} \asymp \{\!\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{-1,-1,-1,\ldots}_{\aleph_0}}, \text{ i. e.,}$$

$$\bar{\mathcal{M}}_1^1 < \bar{\mathcal{M}}_2^1 < \bar{\mathcal{M}}_3^1 < \cdots < -\aleph_0$$

and

$$\bar{\mathcal{M}}_1^1 < \mathcal{M}_1^1, \bar{\mathcal{M}}_2^1 < \mathcal{M}_2^1, \bar{\mathcal{M}}_3^1 < \mathcal{M}_3^1, \ldots \tag{22}$$

Proof. This follows from Properties III and IV. □

Therefore, the smallest possible infinity in this hierarchy will be the infinity $\bar{\mathcal{M}}_1^1$, corresponding to mset $\{\!\{1\}\!\}_{-\aleph_0}$.

Let us denote $\bar{\mathcal{M}}_s^2 = \left|\{\!\{1\}\!\}_{-\mathcal{M}_s^1}\right|, \bar{\mathcal{M}}_s^3 = \left|\{\!\{1\}\!\}_{-\mathcal{M}_s^2}\right|, \bar{\mathcal{M}}_s^4 = \left|\{\!\{1\}\!\}_{-\mathcal{M}_s^3}\right|, \ldots$. Here s is an arbitrary finite positive integer.

Theorem 3.12. *It holds that:*

$$\bar{\mathcal{M}}_1^1 < \bar{\mathcal{M}}_s^2 < \bar{\mathcal{M}}_s^3 < \bar{\mathcal{M}}_s^4 < \cdots. \tag{23}$$

Proof. Indeed, $\left|\{\!\{1\}\!\}_{\aleph_0}{}^*\right| = \left|\{\!\{1\}\!\}_{\mathcal{M}_s^1}{}^*\right|$ and $\mu\left(\{\!\{1\}\!\}_{\aleph_0}\right) = \aleph_0 > \mu\left(\{\!\{1\}\!\}_{\mathcal{M}_s^1}\right) = \mathcal{M}_s^1$, therefore $\mathcal{M}_1^1 > \mathcal{M}_s^2$. Furthermore, $\mathcal{M}_1^1 > \mathcal{M}_s^2 > \mathcal{M}_s^3 > \mathcal{M}_s^4 > \cdots$ (see formula (14)) and $\left|\{\!\{1\}\!\}_{-\aleph_0}{}^*\right| = \left|\{\!\{1\}\!\}_{-\mathcal{M}_s^1}{}^*\right| = \left|\{\!\{1\}\!\}_{-\mathcal{M}_s^2}{}^*\right| = \left|\{\!\{1\}\!\}_{-\mathcal{M}_s^3}{}^*\right| = \cdots = 1$. According to Property IV, we will have $\bar{\mathcal{M}}_1^1 < \bar{\mathcal{M}}_s^2 < \bar{\mathcal{M}}_s^3 < \bar{\mathcal{M}}_s^4 < \cdots$. □

Theorem 3.13. *It holds that for an arbitrary finite positive integer* t,

$$\bar{\mathcal{M}}_1^{t+1} > \bar{\mathcal{M}}_2^{t+1} > \bar{\mathcal{M}}_3^{t+1} > \cdots. \tag{24}$$

Proof. This follows from Property IV and formula (15). ☐

Let us denote with $\bar{\mathcal{M}}_{s,k-1}^{t+1} = \left| \{\!\{1,2,3,\dots,k\}\!\}_{(-\mathcal{M}_s^t),\underbrace{-1,-1,\dots,-1}_{k-1}} \right|$ the transfinite m-cardinal numbers corresponding to the respective msets. (Here t, s, k are arbitrary finite positive integers.)

Example 3.2.

$$\bar{\mathcal{M}}_{1,0}^2 = \bar{\mathcal{M}}_1^2 = \left|\{\!\{1\}\!\}_{-\mathcal{M}_1^1}\right|, \dots,$$

$$\bar{\mathcal{M}}_{1,1}^2 = \left|\{\!\{1,2\}\!\}_{-\mathcal{M}_1^1,-1}\right|, \bar{\mathcal{M}}_{3,2}^2 = \left|\{\!\{1,2,3\}\!\}_{-\mathcal{M}_3^1,-1,-1}\right|, \dots$$

Theorem 3.14. *We will have:*

$$-\aleph_0 > \bar{\mathcal{M}}_k^1 < \bar{\mathcal{M}}_{s,k-1}^2 < \bar{\mathcal{M}}_{s,k-1}^3 < \bar{\mathcal{M}}_{s,k-1}^4 < \cdots < -k+1, \tag{25}$$

$$-k+1 > \bar{\mathcal{M}}_{1,k-1}^{t+1} > \bar{\mathcal{M}}_{2,k-1}^{t+1} > \bar{\mathcal{M}}_{3,k-1}^{t+1} > \bar{\mathcal{M}}_{4,k-1}^{t+1} > \cdots \tag{26}$$

Proof. Indeed,

$$\left| \{\!\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{-1,-1,-1,\dots}_{\aleph_0}}{}^* \right| = \aleph_0 > \left| \{\!\{1,2,3,\dots,k\}\!\}_{(-\aleph_0),\underbrace{-1,-1,\dots,-1}_{k-1}}{}^* \right| =$$

$$\left| \{\!\{1,2,3,\dots,k\}\!\}_{(-\mathcal{M}_s^1),\underbrace{-1,-1,\dots,-1}_{k-1}}{}^* \right| = \left| \{\!\{1,2,3,\dots,k\}\!\}_{(-\mathcal{M}_s^2),\underbrace{-1,-1,\dots,-1}_{k-1}}{}^* \right| = \cdots = |\{1,2,3,\dots,k\}| = k\,,$$

and

$$\mu\left(\{\!\{\underbrace{1,2,3,\dots}_{\mathbb{Z}^+}\}\!\}_{\underbrace{-1,-1,-1,\dots}_{\aleph_0}} \right) = -\aleph_0 = \mu\left(\{\!\{1,2,3,\dots,k\}\!\}_{(-\aleph_0),\underbrace{-1,-1,\dots,-1}_{k-1}} \right) = -\aleph_0 - k + 1$$

$$< \mu\left(\{\!\{1,2,3,\dots,k\}\!\}_{(-\mathcal{M}_s^1),\underbrace{-1,-1,\dots,-1}_{k-1}} \right) = -\mathcal{M}_s^1 - k + 1$$

$$< \mu\left(\{\!\{1,2,3,\dots,k\}\!\}_{(-\mathcal{M}_s^2),\underbrace{-1,-1,\dots,-1}_{k-1}} \right) = -\mathcal{M}_s^2 - k + 1 < \cdots < -k+1.$$

Above, we used formula (14) and Properties I and II. Therefore, formula (25) is fulfilled. Furthermore,

$$\left| \{\!\{1,2,3,\dots,k\}\!\}_{(-\mathcal{M}_1^t),\underbrace{-1,-1,\dots,-1}_{k-1}}{}^* \right| = \left| \{\!\{1,2,3,\dots,k\}\!\}_{(-\mathcal{M}_2^t),\underbrace{-1,-1,\dots,-1}_{k-1}}{}^* \right| = \cdots = |\{1,2,3,\dots,k\}| = k$$

and

$$-k+1 > \mu\left(\{\!\{1,2,3,\dots,k\}\!\}_{(-\mathcal{M}_1^t),\underbrace{-1,-1,\dots,-1}_{k-1}} \right) = -\mathcal{M}_1^t - k + 1 >$$

$$\mu\left(\{\!\{1,2,3,\dots,k\}\!\}_{(-\mathcal{M}_2^t),\underbrace{-1,-1,\dots,-1}_{k-1}} \right) = -\mathcal{M}_2^t - k + 1 > \cdots.$$

The last equation used Formula (15) and Properties I and II. Therefore, Formula (26) is fulfilled. ☐

Corollary 3.1. *Since $k \geq 1$, then according to formulae (25) and (26) we will have:*

$$\bar{\mathcal{M}}_k^1 < \bar{\mathcal{M}}_{s,k-1}^2 < \bar{\mathcal{M}}_{s,k-1}^3 < \bar{\mathcal{M}}_{s,k-1}^4 < \cdots < 0, \tag{27}$$

$$0 > \bar{\mathcal{M}}_{1,k-1}^{t+1} > \bar{\mathcal{M}}_{2,k-1}^{t+1} > \bar{\mathcal{M}}_{3,k-1}^{t+1} > \bar{\mathcal{M}}_{4,k-1}^{t+1} > \cdots. \tag{28}$$

We obtained an interesting result: there are infinite msets with negative multiplicity (for example, $\bar{\mathcal{M}}_1^1 = \left|\{\!\{1\}\!\}_{-\aleph_0}\right|$), which have m-cardinality less than zero.

Theorem 3.15. *Using Property II, we can prove the following relations:*

$$\{\!\{1\}\!\}_{-\aleph_0} \succ \{\!\{1\}\!\}_{-\aleph_1} \succ \{\!\{1\}\!\}_{-\aleph_2} \succ \cdots \tag{29}$$

Proof. Indeed,

$$\left|\{\!\{1\}\!\}_{-\aleph_0}{}^*\right| = \left|\{\!\{1\}\!\}_{-\aleph_1}{}^*\right| = \left|\{\!\{1\}\!\}_{-\aleph_2}{}^*\right| = \cdots = 1$$

and, furthermore,

$$\mu\left(\{\!\{1\}\!\}_{-\aleph_0}\right) = -\aleph_0 > \mu\left(\{\!\{1\}\!\}_{-\aleph_1}\right) = -\aleph_1 > \mu\left(\{\!\{1\}\!\}_{-\aleph_2}\right) = -\aleph_2 > \cdots$$ □

Let us substitute:

$$\bar{\mathcal{M}}_{\aleph_1}^1 = \left|\{\!\{1\}\!\}_{-\aleph_1}\right|, \bar{\mathcal{M}}_{\aleph_2}^1 = \left|\{\!\{1\}\!\}_{-\aleph_2}\right|, \ldots$$

We will have:

$$\bar{\mathcal{M}}_1^1 > \bar{\mathcal{M}}_{\aleph_1}^1 > \bar{\mathcal{M}}_{\aleph_2}^1 > \cdots \tag{30}$$

We obtained a decreasing sequence of transfinite m-cardinal numbers, corresponding to infinite msets with negative multiplicity, with no smallest transfinite m-cardinal number in this sequence.

Theorem 3.16. Acc*ording to Property I, also valid is the relation:*

$$\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots} \succ \{\!\{1\}\!\}_1 . \tag{31}$$

Proof. Here

$$f\colon \{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots} \to \{\!\{1\}\!\}_1 ,$$

$$f = \{\!\{1,1,(1,1)\}\!\}_{\frac{1}{2},1,\frac{1}{2}}, \{\!\{2,1,(2,1)\}\!\}_{\frac{1}{4},1,\frac{1}{4}}, \{\!\{3,1,(3,1)\}\!\}_{\frac{1}{8},1,\frac{1}{8}}, \ldots = \{\!\{k,1,(k,1)\}\!\}_{\frac{1}{2^k},1,\frac{1}{2^k}}, \qquad k \in \mathbb{Z}^+ .$$

We will have

$$\mu(\{\!\{1\}\!\}_1) = \mu\left(\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots}\right) = 1$$

And, furthermore,

$$|\{\!\{1\}\!\}_1{}^*| = 1 < \left|\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots}{}^*\right| = \aleph_0.$$

Therefore, the mset function $f\colon \{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots} \to \{\!\{1\}\!\}_1$ is an m-surjection. These msets have equal total multiplicity, which is equal to 1, but they have different m-cardinality. □

Using Property II, we can prove the following relations (32)–(36):

$$\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots} \prec \{\!\{1,2,3,4,5,6,\ldots\}\!\}_{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\ldots} \asymp \{\underbrace{1,2,3,4,5,6,\ldots}_{\mathbb{Z}^+}\}. \tag{32}$$

(Here we used that $\mu\left(\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots}\right) = 1 < \mu\left(\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\ldots}\right) = \mu\left(\{\underbrace{1,2,3,\ldots}_{\mathbb{Z}^+}\}\right) = \aleph_0.$)

$$\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\frac{1}{5},-\frac{1}{6},\ldots} \prec \{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots} \tag{33}$$

(Here we used that

$$\mu\left(\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\frac{1}{5},-\frac{1}{6},\ldots}\right) = \ln(2) < \mu\left(\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots}\right) = 1.\ )$$

$$\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{-1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\ldots} \prec \{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots} \tag{34}$$

(Here we used that

$$\mu\left(\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{-1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\ldots}\right) = 0 < \mu\left(\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\frac{1}{128},\ldots}\right) = 1.\ )$$

$$\{\!\{1\}\!\}_{\frac{1}{2}} + \{\!\{1\}\!\}_{\frac{1}{4}} + \{\!\{1\}\!\}_{\frac{1}{8}} + \{\!\{1\}\!\}_{\frac{1}{16}} + \{\!\{1\}\!\}_{\frac{1}{64}} + \cdots = \{\!\{1,1,1,1,1,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\ldots} = \{\!\{1\}\!\}_1. \tag{35}$$

(Here we used that $\mu(\{\!\{1\}\!\}_1) = 1 = \mu\left(\{\!\{1,1,1,1,1,\ldots\}\!\}_{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\ldots}\right) = 1.$)

Using Property III, we can easily prove the following relations (36)–(40):

$$\{\!\{1,2\}\!\}_{1,-1} \asymp \{\!\{1,2\}\!\}_{-1,1} \tag{36}$$

$$\{\!\{1,2,3,4\}\!\}_{1,-1,1,-1} \asymp \{\!\{1,2,3,4\}\!\}_{-1,1,-1,1} \tag{37}$$

$$\ldots$$

$$\{\!\{1,2,3,4,\ldots,2n\}\!\}_{\underbrace{1,-1,1,-1,\ldots,-1}_{2n}} \asymp \{\!\{1,2,3,4,\ldots,2n\}\!\}_{\underbrace{-1,1,-1,1,\ldots,1}_{2n}} \tag{38}$$

$$\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{-1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\ldots} \asymp \{\!\{1,2,3,4,5,6,\ldots\}\!\}_{1,-\frac{1}{2},-\frac{1}{4},-\frac{1}{8},-\frac{1}{16},-\frac{1}{64},\ldots}. \tag{39}$$

(In the last formula, we used that $\mu\left(\{\!\{1,2,3,4,5,6,\ldots\}\!\}_{-1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{64},\ldots}\right) = 0.$)

We will present two interesting formulae by which real-valued msets are obtained. The multiplicities of these msets are equal to the well-known mathematical constants π and e:

$$\{\!\{1\}\!\}_{\frac{4}{1}} + \{\!\{1\}\!\}_{-\frac{4}{3}} + \{\!\{1\}\!\}_{\frac{4}{5}} + \{\!\{1\}\!\}_{-\frac{4}{7}} + \{\!\{1\}\!\}_{\frac{4}{9}} + \cdots = \{\!\{1,1,1,1,1,\ldots\}\!\}_{\frac{4}{1},-\frac{4}{3},\frac{4}{5},-\frac{4}{7},\frac{4}{9},\ldots} = \{\!\{1\}\!\}_{\pi}, \tag{40}$$

$$\{\!\{1\}\!\}_1 + \{\!\{1\}\!\}_{\frac{1}{1!}} + \{\!\{1\}\!\}_{\frac{1}{2!}} + \{\!\{1\}\!\}_{\frac{1}{3!}} + \{\!\{1\}\!\}_{\frac{1}{4!}} + \cdots = \{\!\{1,1,1,1,1,\ldots\}\!\}_{1,\frac{1}{1!},\frac{1}{2!},\frac{1}{3!},\frac{1}{4!},\ldots} = \{\!\{1\}\!\}_e. \tag{41}$$

(In the last formulae we used that $\mu\left(\{\!\{1,1,1,1,1,\ldots\}\!\}_{\frac{4}{1},-\frac{4}{3},\frac{4}{5},-\frac{4}{7},\frac{4}{9},\ldots}\right) = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots = \pi$ and $\mu\left(\{\!\{1,1,1,1,1,\ldots\}\!\}_{1,\frac{1}{1!},\frac{1}{2!},\frac{1}{3!},\frac{1}{4!},\ldots}\right) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e.$)

Finally, we will present a formula that illustrates the relationship between infinite msets and infinite series:

$$\{\!\{1\}\!\}_1 + \{\!\{1\}\!\}_{\frac{x}{1!}} + \{\!\{1\}\!\}_{\frac{x^2}{2!}} + \{\!\{1\}\!\}_{\frac{x^3}{3!}} + \{\!\{1\}\!\}_{\frac{x^4}{4!}} + \cdots = \{\!\{1,1,1,1,1,\ldots\}\!\}_{1,\frac{x}{1!},\frac{x^2}{2!},\frac{x^3}{3!},\frac{x^4}{4!},\ldots} = \{\!\{1\}\!\}_{e^x}. \quad (42)$$

4 Conclusion

The present work examines and characterizes infinite msets. It presents the possibility of obtaining msets with elements of negative or fraction multiplicity. A new concept of 'm-cardinal number' is introduced, which is a generalization of the concept of cardinal number in the context of multisets. So far it has been assumed that the smallest transfinite cardinal number is $\aleph_0$, the cardinal number corresponding to a countably infinite set. Concrete examples of transfinite m-cardinal numbers are given in the paper, corresponding to infinite msets, less than $\aleph_0$.

Upon investigating infinite msets, it has been established that there is an infinite number of different sequences of decreasing transfinite m-cardinal numbers which are less than $\aleph_0$. Therefore, between the finite numbers and $\aleph_0$ there are hierarchies of infinities which have not been investigated so far.

Furthermore, there are infinite msets with negative multiplicity which have an m-cardinality less than zero. It has been proved that there is a decreasing sequence of transfinite m-cardinal numbers, corresponding to infinite msets with negative multiplicity, without a smallest transfinite m-cardinal number in this sequence.

The concepts and methods presented can find interesting applications and help the improvement and development of a variety of spheres, such as the set theory, mathematical logic, probability and statistics, combinatorics, order theory, computer science, data science, artificial intelligence, etc.

References

[1] Blizard, W. D. (1989). Multiset theory. *Notre Dame Journal of Formal Logic*, 30(1), 36–66.

[2] Blizard, W. D. (1989). Real-valued multisets and fuzzy sets. *Fuzzy Sets and Systems*, 33, 77–97.

[3] Blizard, W. D. (1990). Negative membership. *Notre Dame Journal of Formal Logic*, 31(1), 346–368.

[4] Blizard, W. D. (1991). The development of multiset theory. *Modern Logic*, 1(4), 319–352.

[5] Da F. Costa, L. (2021). *Multisets*. Available online at: https://hal.science/hal-03388935/.

[6] Girish, K. P., & John, S. J. (2009). Relations and functions in multiset context. *Information Sciences*, 179(6), 758–768.

[7] Singh, D., Ibrahim, A. M., Yohanna, T., Singh, J. N. (2007). An overview of the applications of multisets. *Novi Sad Journal of Mathematics*, 37(2), 73–92.

[8] Singh, D., Ibrahim, A. M., Yohana, T., Singh, J. N. (2011). Complementation in Multiset Theory. *International Mathematical Forum*, 6(38), 1877–1884.

[9] Syropoulos, A. (2001). Mathematics of Multisets. In: Calude, C.S., Păun, G., Rozenberg, G., Salomaa, A. (eds) *Multiset Processing*. WMC 2000. Lecture Notes in Computer Science, vol 2235. Springer, Berlin, Heidelberg. Available online at: `https://doi.org/10.1007/3-540-45523-X_17`.