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On some series involving the binomial coefficients $\binom{3n}{n}$

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Dedicated to the memory of Professor Khristo N. Boyadzhiev
who passed away on June 28, 2023,
and who loved infinite series.

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Abstract: Using a simple transformation, we obtain much simpler forms for some series involving binomial coefficients $\binom{3n}{n}$ derived by Necdet Batır. New evaluations are given and connections with Fibonacci numbers and the golden ratio are established. Finally we derive some Fibonacci and Lucas series involving the reciprocals of $\binom{3n}{n}$.

Keywords: Binomial coefficient, Binomial sum, Fibonacci number, Lucas number.
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1 Introduction

In an article published in the year 2005, Batır [1], inspired by the results of Lehmer [6], studied the series $\sum_{k=1}^{\infty} \binom{3k}{k}^{-1} \frac{z^k}{k^n}$, giving particular attention to the special cases $n \in \mathbb{N} \cup \{0\}$, for which he derived explicit closed formulas. He obtained many interesting formulas by evaluating the closed forms at appropriate arguments. Some of his results had earlier been obtained experimentally by Borwein and Girgensohn [2]. In [4], D'Aurizio and Di Trani studied this kind of series using hypergeometric functions. In the recent paper [3], Chu evaluated many series having the form

$$\sum_{k=1}^{\infty} \frac{z^k}{k^{a+1}\binom{3k+b}{k}},$$

where $a \in \{0; \pm 1; \pm 2\}$ and $b \in \{0; 1; \pm 2\}$.

The purpose of this note is to derive equivalent but much simpler expressions for the special cases and thereby obtain new evaluations.

Batır [1, Identity (3.1)] showed, for $|z| \le \frac{27}{4}$, that

$$\sum_{k=1}^{\infty} \frac{z^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\phi(z)-1}\right) - \frac{1}{2}\ln^2\left(\frac{\phi^3(z)+1}{\left(\phi(z)+1\right)^3}\right), \tag{1}$$

where

$$\phi(z) = \sqrt[3]{\frac{27-2z+3\sqrt{81-12z}}{2z}}. \tag{2}$$

At $z = \frac{27}{4}$, $z = \frac{20}{3}$, $z = \frac{77}{12}$, $z = 6$, $z = \frac{65}{12}$, $z = \frac{14}{3}$, $z = \frac{15}{4}$, $z = \frac{8}{3}$, and $z = \frac{17}{12}$ (then the expression $81 - 12z$ in (2) will be a perfect square), from (1) we immediately obtain, respectively, such series:

$$\sum_{k=1}^{\infty} \frac{\left(\frac{27}{4}\right)^k}{k^2\binom{3k}{k}} = \frac{2\pi^2}{3} - 2\ln^2 2, \tag{3}$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{20}{3}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{\sqrt[3]{10}-1}\right) - \frac{1}{2}\ln^2\left(\frac{18}{\left(\sqrt[3]{10}+2\right)^3}\right),$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{77}{12}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{7\sqrt{3}}{2\sqrt[3]{539}-7}\right) - \frac{1}{2}\ln^2\left(\frac{882}{\left(\sqrt[3]{539}+7\right)^3}\right),$$

$$\sum_{k=1}^{\infty} \frac{6^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{2}-1}\right) - \frac{1}{2}\ln^2\left(\frac{3}{\left(\sqrt[3]{2}+1\right)^3}\right), \tag{4}$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{65}{12}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{5\sqrt{3}}{2\sqrt[3]{325}-5}\right) - \frac{1}{2}\ln^2\left(\frac{450}{\left(\sqrt[3]{325}+5\right)^3}\right),$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{14}{3}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{\sqrt[3]{28}-1}\right) - \frac{1}{2}\ln^2\left(\frac{36}{\left(\sqrt[3]{28}+2\right)^3}\right),$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{15}{4}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{5}-1}\right) - \frac{1}{2}\ln^2\left(\frac{6}{\left(\sqrt[3]{5}+1\right)^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{8}{3}\right)^k}{k^2\binom{3k}{k}}=\frac{\pi^2}{6}-\frac{\ln^2 3}{2}, \tag{5}$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{17}{12}\right)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{17}-1}\right)-\frac{1}{2}\ln^2\left(\frac{18}{\left(\sqrt[3]{17}+1\right)^3}\right).$$

Series (3) and (4) one can find in [1, Identities (3.4) and (3.5)]. Series (5) was obtained by D'Aurizio and Di Trani [4, Formula (8)] using the hypergeometric function ${}_4F_3$.

Similarly, we have the corresponding alternating series:

$$\sum_{k=1}^{\infty}\frac{\left(\frac{-27}{4}\right)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{3+2\sqrt{2}}+1}\right)-\frac{1}{2}\ln^2\left(\frac{2+2\sqrt{2}}{\left(\sqrt[3]{3+2\sqrt{2}}-1\right)^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{-20}{3}\right)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{40}}{2\sqrt[3]{121+9\sqrt{161}}+\sqrt[3]{40}}\right)-\frac{1}{2}\ln^2\left(\frac{81+9\sqrt{161}}{\left(\sqrt[3]{121+9\sqrt{161}}-\sqrt[3]{40}\right)^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{-77}{12}\right)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{77}}{2\sqrt[3]{239+36\sqrt{158}}+\sqrt[3]{77}}\right)-\frac{1}{2}\ln^2\left(\frac{162+18\sqrt{158}}{\left(\sqrt[3]{239+18\sqrt{158}}-\sqrt[3]{77}\right)^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{(-6)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{\sqrt[3]{26+6\sqrt{17}}+1}\right)-\frac{1}{2}\ln^2\left(\frac{9+3\sqrt{17}}{\left(\sqrt[3]{13+3\sqrt{17}}-\sqrt[3]{4}\right)^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{-65}{12}\right)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{65}}{2\sqrt[3]{227+18\sqrt{146}}+\sqrt[3]{65}}\right)-\frac{1}{2}\ln^2\left(\frac{162+18\sqrt{146}}{\left(\sqrt[3]{227+18\sqrt{146}}-\sqrt[3]{65}\right)^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{-14}{3}\right)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{7}}{\sqrt[3]{218+18\sqrt{137}}+\sqrt[3]{7}}\right)-\frac{1}{2}\ln^2\left(\frac{81+9\sqrt{137}}{\left(\sqrt[3]{109+9\sqrt{137}}-\sqrt[3]{28}\right)^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{-15}{4}\right)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{5}}{2\sqrt[3]{23+6\sqrt{14}}+\sqrt[3]{5}}\right)-\frac{1}{2}\ln^2\left(\frac{18+6\sqrt{14}}{\left(\sqrt[3]{23+6\sqrt{14}}-\sqrt[3]{5}\right)^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{-8}{3}\right)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{2}}{\sqrt[3]{97+9\sqrt{113}}+\sqrt[3]{2}}\right)-\frac{1}{2}\ln^2\left(\frac{81+9\sqrt{113}}{\left(\sqrt[3]{97+9\sqrt{113}}-\sqrt[3]{16}\right)^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{-17}{12}\right)^k}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{17}}{2\sqrt[3]{179+126\sqrt{2}}+\sqrt[3]{17}}\right)-\frac{1}{2}\ln^2\left(\frac{162+126\sqrt{2}}{\left(\sqrt[3]{179+126\sqrt{2}}-\sqrt[3]{17}\right)^3}\right).$$

The substitution $z=\frac{27xy}{(x+y)^2}$ in (1) reduces function $\phi\big(z(x,y)\big)$ to $\sqrt[3]{x/y}$, thereby yielding the following more manageable identity:

$$\sum_{k=1}^{\infty}\frac{(27xy)^k}{k^2(x+y)^{2k}\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{y}}{2\sqrt[3]{x}-\sqrt[3]{y}}\right)-\frac{1}{2}\ln^2\left(\frac{x+y}{(\sqrt[3]{x}+\sqrt[3]{y})^3}\right), \tag{A}$$

which is valid for $\frac{x}{y}\geq 1$ or $\frac{x}{y}\leq -(\sqrt{2}+1)^2=-\cot^2\left(\frac{\pi}{8}\right)$.

Differentiating twice identity (A) with respect to x, we find, for $\frac{x}{y}>1$ or $\frac{x}{y}\leq -(\sqrt{2}+1)^2$, the following identities:

$$\sum_{k=1}^{\infty}\frac{(27xy)^k}{k(x+y)^{2k}\binom{3k}{k}}$$
$$=\frac{\sqrt[3]{xy}}{x-y}\left(2\sqrt{3}\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\arctan\left(\frac{\sqrt{3}\sqrt[3]{y}}{2\sqrt[3]{x}-\sqrt[3]{y}}\right)+\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\ln\left(\frac{x+y}{\left(\sqrt[3]{x}+\sqrt[3]{y}\right)^3}\right)\right) \quad \text{(B)}$$

and

$$\sum_{k=1}^{\infty}\frac{(27xy)^k}{(x+y)^{2k}\binom{3k}{k}}$$
$$=\frac{4xy}{(x-y)^2}+\frac{\sqrt[3]{xy}}{3}\frac{x+y}{(x-y)^3}\left(2\sqrt{3}\left(2\sqrt[3]{xy}\left(\sqrt[3]{x^2}+\sqrt[3]{y^2}\right)+\sqrt[3]{x^4}+\sqrt[3]{y^4}\right)\arctan\left(\frac{\sqrt{3}\sqrt[3]{y}}{2\sqrt[3]{x}-\sqrt[3]{y}}\right)\right.$$
$$\left.-\left(2\sqrt[3]{xy}\left(\sqrt[3]{x^2}-\sqrt[3]{y^2}\right)-\sqrt[3]{x^4}+\sqrt[3]{y^4}\right)\ln\left(\frac{x+y}{\left(\sqrt[3]{x}+\sqrt[3]{y}\right)^3}\right)\right). \quad \text{(C)}$$

Setting $(x,y)=(8,1)$, $(x,y)=(8,-1)$, $(x,y)=(8,\frac{1}{8})$, $(x,y)=(8,-\frac{1}{8})$, $(x,y)=(1,\frac{1}{27})$ and $(x,y)=(1,-\frac{1}{27})$ in (A), (B), and (C) we have the following series list:

$$\sum_{k=1}^{\infty}\frac{\left(\frac{8}{3}\right)^k}{k\binom{3k}{k}}=\frac{2\sqrt{3}\,\pi}{7}-\frac{2}{7}\ln 3,$$
$$\sum_{k=1}^{\infty}\frac{\left(\frac{8}{3}\right)^k}{\binom{3k}{k}}=\frac{32}{49}+\frac{74\sqrt{3}\,\pi}{343}-\frac{18}{343}\ln 3,$$
$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{6\sqrt{6}}{7}\right)^{2k}}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{5}\right)-\frac{1}{2}\ln^2 7,$$
$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{6\sqrt{6}}{7}\right)^{2k}}{k\binom{3k}{k}}=\frac{4\sqrt{3}}{9}\arctan\left(\frac{\sqrt{3}}{5}\right)-\frac{2}{3}\ln 7,$$
$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{6\sqrt{6}}{7}\right)^{2k}}{\binom{3k}{k}}=-\frac{32}{81}-\frac{28\sqrt{3}}{729}\arctan\left(\frac{\sqrt{3}}{5}\right)-\frac{14}{81}\ln 7,$$
$$\sum_{k=1}^{\infty}\frac{\left(\frac{24\sqrt{3}}{65}\right)^{2k}}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{7}\right)-\frac{1}{2}\ln^2\left(\frac{25}{13}\right),$$
$$\sum_{k=1}^{\infty}\frac{\left(\frac{24\sqrt{3}}{65}\right)^{2k}}{k\binom{3k}{k}}=\frac{40\sqrt{3}}{63}\arctan\left(\frac{\sqrt{3}}{7}\right)-\frac{4}{21}\ln\left(\frac{25}{13}\right),$$
$$\sum_{k=1}^{\infty}\frac{\left(\frac{24\sqrt{3}}{65}\right)^{2k}}{\binom{3k}{k}}=\frac{256}{3969}+\frac{68120\sqrt{3}}{250047}\arctan\left(\frac{\sqrt{3}}{7}\right)-\frac{1300}{27783}\ln\left(\frac{25}{13}\right),$$
$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{8\sqrt{3}}{21}\right)^{2k}}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{9}\right)-\frac{1}{2}\ln^2\left(\frac{7}{3}\right),$$
$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{8\sqrt{3}}{21}\right)^{2k}}{k\binom{3k}{k}}=\frac{24\sqrt{3}}{65}\arctan\left(\frac{\sqrt{3}}{9}\right)-\frac{4}{13}\ln\left(\frac{7}{3}\right),$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{8\sqrt{3}}{21}\right)^{2k}}{\binom{3k}{k}}=-\frac{256}{4225}+\frac{20328\sqrt{3}}{274625}\arctan\left(\frac{\sqrt{3}}{9}\right)-\frac{252}{2197}\ln\left(\frac{7}{3}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{27}{28}\right)^{2k}}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{5}\right)-\frac{1}{2}\ln^2\left(\frac{16}{7}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{27}{28}\right)^{2k}}{k\binom{3k}{k}}=\frac{12\sqrt{3}}{13}\arctan\left(\frac{\sqrt{3}}{5}\right)-\frac{3}{13}\ln\left(\frac{16}{7}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{27}{28}\right)^{2k}}{\binom{3k}{k}}=\frac{27}{169}+\frac{994\sqrt{3}}{2197}\arctan\left(\frac{\sqrt{3}}{5}\right)-\frac{112}{2197}\ln\left(\frac{16}{7}\right),$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{27}{26}\right)^{2k}}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{7}\right)-\frac{1}{2}\ln^2\left(\frac{13}{4}\right),$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{27}{26}\right)^{2k}}{k\binom{3k}{k}}=\frac{3\sqrt{3}}{7}\arctan\left(\frac{\sqrt{3}}{7}\right)-\frac{3}{7}\ln\left(\frac{13}{4}\right),$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{27}{26}\right)^{2k}}{\binom{3k}{k}}=-\frac{27}{196}+\frac{143\sqrt{3}}{2744}\arctan\left(\frac{\sqrt{3}}{7}\right)-\frac{52}{343}\ln\left(\frac{13}{4}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{54\sqrt{2}}{35}\right)^{2k}}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\ln^2\left(\frac{25}{7}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{54\sqrt{2}}{35}\right)^{2k}}{k\binom{3k}{k}}=\frac{60\sqrt{3}}{19}\arctan\left(\frac{\sqrt{3}}{2}\right)-\frac{6}{19}\ln\left(\frac{25}{7}\right),$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{54\sqrt{2}}{35}\right)^{2k}}{\binom{3k}{k}}=\frac{864}{361}+\frac{35420\sqrt{3}}{6859}\arctan\left(\frac{\sqrt{3}}{2}\right)-\frac{350}{6859}\ln\left(\frac{25}{7}\right),$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{54\sqrt{2}}{19}\right)^{2k}}{k^2\binom{3k}{k}}=6\arctan^2\left(\frac{\sqrt{3}}{4}\right)-\frac{1}{2}\ln^2 19,$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{54\sqrt{2}}{19}\right)^{2k}}{k\binom{3k}{k}}=\frac{12\sqrt{3}}{35}\arctan\left(\frac{\sqrt{3}}{4}\right)-\frac{6}{7}\ln 19,$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{\left(\frac{54\sqrt{2}}{19}\right)^{2k}}{\binom{3k}{k}}=-\frac{864}{1225}-\frac{4484\sqrt{3}}{42875}\arctan\left(\frac{\sqrt{3}}{4}\right)-\frac{38}{343}\ln 19.$$

The first two series from this list can be found in [3, Corollaries 2.3 and 3.3].

2 Evaluations at selected arguments

In this section we will evaluate identities (A), (B) and (C) at carefully selected values of x and y. Some of the resulting summation identities will involve Fibonacci and Lucas numbers in the summand and possibly in the evaluations.

Let F_n and L_n denote the n-th Fibonacci and Lucas numbers, both satisfying the recurrence relation $X_n = X_{n-1} + X_{n-2}$, $n \geq 2$, but with the initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$,

$L_1 = 1$. Extending Fibonacci and Lucas numbers to negative subscripts gives $F_{-j} = (-1)^{j-1}F_j$ and $L_{-j} = (-1)^j L_j$. Throughout this paper, we denote the golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$ and write $\beta = \frac{1-\sqrt{5}}{2}$, so that $\alpha\beta = -1$ and $\alpha + \beta = 1$. For any integer j, the explicit formulas (Binet formulas) for Fibonacci and Lucas numbers are

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}, \qquad L_j = \alpha^j + \beta^j. \tag{6}$$

We will often require the following identities, valid for any integer r, which are straightforward consequences of (6):

$$\alpha^{2r} + (-1)^{r+1} = \alpha^r F_r \sqrt{5}, \tag{7}$$

$$\alpha^{2r} + (-1)^r = \alpha^r L_r, \tag{8}$$

$$\beta^{2r} + (-1)^{r+1} = -\beta^r F_r \sqrt{5}, \tag{9}$$

$$\beta^{2r} + (-1)^r = \beta^r L_r. \tag{10}$$

We also require the following well-known identities [5, 7]:

$$F_n^2 + (-1)^{n+m-1} F_m^2 = F_{n-m} F_{n+m}, \tag{11}$$

$$F_{n+m} + (-1)^m F_{n-m} = L_m F_n, \tag{12}$$

$$F_{n+m} + (-1)^{m-1} F_{n-m} = F_m L_n, \tag{13}$$

$$L_n F_m + F_n L_m = 2F_{n+m}, \tag{14}$$

$$L_{n+m} + (-1)^m L_{n-m} = L_m L_n, \tag{15}$$

$$L_{n+m} + (-1)^{m-1} L_{n-m} = 5F_m F_n. \tag{16}$$

2.1 Results from identity (A)

Theorem 2.1. *If r is a non-negative integer, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k(r-1)} \left(\frac{27}{5}\right)^k}{k^2 \binom{3k}{k} F_r^{2k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^{2r}} + (-1)^r}\right) - \frac{1}{2}\ln^2\left(\frac{\sqrt{5}\alpha^r F_r}{\left(\sqrt[3]{\alpha^{2r}} - (-1)^r\right)^3}\right), \quad r \neq 0, \tag{17}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{kr} 27^k}{k^2 \binom{3k}{k} L_r^{2k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^{2r}} - (-1)^r}\right) - \frac{1}{2}\ln^2\left(\frac{\alpha^r L_r}{\left(\sqrt[3]{\alpha^{2r}} + (-1)^r\right)^3}\right), \quad r \neq 1. \tag{18}$$

Proof. Identity (17) is proved by setting $x = \alpha^{2r}$, $y = (-1)^{r+1}$ in (A) and making use of identity (7). Identity (18) follows from setting $x = \alpha^{2r}$, $y = (-1)^r$ in (A) and using (8). □

Example 2.1.1. *Evaluation at* $r = 1, 2, 3$ *in* (17) *and* (18)*, respectively, gives*

$$\sum_{k=1}^{\infty} \frac{\left(\frac{27}{5}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^2}-1}\right) - \frac{1}{2}\ln^2\left(\frac{\alpha\sqrt{5}}{\left(\sqrt[3]{\alpha^2}+1\right)^3}\right),$$
$$\sum_{k=1}^{\infty} \frac{(-27)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^2}+1}\right) - \frac{1}{2}\ln^2\left(\frac{\alpha}{\left(\sqrt[3]{\alpha^2}-1\right)^3}\right),$$
$$\sum_{k=1}^{\infty} \frac{\left(-\frac{27}{5}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^4}+1}\right) - \frac{1}{2}\ln^2\left(\frac{\alpha^2\sqrt{5}}{\left(\sqrt[3]{\alpha^4}-1\right)^3}\right),$$
$$\sum_{k=1}^{\infty} \frac{3^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^4}-1}\right) - \frac{1}{2}\ln^2\left(\frac{3\alpha^2}{\left(\sqrt[3]{\alpha^4}+1\right)^3}\right),$$
$$\sum_{k=11}^{\infty} \frac{\left(\frac{27}{20}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\sqrt{15}-\sqrt{12}\right) - \frac{1}{2}\ln^2\left(\frac{5}{2}\right),$$
$$\sum_{k=1}^{\infty} \frac{\left(-\frac{27}{16}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{4\sqrt{3}-\sqrt{15}}{11}\right) - 2\ln^2 2.$$

Corollary 2.1.1. *If* r *is a non-negative integer, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k(r-1)}\left(\frac{27}{5}\right)^k}{k^2\binom{3k}{k}F_{3r}^{2k}} = 6\arctan^2\left(\frac{\sqrt{3}}{\alpha^{2r}+\alpha^r L_r}\right) - \frac{1}{2}\ln^2\left(\frac{F_{3r}}{5F_r^3}\right), \quad r \neq 0,$$
$$\sum_{k=1}^{\infty} \frac{(-1)^{kr}27^k}{k^2\binom{3k}{k}L_{3r}^{2k}} = 6\arctan^2\left(\frac{\sqrt{3}}{\alpha^{2r}+\sqrt{5}\alpha^r F_r}\right) - \frac{1}{2}\ln^2\left(\frac{L_{3r}}{L_r^3}\right).$$

Proof. Replace r with $3r$ in (17) and (18) and use (7), (8). □

Theorem 2.2. *Let* m *and* n *be positive integers such that* $n \geq m$ *unless stated otherwise. Then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k(n-m-1)}}{k^2\binom{3k}{k}}\left(\frac{27F_n^2F_m^2}{F_{n-m}^2F_{n+m}^2}\right)^k$$
$$= 6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{F_m^2}}{2\sqrt[3]{F_n^2}+(-1)^{n-m}\sqrt[3]{F_m^2}}\right) - \frac{1}{2}\ln^2\left(\frac{F_{n-m}F_{n+m}}{\left(\sqrt[3]{F_n^2}-(-1)^{n-m}\sqrt[3]{F_m^2}\right)^3}\right), \quad n > m,$$
$$\sum_{k=1}^{\infty} \frac{(-1)^{km}}{k^2\binom{3k}{k}}\left(\frac{27F_{n+m}F_{n-m}}{L_m^2F_n^2}\right)^k$$
$$= 6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{F_{n-m}}}{2\sqrt[3]{F_{n+m}}-(-1)^m\sqrt[3]{F_{n-m}}}\right) - \frac{1}{2}\ln^2\left(\frac{L_mF_n}{\left(\sqrt[3]{F_{n+m}}+(-1)^m\sqrt[3]{F_{n-m}}\right)^3}\right),$$
$$\sum_{k=1}^{\infty} \frac{(-1)^{k(m-1)}}{k^2\binom{3k}{k}}\left(\frac{27F_{n+m}F_{n-m}}{F_m^2L_n^2}\right)^k$$
$$= 6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{F_{n-m}}}{2\sqrt[3]{F_{n+m}}+(-1)^m\sqrt[3]{F_{n-m}}}\right) - \frac{1}{2}\ln^2\left(\frac{F_mL_n}{\left(\sqrt[3]{F_{n+m}}-(-1)^m\sqrt[3]{F_{n-m}}\right)^3}\right),$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2\binom{3k}{k}} \left(\frac{27F_{2m}F_{2n}}{4F_{n+m}^2}\right)^k$$
$$= 6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{L_mF_n}}{2\sqrt[3]{L_nF_m} - \sqrt[3]{L_mF_n}}\right) - \frac{1}{2}\ln^2\left(\frac{2F_{n+m}}{\left(\sqrt[3]{L_nF_m} + \sqrt[3]{L_mF_n}\right)^3}\right), \quad L_nF_m > L_mF_n,$$
$$\sum_{k=1}^{\infty} \frac{(-1)^{km}}{k^2\binom{3k}{k}} \left(\frac{27L_{n+m}L_{n-m}}{L_m^2L_n^2}\right)^k$$
$$= 6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{L_{n-m}}}{2\sqrt[3]{L_{n+m}} - (-1)^m\sqrt[3]{L_{n-m}}}\right) - \frac{1}{2}\ln^2\left(\frac{L_mL_n}{\left(\sqrt[3]{L_{n+m}} + (-1)^m\sqrt[3]{L_{n-m}}\right)^3}\right), \tag{19}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k(m-1)}}{k^2\binom{3k}{k}} \left(\frac{27L_{n+m}L_{n-m}}{25F_m^2F_n^2}\right)^k$$
$$= 6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{L_{n-m}}}{2\sqrt[3]{L_{n+m}} + (-1)^m\sqrt[3]{L_{n-m}}}\right) - \frac{1}{2}\ln^2\left(\frac{5F_mF_n}{\left(\sqrt[3]{L_{n+m}} - (-1)^m\sqrt[3]{L_{n-m}}\right)^3}\right). \tag{20}$$

Proof. Straightforward using identities (11) to (16) and identity (A). □

Example 2.2.1. *Identities* (19) *and* (20) *yield*

$$\sum_{k=1}^{\infty} \frac{(-1)^{kn}}{k^2\binom{3k}{k}} \left(\frac{54L_{2n}}{L_n^4}\right)^k = 6\arctan^2\left(\frac{\sqrt{3}}{\sqrt[3]{4L_{2n}} - (-1)^n}\right) - \frac{1}{2}\ln^2\left(\frac{L_n^2}{\left(\sqrt[3]{L_{2n}} + (-1)^n\sqrt[3]{2}\right)^3}\right),$$
$$\sum_{k=1}^{\infty} \frac{(-1)^{k(n-1)}}{k^2\binom{3k}{k}} \left(\frac{54L_{2n}}{25F_n^4}\right)^k = 6\arctan^2\left(\frac{\sqrt{3}}{\sqrt[3]{4L_{2n}} + (-1)^n}\right) - \frac{1}{2}\ln^2\left(\frac{5F_n^2}{\left(\sqrt[3]{L_{2n}} - (-1)^n\sqrt[3]{2}\right)^3}\right).$$

By writing $\cot^2 x$ for x and setting $y = 1$, a useful trigonometric version of identity (A) is obtained, namely,

$$\sum_{k=1}^{\infty} \frac{\left(\frac{27}{4}\right)^k}{k^2\binom{3k}{k}} \sin^{2k} 2x = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{\cot^2 x} - 1}\right) - \frac{1}{2}\ln^2\left(\frac{\csc^2 x}{\left(\sqrt[3]{\cot^2 x} + 1\right)^3}\right). \tag{21}$$

Identity (21) is valid for $x \in (0, \frac{\pi}{4}]$.

Evaluation of identity (21) at $x = \frac{\pi}{12}$, $x = \frac{\pi}{8}$ and $x = \frac{\pi}{6}$, respectively, gives

$$\sum_{k=1}^{\infty} \frac{\left(\frac{27}{16}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{7+4\sqrt{3}} - 1}\right) - \frac{1}{2}\ln^2\left(\frac{4+2\sqrt{3}}{\left(\sqrt[3]{7+4\sqrt{3}} + 1\right)^3}\right),$$
$$\sum_{k=1}^{\infty} \frac{\left(\frac{27}{8}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{3+2\sqrt{2}} - 1}\right) - \frac{1}{2}\ln^2\left(\frac{4+2\sqrt{2}}{\left(\sqrt[3]{3+2\sqrt{2}} + 1\right)^3}\right),$$
$$\sum_{k=1}^{\infty} \frac{\left(\frac{81}{16}\right)^k}{k^2\binom{3k}{k}} = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{3} - 1}\right) - \frac{1}{2}\ln^2\left(\frac{4}{(\sqrt[3]{3} + 1)^3}\right).$$

Writing $-\cot^2 x$ for x and setting $y = 1$ in (A) and noting that $1 - \cot^2 x = -\frac{\cos 2x}{\sin^2 x}$, we obtain another useful trigonometric version of (A):

$$\sum_{k=1}^{\infty} \frac{\left(-\frac{27}{4}\right)^k}{k^2\binom{3k}{k}} \tan^{2k} 2x = 6\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{\cot^2 x} + 1}\right) - \frac{1}{2}\ln^2\left(\frac{\csc^2 x\,\cos 2x}{\left(\sqrt[3]{\cot^2 x} - 1\right)^3}\right), \tag{22}$$

valid for $x \in (0, \frac{\pi}{8}]$. At $x = \frac{\pi}{12}$ in (22) we obtain

$$\sum_{k=1}^{\infty} \frac{\left(-\frac{9}{4}\right)^k}{k^2 \binom{3k}{k}} = 6 \arctan^2 \left(\frac{\sqrt{3}}{2\sqrt[3]{7+4\sqrt{3}}+1} \right) - \frac{1}{2} \ln^2 \left(\frac{6+4\sqrt{3}}{\left(\sqrt[3]{7+4\sqrt{3}}-1\right)^3} \right).$$

2.2 Results from identity (B)

Theorem 2.3. *If r is a positive integer, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k(r-1)} \left(\frac{27}{5}\right)^k}{k\binom{3k}{k} F_r^{2k}} = \frac{1}{\sqrt[3]{\alpha^r} L_r} \Bigg(2\sqrt{3} \left(\sqrt[3]{\alpha^{2r}} - (-1)^r\right) \arctan \left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^{2r}} + (-1)^r} \right)$$
$$- (-1)^r \left(\sqrt[3]{\alpha^{2r}} + (-1)^r\right) \ln \left(\frac{\sqrt{5}\, \alpha^r F_r}{\left(\sqrt[3]{\alpha^{2r}} - (-1)^r\right)^3} \right) \Bigg) \tag{23}$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{kr} 27^k}{k\binom{3k}{k} L_r^{2k}} = \frac{1}{\sqrt{5}\sqrt[3]{\alpha^r} F_r} \Bigg(2\sqrt{3} \left(\sqrt[3]{\alpha^{2r}} + (-1)^r\right) \arctan \left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^{2r}} - (-1)^r} \right)$$
$$+ (-1)^r \left(\sqrt[3]{\alpha^{2r}} - (-1)^r\right) \ln \left(\frac{\alpha^r L_r}{\left(\sqrt[3]{\alpha^{2r}} + (-1)^r\right)^3} \right) \Bigg). \tag{24}$$

Proof. Identity (23) is proved by setting $x = \alpha^{2r}$, $y = (-1)^{r+1}$ in (B) and making use of identity (7). Identity (24) follows from setting $x = \alpha^{2r}$, $y = (-1)^r$ in (B) and using identity (8). □

Example 2.3.1. *Evaluation of* (23) *at $r = 1$, $r = 2$, $r = 3$ and* (24) *at $r = 2$ and $r = 3$ respectively gives*

$$\sum_{k=1}^{\infty} \frac{\left(\frac{27}{5}\right)^k}{k\binom{3k}{k}} = 2\sqrt{3}\, \frac{\sqrt[3]{\alpha^2}+1}{\sqrt[3]{\alpha}} \arctan \left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^2}-1} \right) + \frac{\sqrt[3]{\alpha^2}-1}{\sqrt[3]{\alpha}} \ln \left(\frac{\alpha\sqrt{5}}{(\sqrt[3]{\alpha^2}+1)^3} \right),$$
$$\sum_{k=1}^{\infty} \frac{\left(-\frac{27}{5}\right)^k}{k\binom{3k}{k}} = \frac{2\sqrt{3}}{3} \frac{\sqrt[3]{\alpha^4}-1}{\sqrt[3]{\alpha^2}} \arctan \left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^4}+1} \right) - \frac{\sqrt[3]{\alpha^4}+1}{3\sqrt[3]{\alpha^2}} \ln \left(\frac{\alpha^2\sqrt{5}}{(\sqrt[3]{\alpha^4}-1)^3} \right),$$
$$\sum_{k=1}^{\infty} \frac{\left(\frac{27}{20}\right)^k}{k\binom{3k}{k}} = \frac{\sqrt{15}}{2} \arctan \left(\sqrt{15}-\sqrt{12}\right) - \frac{1}{4} \ln \left(\frac{5}{2}\right)$$

and

$$\sum_{k=1}^{\infty} \frac{3^k}{k\binom{3k}{k}} = \frac{2\sqrt{15}}{5} \frac{\sqrt[3]{\alpha^4}+1}{\sqrt[3]{\alpha^2}} \arctan \left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^4}-1} \right) + \frac{\sqrt[3]{\alpha^4}-1}{\sqrt{5}\sqrt[3]{\alpha^2}} \ln \left(\frac{3\alpha^2}{(\sqrt[3]{\alpha^4}+1)^3} \right),$$
$$\sum_{k=1}^{\infty} \frac{\left(-\frac{27}{16}\right)^k}{k\binom{3k}{k}} = \frac{\sqrt{15}}{5} \arctan \left(\frac{4\sqrt{3}-\sqrt{15}}{11} \right) - \ln 2.$$

Corollary 2.3.1. *If r is a positive integer, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{rk}\left(-\frac{27}{5}\right)^k}{k\binom{3k}{k}F_{3r}^{2k}}=2\sqrt{15}\,\frac{F_r}{L_{3r}}\arctan\left(\frac{\sqrt{3}}{\alpha^r\big(\alpha^r+L_r\big)}\right)-(-1)^r\frac{L_r}{L_{3r}}\ln\left(\frac{F_{3r}}{5F_r^3}\right),$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{rk}27^k}{k\binom{3k}{k}L_{3r}^{2k}}=\frac{2\sqrt{15}}{5}\frac{L_r}{F_{3r}}\arctan\left(\frac{\sqrt{3}}{\alpha^r\big(\alpha^r+\sqrt{5}F_r\big)}\right)+(-1)^r\frac{F_r}{F_{3r}}\ln\left(\frac{L_{3r}}{L_r^3}\right).$$

Replacing x with $\cot^2 x$ and setting $y=1$ in identity (B) gives

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\left(\frac{27}{4}\right)^k}{k\binom{3k}{k}}\sin^{2k}2x&=\frac{2\sqrt{3}\sin^2 x}{\cos 2x}\sqrt[3]{\cot^2 x}\left(\sqrt[3]{\cot^2 x}+1\right)\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{\cot^2 x}-1}\right)\\&\quad+\frac{\sin^2 x}{\cos 2x}\sqrt[3]{\cot^2 x}\left(\sqrt[3]{\cot^2 x}-1\right)\ln\left(\frac{\csc^2 x}{\left(\sqrt[3]{\cot^2 x}+1\right)^3}\right),\end{aligned}\tag{25}$$

valid for $x\in(0,\frac{\pi}{4})$. At $x=\frac{\pi}{12}$, $x=\frac{\pi}{8}$, and $x=\frac{\pi}{6}$, from (25) we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\left(\frac{27}{16}\right)^k}{k\binom{3k}{k}}&=\left(\sqrt[3]{2+\sqrt{3}}+\sqrt[3]{2-\sqrt{3}}\right)\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{7+4\sqrt{3}}-1}\right)\\&\quad+\frac{\sqrt{3}}{6}\left(\sqrt[3]{2+\sqrt{3}}-\sqrt[3]{2-\sqrt{3}}\right)\ln\left(\frac{8+4\sqrt{3}}{\left(\sqrt[3]{7+4\sqrt{3}}+1\right)^3}\right),\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\left(\frac{27}{8}\right)^k}{k\binom{3k}{k}}&=\sqrt{3}\left(\sqrt[3]{1+\sqrt{2}}-\sqrt[3]{1-\sqrt{2}}\right)\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{3+2\sqrt{2}}-1}\right)\\&\quad+\frac{1}{2}\left(\sqrt[3]{1+\sqrt{2}}+\sqrt[3]{1-\sqrt{2}}\right)\ln\left(\frac{4+2\sqrt{2}}{\left(\sqrt[3]{3+2\sqrt{2}}+1\right)^3}\right),\end{aligned}$$

$$\sum_{k=1}^{\infty}\frac{\left(\frac{81}{16}\right)^k}{k\binom{3k}{k}}=\sqrt[6]{3^5}\left(\sqrt[3]{3}+1\right)\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{3}-1}\right)+\frac{\sqrt[3]{3}}{2}\left(\sqrt[3]{3}-1\right)\ln\left(\frac{4}{(\sqrt[3]{3}+1)^3}\right).$$

2.3 Results from identity (C)

Theorem 2.4. *If r is a positive integer, then*

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{(k-1)(r-1)}\left(\frac{27}{5}\right)^k}{F_r^{2k}\binom{3k}{k}}&=\frac{4}{L_r^2}\\&\quad+\frac{2\sqrt{15}F_r}{3L_r^3}\left(2\left(\sqrt[3]{\alpha^{2r}}+\sqrt[3]{\beta^{2r}}\right)-(-1)^r\left(\sqrt[3]{\alpha^{4r}}+\sqrt[3]{\beta^{4r}}\right)\right)\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^{2r}}+(-1)^r}\right)\\&\quad+\frac{(-1)^r\sqrt{5}F_r}{3L_r^3}\left(2\left(\sqrt[3]{\alpha^{2r}}-\sqrt[3]{\beta^{2r}}\right)+(-1)^r\left(\sqrt[3]{\alpha^{4r}}-\sqrt[3]{\beta^{4r}}\right)\right)\ln\left(\frac{\sqrt{5}\alpha^r F_r}{\left(\sqrt[3]{\alpha^{2r}}-(-1)^r\right)^3}\right),\end{aligned}\tag{26}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{r(k-1)}27^k}{L_r^{2k}\binom{3k}{k}}&=\frac{4}{5F_r^2}\\&\quad+\frac{2\sqrt{15}L_r}{75F_r^3}\left(2\left(\sqrt[3]{\alpha^{2r}}+\sqrt[3]{\beta^{2r}}\right)+(-1)^r\left(\sqrt[3]{\alpha^{4r}}+\sqrt[3]{\beta^{4r}}\right)\right)\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^{2r}}-(-1)^r}\right)\\&\quad-\frac{(-1)^r\sqrt{5}L_r}{75F_r^3}\left(2\left(\sqrt[3]{\alpha^{2r}}-\sqrt[3]{\beta^{2r}}\right)-(-1)^r\left(\sqrt[3]{\alpha^{4r}}-\sqrt[3]{\beta^{4r}}\right)\right)\ln\left(\frac{L_r}{\left(\sqrt[3]{\alpha^r}+\sqrt[3]{\beta^r}\right)^3}\right).\end{aligned}\tag{27}$$

Proof. Identities (26) and (27) are proved by setting, respectively, $x = \alpha^r$, $y = -\beta^r$ and $x = \alpha^{2r}$, $y = (-1)^r$ in (C) and making use of (7) and (8). $\square$

Example 2.4.1. *Evaluation of* (26) *at* $r = 1, 2, 3$ *and* (27) *at* $r = 2, 3, 6$*, respectively, gives*

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\left(\frac{27}{5}\right)^k}{\binom{3k}{k}} &= 4 + \frac{2\sqrt{15}}{3}\Big(2\left(\sqrt[3]{\alpha^2} + \sqrt[3]{\beta^2}\right) + \sqrt[3]{\alpha^4} + \sqrt[3]{\beta^4}\Big) \arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^2} - 1}\right) \\ &\quad - \frac{\sqrt{5}}{3}\Big(2\left(\sqrt[3]{\alpha^2} - \sqrt[3]{\beta^2}\right) - \left(\sqrt[3]{\alpha^4} - \sqrt[3]{\beta^4}\right)\Big) \ln\left(\frac{\sqrt{5}\alpha}{(\sqrt[3]{\alpha^2} + 1)^3}\right),\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\left(-\frac{27}{5}\right)^k}{\binom{3k}{k}} &= -\frac{4}{9} - \frac{2\sqrt{15}}{81}\Big(2\big(\sqrt[3]{\alpha^4} + \sqrt[3]{\beta^4}\big) - \big(\sqrt[3]{\alpha^8} + \sqrt[3]{\beta^8}\big)\Big) \arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^4} + 1}\right) \\ &\quad - \frac{\sqrt{5}}{81}\Big(2\big(\sqrt[3]{\alpha^4} - \sqrt[3]{\beta^4}\big) + \sqrt[3]{\alpha^8} - \sqrt[3]{\beta^8}\Big) \ln\left(\frac{\sqrt{5}\alpha^2}{(\sqrt[3]{\alpha^4} - 1)^3}\right),\end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{27}{20}\right)^k}{\binom{3k}{k}} = \frac{1}{4} + \frac{13\sqrt{15}}{48} \arctan\left(\sqrt{15} - \sqrt{12}\right) - \frac{5}{96} \ln\left(\frac{5}{2}\right),$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{3^k}{\binom{3k}{k}} &= \frac{4}{5} + \frac{2\sqrt{15}}{25}\Big(2\big(\sqrt[3]{\alpha^4} + \sqrt[3]{\beta^4}\big) + \sqrt[3]{\alpha^8} + \sqrt[3]{\beta^8}\Big) \arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^4} - 1}\right) \\ &\quad + \frac{\sqrt{5}}{25}\Big(2\big(\sqrt[3]{\alpha^4} - \sqrt[3]{\beta^4}\big) - \big(\sqrt[3]{\alpha^8} - \sqrt[3]{\beta^8}\big)\Big) \ln\left(1 + \sqrt[3]{\alpha^2} + \sqrt[3]{\beta^2}\right),\end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{\left(-\frac{27}{16}\right)^k}{\binom{3k}{k}} = -\frac{1}{5} + \frac{\sqrt{15}}{75} \arctan\left(\frac{4\sqrt{3} - \sqrt{15}}{11}\right) - \frac{1}{6} \ln 4,$$

and

$$\sum_{k=1}^{\infty} \frac{\left(\frac{1}{12}\right)^k}{\binom{3k}{k}} = \frac{1}{80} + \frac{183\sqrt{15}}{3200} \arctan\left(\frac{\sqrt{15} - \sqrt{12}}{3}\right) - \frac{9}{256} \ln\left(\frac{3}{2}\right).$$

3 Fibonacci and Lucas series involving inverses of the binomial coefficients $\binom{3n}{n}$

In this section we will derive Fibonacci and Lucas identities which contain reciprocals of the binomial coefficients $\binom{3n}{n}$.

Lemma 3.1. [5] *If* p *and* q *are integers, then*

$$F_{p+q} = \alpha^q F_p + \beta^p F_q, \tag{28}$$

$$F_{p+q} = \alpha^p F_q + \beta^q F_p. \tag{29}$$

3.1 Fibonacci series associated with identity (A)

Theorem 3.2. *Let p and q be integers such that $p \le -2$, $q \ge 4$ with $q > |p| + 1$. Then*

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{-27F_pF_{p+q}}{F_q^2}\right)^k\frac{F_{(2p+q)k}}{k^2\binom{3k}{k}}\\&=\frac{6}{\sqrt{5}}\left(\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{F_{p+q}}}{2\sqrt[3]{\alpha^qF_p}+\sqrt[3]{F_{p+q}}}\right)-\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{F_p}}{2\sqrt[3]{\alpha^qF_{p+q}}+(-1)^q\sqrt[3]{F_p}}\right)\right)\\&\quad-\frac{\sqrt{5}}{10}\left(\ln^2\left(\frac{(-1)^pF_q}{\left(\sqrt[3]{\alpha^pF_{p+q}}-\sqrt[3]{\alpha^{p+q}F_p}\right)^3}\right)-\ln^2\left(\frac{\alpha^{p+q}F_q}{\left(\sqrt[3]{\alpha^qF_{p+q}}+(-1)^q\sqrt[3]{F_p}\right)^3}\right)\right)\end{aligned}\tag{30}$$

and

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{-27F_pF_{p+q}}{F_q^2}\right)^k\frac{L_{(2p+q)k}}{k^2\binom{3k}{k}}\\&=6\left(\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{F_{p+q}}}{2\sqrt[3]{\alpha^qF_p}+\sqrt[3]{F_{p+q}}}\right)+\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{F_p}}{2\sqrt[3]{\alpha^qF_{p+q}}+(-1)^q\sqrt[3]{F_p}}\right)\right)\\&\quad-\frac{1}{2}\left(\ln^2\left(\frac{(-1)^pF_q}{\left(\sqrt[3]{\alpha^pF_{p+q}}-\sqrt[3]{\alpha^{p+q}F_p}\right)^3}\right)+\ln^2\left(\frac{\alpha^{p+q}F_q}{\left(\sqrt[3]{\alpha^qF_{p+q}}-(-1)^q\sqrt[3]{F_p}\right)^3}\right)\right).\end{aligned}\tag{31}$$

Proof. Set $(x, y) = (F_p\alpha^q, -F_{p+q})$ in identity (A) and use (28) to obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{-27F_pF_{p+q}}{F_q^2}\right)^k\frac{\alpha^{(2p+q)k}}{k^2\binom{3k}{k}}\\&=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{F_{p+q}}}{2\sqrt[3]{\alpha^qF_p}+\sqrt[3]{F_{p+q}}}\right)-\frac{1}{2}\ln^2\left(\frac{(-1)^pF_q}{\left(\sqrt[3]{\alpha^pF_{p+q}}-\sqrt[3]{\alpha^{p+q}F_p}\right)^3}\right).\end{aligned}\tag{32}$$

Similarly, $(x, y) = (F_{p+q}, -\beta^qF_p)$ in identity (A) and the use of (29) gives

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{-27F_pF_{p+q}}{F_q^2}\right)^k\frac{\beta^{(2p+q)k}}{k^2\binom{3k}{k}}\\&=6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{F_p}}{2\sqrt[3]{\alpha^qF_{p+q}}+(-1)^q\sqrt[3]{F_p}}\right)-\frac{1}{2}\ln^2\left(\frac{\alpha^{p+q}F_q}{\left(\sqrt[3]{\alpha^qF_{p+q}}-(-1)^q\sqrt[3]{F_p}\right)^3}\right).\end{aligned}\tag{33}$$

Identities (30) and (31) follow from the subtraction and addition of (32) and (33) with the use of the Binet formulas (6). □

Example 3.2.1. *At $p = -2$ and $q = 5$ from* (30) *and* (31) *we have the following series:*

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\left(\frac{54}{25}\right)^kF_k}{k^2\binom{3k}{k}}&=\frac{6\sqrt{5}}{5}\left(\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{2}}{2\sqrt[3]{\alpha^5}-\sqrt[3]{2}}\right)-\arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{2\alpha^5}+1}\right)\right)\\&\quad+\frac{\sqrt{5}}{10}\left(\ln^2\left(\frac{5\alpha^3}{\left(\sqrt[3]{2\alpha^5}-1\right)^3}\right)-\ln^2\left(\frac{5\alpha^2}{\left(\sqrt[3]{\alpha^5}+\sqrt[3]{2}\right)^3}\right)\right),\end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{54}{25}\right)^k L_k}{k^2\binom{3k}{k}} = 6\left(\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{2}}{2\sqrt[3]{\alpha^5}-\sqrt[3]{2}}\right) + \arctan^2\left(\frac{\sqrt{3}}{2\sqrt[3]{2\alpha^5}+1}\right)\right)$$
$$- \frac{1}{2}\left(\ln^2\left(\frac{5\alpha^3}{\left(\sqrt[3]{2\alpha^5}-1\right)^3}\right) + \ln^2\left(\frac{5\alpha^2}{\left(\sqrt[3]{\alpha^5}+\sqrt[3]{2}\right)^3}\right)\right).$$

3.2 Fibonacci series associated with identity (B)

Theorem 3.3. *Let p and q be integers such that $p \leq -2$, $q \geq 4$, and $q > |p| + 1$. Then*

$$\frac{1}{\sqrt[3]{F_p F_{p+q}}} \sum_{k=1}^{\infty} \left(\frac{-27F_pF_{p+q}}{F_q^2}\right)^k \frac{F_{(2p+q)k}}{k\binom{3k}{k}}$$
$$= \frac{2\sqrt{15}}{5}\left(A_\alpha^- \arctan\left(\frac{\sqrt{3}\sqrt[3]{F_{p+q}}}{2\sqrt[3]{\alpha^q F_p}+\sqrt[3]{F_{p+q}}}\right) + A_\beta^- \arctan\left(\frac{\sqrt{3}\sqrt[3]{F_p}}{2(-1)^q\sqrt[3]{\alpha^q F_{p+q}}+\sqrt[3]{F_p}}\right)\right)$$
$$- \frac{\sqrt{5}}{5}\left(A_\alpha^+ \ln\left(\frac{\beta^p F_q}{\left(\sqrt[3]{F_{p+q}}-\sqrt[3]{\alpha^q F_p}\right)^3}\right) - A_\beta^+ \ln\left(\frac{\alpha^p F_q}{\left(\sqrt[3]{F_{p+q}}-\sqrt[3]{\beta^q F_p}\right)^3}\right)\right)$$

and

$$\frac{1}{\sqrt[3]{F_p F_{p+q}}} \sum_{k=1}^{\infty} \left(\frac{-27F_pF_{p+q}}{F_q^2}\right)^k \frac{L_{(2p+q)k}}{k\binom{3k}{k}}$$
$$= 2\sqrt{3}\left(A_\alpha^- \arctan\left(\frac{\sqrt{3}\sqrt[3]{F_{p+q}}}{2\sqrt[3]{\alpha^q F_p}+\sqrt[3]{F_{p+q}}}\right) - A_\beta^- \arctan\left(\frac{\sqrt{3}\sqrt[3]{F_p}}{2(-1)^q\sqrt[3]{\alpha^q F_{p+q}}+\sqrt[3]{F_p}}\right)\right)$$
$$- \left(A_\alpha^+ \ln\left(\frac{\beta^p F_q}{\left(\sqrt[3]{F_{p+q}}-\sqrt[3]{\alpha^q F_p}\right)^3}\right) + A_\beta^+ \ln\left(\frac{\alpha^p F_q}{\left(\sqrt[3]{F_{p+q}}-\sqrt[3]{\beta^q F_p}\right)^3}\right)\right),$$

where

$$A_s^{\pm} = \sqrt[3]{s^q}\,\frac{\sqrt[3]{s^q F_p} \pm \sqrt[3]{F_{p+q}}}{s^q F_p + F_{p+q}}.$$

Proof. The proof is similar to that one given for Theorem 3.2 and omitted. □

Example 3.3.1. *At $p = -2$ and $q = 5$ from Theorem 3.3 we obtain the series:*

$$\sum_{k=1}^{\infty} \frac{\left(\frac{54}{25}\right)^k F_k}{k\binom{3k}{k}}$$
$$= \frac{2\sqrt{15}}{5}\sqrt[3]{2\alpha^5}\left(\frac{\sqrt[3]{2}+\sqrt[3]{\alpha^5}}{2-\alpha^5}\arctan\left(\frac{\sqrt{3}}{1-\sqrt[3]{4\alpha^5}}\right) + \frac{1-\sqrt[3]{2\alpha^5}}{1+2\alpha^5}\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{2\alpha^5}+1}\right)\right)$$
$$+ \frac{\sqrt{5}}{5}\sqrt[3]{2\alpha^5}\left(\frac{\sqrt[3]{2}-\sqrt[3]{\alpha^5}}{2-\alpha^5}\ln\left(\frac{5\alpha^2}{\left(\sqrt[3]{2}+\sqrt[3]{\alpha^5}\right)^3}\right) + \frac{1+\sqrt[3]{2\alpha^5}}{1+2\alpha^5}\ln^2\left(\frac{5\alpha^3}{\left(\sqrt[3]{2\alpha^5}-1\right)^3}\right)\right),$$

$$\sum_{k=1}^{\infty} \frac{\left(\frac{54}{25}\right)^k L_k}{k\binom{3k}{k}}$$
$$= 2\sqrt{3}\sqrt[3]{2\alpha^5}\left(\frac{\sqrt[3]{\alpha^5}+\sqrt[3]{2}}{2-\alpha^5}\arctan\left(\frac{\sqrt{3}}{1-\sqrt[3]{4\alpha^5}}\right)-\frac{1-\sqrt[3]{2\alpha^5}}{1+2\alpha^5}\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{2\alpha^5}+1}\right)\right)$$
$$+\sqrt[3]{2\alpha^5}\left(\frac{\sqrt[3]{2}-\sqrt[3]{\alpha^5}}{2-\alpha^5}\ln\left(\frac{5\alpha^2}{\left(\sqrt[3]{2}+\sqrt[3]{\alpha^5}\right)^3}\right)-\frac{1+\sqrt[3]{2\alpha^5}}{1+2\alpha^5}\ln^2\left(\frac{5\alpha^3}{\left(\sqrt[3]{2\alpha^5}-1\right)^3}\right)\right).$$

3.3 Fibonacci series associated with identity (C)

Theorem 3.4. *Let p and q be integers such that $p \le -2$, $q \ge 4$, and $|q| > |p|+1$. Then we have*

$$\frac{(-1)^p}{F_q\sqrt[3]{F_pF_{p+q}}}\sum_{k=1}^{\infty}\left(\frac{-27F_pF_{p+q}}{F_q^2}\right)^k\frac{F_{(2p+q)k}}{\binom{3k}{k}}$$
$$= \frac{4(-1)^{p-1}\sqrt[3]{F_{p+q}^2F_p^2}\left(F_{p+q}^2-(-1)^qF_p^2\right)}{(F_{p+q}+\alpha^qF_p)^2(F_{p+q}+\beta^qF_p)^2}$$
$$-\frac{2\sqrt{15}}{15}\left(B_\alpha^+\arctan\left(\frac{\sqrt{3}\sqrt[3]{F_{p+q}}}{2\sqrt[3]{\alpha^qF_p}+\sqrt[3]{F_{p+q}}}\right)+B_\beta^+\arctan\left(\frac{\sqrt{3}\sqrt[3]{F_p}}{2(-1)^q\sqrt[3]{\alpha^qF_{p+q}}+\sqrt[3]{F_p}}\right)\right)$$
$$+\frac{\sqrt{5}}{15}\left(B_\alpha^-\ln\left(\frac{(-1)^pF_q}{\left(\sqrt[3]{\alpha^pF_{p+q}}-\sqrt[3]{\alpha^{p+q}F_p}\right)^3}\right)-B_\beta^-\ln\left(\frac{\alpha^{p+q}F_q}{\left(\sqrt[3]{\alpha^qF_{p+q}}-(-1)^q\sqrt[3]{F_p}\right)^3}\right)\right)$$

and

$$\frac{1}{F_q\sqrt[3]{F_pF_{p+q}}}\sum_{k=1}^{\infty}\left(\frac{-27F_pF_{p+q}}{F_q^2}\right)^k\frac{L_{(2p+q)k}}{\binom{3k}{k}}$$
$$= \frac{4(-1)^{q-1}\sqrt[3]{F_{p+q}^2F_p^2}\left(F_p^2L_q+(-1)^qF_{p+q}^2L_q+4F_pF_{p+q}\right)}{F_q(F_{p+q}+\alpha^qF_p)^2(F_{p+q}+\beta^qF_p)^2}$$
$$-\frac{2(-1)^p}{\sqrt{3}}\left(B_\alpha^+\arctan\left(\frac{\sqrt{3}\sqrt[3]{F_{p+q}}}{2\sqrt[3]{\alpha^qF_p}+\sqrt[3]{F_{p+q}}}\right)-B_\beta^+\arctan\left(\frac{\sqrt{3}\sqrt[3]{F_p}}{2(-1)^q\sqrt[3]{\alpha^qF_{p+q}}+\sqrt[3]{F_p}}\right)\right)$$
$$+\frac{(-1)^p}{3}\left(B_\alpha^-\ln\left(\frac{(-1)^pF_q}{\left(\sqrt[3]{\alpha^pF_{p+q}}-\sqrt[3]{\alpha^{p+q}F_p}\right)^3}\right)+B_\beta^-\ln\left(\frac{\alpha^{p+q}F_q}{\left(\sqrt[3]{\alpha^qF_{p+q}}-(-1)^q\sqrt[3]{F_p}\right)^3}\right)\right),$$

where

$$B_s^{\pm}=\frac{\sqrt[3]{s^{q-3p}}}{(s^qF_p+F_{p+q})^3}\left(\sqrt[3]{s^{4q}F_p^4}\pm\sqrt[3]{F_{p+q}^4}\mp 2\sqrt[3]{s^qF_pF_{p+q}}\left(\sqrt[3]{s^{2q}F_p^2}\pm\sqrt[3]{F_{p+q}^2}\right)\right).$$

Proof. The proof is similar to the previous two proofs. □

Example 3.4.1. *At $p = -2$ and $q = 5$ from Theorem 3.4 we obtain the following series:*

$$\sum_{k=1}^{\infty} \frac{\left(\frac{54}{25}\right)^k F_k}{k\binom{3k}{k}} = \frac{200}{361} + \frac{2\sqrt{15}\sqrt[3]{2\alpha^{11}}}{3}\left(\frac{\sqrt[3]{16}(1+\alpha^5) + \sqrt[3]{\alpha^5}(4+\alpha^5)}{(5\alpha+1)^3}\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^5/2}-1}\right)\right.$$
$$\left. + \frac{\sqrt[3]{16\alpha^5}(\alpha^5-1)+1-4\alpha^5}{(\alpha-5)^3\alpha^{11}}\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{2\alpha^5}+1}\right)\right)$$
$$- \frac{\sqrt{5}\sqrt[3]{2\alpha^{11}}}{3}\left(\frac{\left(\sqrt[3]{2}+\sqrt[3]{\alpha^5}\right)\left(\sqrt[3]{2}-\sqrt[3]{\alpha^5}\right)^3}{(5\alpha+1)^3}\ln\left(\frac{5\alpha^2}{\left(\sqrt[3]{\alpha^5}+\sqrt[3]{2}\right)^3}\right)\right.$$
$$\left. + \frac{\left(\sqrt[3]{2\alpha^5}-1\right)\left(\sqrt[3]{2\alpha^5}+1\right)^3}{(\alpha-5)^3\alpha^{11}}\ln\left(\frac{5\alpha^3}{\left(\sqrt[3]{2\alpha^5}-1\right)^3}\right)\right)$$

and

$$\sum_{k=1}^{\infty} \frac{\left(\frac{54}{25}\right)^k L_k}{k\binom{3k}{k}} = \frac{328}{361} + \frac{10\sqrt{3}\sqrt[3]{2\alpha^{11}}}{3}\left(\frac{\sqrt[3]{\alpha^5}(\alpha^5+4) + \sqrt[3]{16}(\alpha^5+1)}{(\alpha^5-2)^3}\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{\alpha^5/2}-1}\right)\right.$$
$$\left. + \alpha\frac{\sqrt[3]{16\alpha^{20}}+1-\sqrt[3]{16\alpha^5}\left(\sqrt[3]{4\alpha^{10}}+1\right)}{(2\alpha^5+1)^3}\arctan\left(\frac{\sqrt{3}}{2\sqrt[3]{2\alpha^5}+1}\right)\right)$$
$$- \frac{5\sqrt[3]{2\alpha^{11}}}{3}\left(\frac{\left(\sqrt[3]{2}+\sqrt[3]{\alpha^5}\right)\left(\sqrt[3]{2}-\sqrt[3]{\alpha^5}\right)^3}{(\alpha^5-2)^3}\ln\left(\frac{5\alpha^2}{\left(\sqrt[3]{\alpha^5}+\sqrt[3]{2}\right)^3}\right)\right.$$
$$\left. - \frac{\alpha\left(\sqrt[3]{2\alpha^5}-1\right)\left(\sqrt[3]{2\alpha^5}+1\right)^3}{(2\alpha^5+1)^3}\ln\left(\frac{5\alpha^3}{\left(\sqrt[3]{2\alpha^5}-1\right)^3}\right)\right).$$

4 Concluding comments

In this paper we presented new closed forms for some types of infinite series involving binomial coefficients $\binom{3n}{n}$. To prove our results, we applied some routine arguments, combining Batır's formula (1) with Binet's formulas for Fibonacci and Lucas numbers. Using similar techniques, we can establish series evaluations involving binomial coefficients $\binom{3n}{n}$ with Fibonacci and Lucas polynomials and other known number and polynomial sequences.

Let us give, for example, a generalization of Theorems 2.1 and 2.3 to the case of the Horadam sequence defined by the recurrence $W_n = pW_{n-1} - qW_{n-2}$, $n \geq 2$, with initial values $W_0 = a$ and $W_1 = b$.

Let

$$\Delta = \sqrt{p^2 - 4q}, \quad \alpha_* = \frac{p+\Delta}{2}, \quad \beta_* = \frac{p-\Delta}{2}, \quad A = b - a\beta_*, \quad B = b - a\alpha_*.$$

Then the following identities hold for positive integer r:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k(r-1)}}{k^2\binom{3k}{k}W_r^{2k}}\left(\frac{27ABq^r}{\Delta^2}\right)^k$$
$$= 6\arctan^2\left(\frac{\sqrt{3}\sqrt[3]{Bq^r}}{2\sqrt[3]{A\alpha_*^{2r}} + \sqrt[3]{B(-q)^r}}\right) - \frac{1}{2}\ln^2\left(\frac{\alpha_*^r \Delta W_r}{\left(\sqrt[3]{A\alpha_*^{2r}} - \sqrt[3]{B(-q)^r}\right)^3}\right)$$

and

$$\frac{A\alpha_*^{2r}+B(-q)^r}{\sqrt[3]{AB\alpha_*^{2r}q^r}}\sum_{k=1}^{\infty}\frac{(-1)^{k(r-1)}}{k\binom{3k}{k}W_r^{2k}}\left(\frac{27ABq^r}{\Delta^2}\right)^k$$
$$=2\sqrt{3}\big(\sqrt[3]{A\alpha_*^{2r}}-\sqrt[3]{B(-q)^r}\big)\arctan\left(\frac{\sqrt{3}\sqrt[3]{Bq^r}}{2\sqrt[3]{A\alpha_*^{2r}}+\sqrt[3]{B(-q)^r}}\right)$$
$$-(-1)^r\big(\sqrt[3]{A\alpha_*^{2r}}+\sqrt[3]{B(-q)^r}\big)\ln\left(\frac{\alpha_*^r\Delta W_r}{\big(\sqrt[3]{A\alpha_*^{2r}}-\sqrt[3]{B(-q)^r}\big)^3}\right).$$

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