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The error term of the sum of digital sum functions, in arbitrary bases

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Abstract: Let k be a non-negative integer and $q > 1$ be a positive integer. Let $s_q(k)$ be the sum of digits of k written in base q. In 1940, Bush proved that $A_q(x) = \sum_{k \le x} s_q(k)$ is asymptotic to $\frac{q-1}{2} x \log_q x$. In 1968, Trollope proved an explicit formula for the error term of $A_2(n-1)$, labeled by $-E_2(n)$, where n is a positive integer. In 1975, Delange extended Trollope's result to an arbitrary base q by another method and labeled the error term $nF_q(\log_q n)$. When $q = 2$, the two formulas of the error term are supposed to be equal, but they look quite different. We proved directly that those two formulas are equal. More interestingly, Cooper and Kennedy in 1999 applied Trollope's method to extend $-E_2(n)$ to $-E_q(n)$ with a general base q, and we also proved directly that $nF_q(\log_q n)$ and $-E_q(n)$ are equal for any q.
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1 Introduction

Let $q > 1$ be a fixed integer and denote by $s_q(k)$ the sum of digits of k written in base q when k is a positive integer, i.e.,

$$s_q(k) = \sum_{i=0}^{m} a_i$$

where $k = \sum_{i=0}^{m} a_i q^i$ for some non-negative integer m and $0 \le a_i < q$, $a_m > 0$. Define $s_q(0) = 0$.

It was L.E. Bush [3] who first showed in 1940 that

$$A_q(x) = \sum_{k \le x} s_q(k) \sim \frac{q-1}{2} x \log_q x \text{ as } x \to \infty. \tag{1}$$

In 1948, R. Bellman and H.N. Shapiro [2] proved that when $q = 2$,

$$\sum_{k \le x} s_2(k) = \frac{1}{2} x \log_2 x + \mathcal{O}(x \log \log x). \tag{2}$$

Their method was extended to an arbitrary base by C. Gadd and the second author [7] recently with the same error term, and they also fixed a major error in Bellman and Shapiro's paper [2], although the error term was not the best possible one.

In 1949, L. Mirsky [8] proved that

$$\sum_{k \le x} s_q(k) = \frac{q-1}{2} x \log_q x + \mathcal{O}(x), \tag{3}$$

where $\mathcal{O}(x)$ is the best possible error term. Many other authors have also proved (3) using different methods ([1], [4], and [9]).

In 1968, J. P. Trollope [10] discovered the following result: Let $g_2(x)$ be periodic of period one and defined on $[0, 1]$ by

$$g_2(x) = \begin{cases} \frac{1}{2}x, & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{1}{2}(1-x), & \text{if } \frac{1}{2} < x \le 1 \end{cases} \tag{4}$$

and put

$$f_2(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} g_2(2^i x).$$

Now, if $n = 2^m(1 + x)$, where $0 \le x < 1$, one has

$$\sum_{k=0}^{n-1} s_2(k) = \frac{1}{2} n \log_2 n - E_2(n), \tag{5}$$

where

$$E_2(n) = 2^{m-1} \left(2 f_2(x) + (1 + x) \log_2(1 + x) - 2x \right). \tag{6}$$

This explicit formula of the error term was derived based on a function ϕ related to a part of an explicit formula of $A_2(x)$, and a continuous extension of ϕ. Trollope said that his result could be generalized for any base q, but that the calculations are much more complicated.

In 1975, H. Delange [6] extended Trollope's results by a much simpler method that even works for any base as follows: There exists a function $F_q : \mathbb{R} \to \mathbb{R}$ of period one, which is continuous and nowhere differentiable, such that

$$\sum_{k=0}^{n-1} s_q(k) = \frac{q-1}{2} n \log_q n + nF_q(\log_q n) \tag{7}$$

for all $n \geq 1$.

In fact,

$$F_q(x) = \frac{q-1}{2}(1 + [x] - x) + q^{1+[x]-x} h_q(q^{x-[x]-1}), \tag{8}$$

where

$$h_q(x) = \sum_{r=0}^{\infty} \frac{j_q(q^r x)}{q^r} \tag{9}$$

and

$$j_q(x) = \int_0^x \left([qt] - q[t] - \frac{q-1}{2} \right) dt. \tag{10}$$

Delange's method was based on turning the coefficients of a q-base expansion of an integer into integrals, followed by evaluating the sum of digital sum functions in terms of those integrals.

Delange even derived a more general formula for the sum of digital sum functions up to a certain real number x. The formula says, for all real numbers $x \geq 1$, we have

$$\sum_{n \leq x} s_q(n) = \frac{q-1}{2} x \log_q x + xF_q(\log_q n) - h_q(x) + (1 + [x] - x)s_q([x]). \tag{11}$$

As Trollope suggested, Cooper and Kennedy [5], in 1999, extended the error term $-E_2(n)$ to $-E_q(n)$ as follows: Let $g_q(x)$ be periodic of period $\frac{1}{q-1}$ and on $\left[0, \frac{1}{q-1}\right]$ be equal to the piecewise linear function connecting the points $\left(\frac{a}{q^2-q}, \frac{(q-a)a}{2q}\right)$ where a is a nonnegative integer and $0 \leq a \leq q$. Let

$$f_q(x) = \sum_{i=0}^{\infty} \frac{1}{q^i} g_q(q^i x).$$

Now, if $n = q^m(1 + x(q-1))$, where $0 \leq x < 1$, one has

$$\sum_{k=0}^{n-1} s_q(k) = \frac{q-1}{2} n \log_q n - E_q(n), \tag{12}$$

where

$$E_q(n) = q^m \left(f_q(x) + \frac{q-1}{2}(1 + (q-1)x) \log_q(1 + (q-1)x) - a_m(1 - a_m + (q-1)x) - \frac{(a_m-1)a_m}{2} \right), \tag{13}$$

with

$$n = \sum_{i=0}^{m} a_i q^i = a_m q^m + n_{m-1}. \tag{14}$$

In Section 2, we will show that $-E_2(n) = nF_2(\log_2 n)$ directly. In Section 3, we will show $-E_q(n) = nF_q(\log_q n)$ without all the details, but point out some differences from the base 2 case.

2 The binary case

Define $\{x\} = x - [x]$, i.e, the fractional part of x. From (8), for $q = 2$, we have

$$F_2(\log_2 n) = \frac{1}{2}(1 - \{\log_2 n\}) + 2^{1-\{\log_2 n\}} h_2\left(2^{\{\log_2 n\}-1}\right). \tag{15}$$

If $n = 2^m(1 + x)$, then we have

$$\log_2 n = \log_2(2^m(1 + x)) = \log_2(2^m) + \log_2(1 + x) = m + \log_2(1 + x). \tag{16}$$

Since $0 \le x < 1$, we have $0 \le \log_2(1+x) < 1$. Thus, we can conclude that $\{\log_2 n\} = \log_2(1+x)$. With this equation, we can get

$$2^{1-\{\log_2 n\}} = 2^{1-\log_2(1+x)} = \frac{2}{1+x} \quad \text{and} \quad 2^{\{\log_2 n\}-1} = \frac{1+x}{2}.$$

Therefore, (15) becomes

$$F_2(\log_2 n) = \frac{1}{2}(1 - \log_2(1 + x)) + \frac{2}{1+x} h_2\left(\frac{1+x}{2}\right). \tag{17}$$

Now, we need to evaluate $h_2\left(\frac{1+x}{2}\right)$. To do that, we need an explicit formula for j_2. From (10), we have

$$\begin{aligned} j_2(x) &= \int_0^x \left([2t] - 2[t] - \frac{1}{2}\right) dt \\ &= \int_0^x \left([2([t] + \{t\})] - 2[t] - \frac{1}{2}\right) dt \\ &= \int_0^x \left([2[t] + 2\{t\}] - 2[t] - \frac{1}{2}\right) dt \\ &= \int_0^x \left(2[t] + [2\{t\}] - 2[t] - \frac{1}{2}\right) dt \\ &= \int_0^x \left([2\{t\}] - \frac{1}{2}\right) dt. \end{aligned}$$

We can see that the integrand function $[2\{t\}] - \dfrac{1}{2}$ is periodic with period 1, so is $j_2(x)$. When $0 \le t < \frac{1}{2}$, $[2\{t\}] - \dfrac{1}{2} = -\dfrac{1}{2}$ and when $\frac{1}{2} \le t < 1$, $[2\{t\}] - \dfrac{1}{2} = \dfrac{1}{2}$. Therefore, on $[0, 1]$, we have

$$j_2(x) = \begin{cases} -\frac{1}{2}x, & \text{if } 0 \le x < \frac{1}{2}, \\ -\frac{1}{2}(1 - x), & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

For simplification, we omit the subscript 2 for the rest of this section. Note that $j = -g$ from (4) and so $h = -f$. From the definition of h (see (9)), we have

$$
\begin{aligned}
h\left(\frac{1+x}{2}\right) &= \sum_{r=0}^{\infty} \frac{j(2^r \cdot \frac{1+x}{2})}{2^r} \\
&= j\left(\frac{1+x}{2}\right) + \sum_{r=1}^{\infty} \frac{j(2^r \cdot \frac{1+x}{2})}{2^r} \\
&= j\left(\frac{1+x}{2}\right) + \sum_{r=0}^{\infty} \frac{j(2^r(1+x))}{2^{r+1}} \\
&= j\left(\frac{1+x}{2}\right) + \frac{1}{2}h(x) \\
&= j\left(\frac{1+x}{2}\right) - \frac{1}{2}f(x).
\end{aligned}
$$

Now we substitute this into (17) and we get

$$
F_2\left(\log_2 n\right) = \frac{1}{2}\left(1 - \log_2(1+x)\right) + \frac{2}{1+x}\left(j\left(\frac{1+x}{2}\right) - \frac{1}{2}f(x)\right). \tag{18}
$$

Since $0 \leq x < 1$, we have $\frac{1}{2} \leq \frac{1+x}{2} < 1$. Thus, $j\left(\frac{1+x}{2}\right) = -\frac{1}{2}\left(1 - \frac{1+x}{2}\right)$. Now, substituting this into (18), we have

$$
\begin{aligned}
F_2\left(\log_2 n\right) &= \frac{1}{2} - \frac{\log_2(1+x)}{2} + \frac{2}{1+x}\left(-\frac{1}{2}\left(1 - \frac{1+x}{2}\right) - \frac{1}{2}f(x)\right) \\
&= -\frac{\log_2(1+x)}{2} - \frac{f(x)}{1+x} + 1 - \frac{1}{1+x}.
\end{aligned}
$$

At the same time, from (6), we get

$$
\begin{aligned}
E_2(n) &= 2^m f(x) + 2^{m-1}(1+x)\log_2(1+x) - 2^m x \\
&= \frac{n}{1+x}f(x) + \frac{n}{2}\log_2(1+x) - \frac{nx}{1+x} \\
&= -n\left(-\frac{1}{2}\log_2(1+x) - \frac{f(x)}{1+x} + 1 - \frac{1}{1+x}\right) \\
&= -nF_2\left(\log_2 n\right).
\end{aligned}
$$

Therefore, we have proven $-E_2(n) = nF_2\left(\log_2 n\right)$.

3 The q-base case

If $n = q^m(1 + x(q-1))$, where $0 \leq x < 1$, we can write

$$
\log_q n = m + \log_q(1 + (q-1)x), \tag{19}
$$

$$
\left[\log_q n\right] = m \quad \text{and} \quad \left\{\log_q n\right\} = \log_q(1 + (q-1)x). \tag{20}
$$

For simplification, we omit the subscript q. Now, we have

$$nF_q\left(\log_q n\right) = n\left[\frac{q-1}{2}\left(1-\log_q(1+(q-1)x)\right) + q^{1-\log_q(1+(q-1)x)}h\left(q^{\log_q(1+(q-1)x)-1}\right)\right] \tag{21}$$

$$= n\left[\frac{q-1}{2}\left(1-\log_q(1+(q-1)x)\right) + \frac{q}{1+(q-1)x}h\left(\frac{1+(q-1)x}{q}\right)\right]. \tag{22}$$

To evaluate h, we first find a relation between j and g. From (10), we have

$$j(x) = \int_0^x \left([qt] - q[t] - \frac{q-1}{2}\right) dt = \int_0^x \left([q\{t\}] - \frac{q-1}{2}\right) dt.$$

When $q = 2$, it was easy to see $j = -g$ with their explicit formulas. However, when $q > 2$, it is not obvious to see the exact relation between j and g. Let us look at some graphs of the integrand function in j and the graphs of j for some small q (Figure 1).

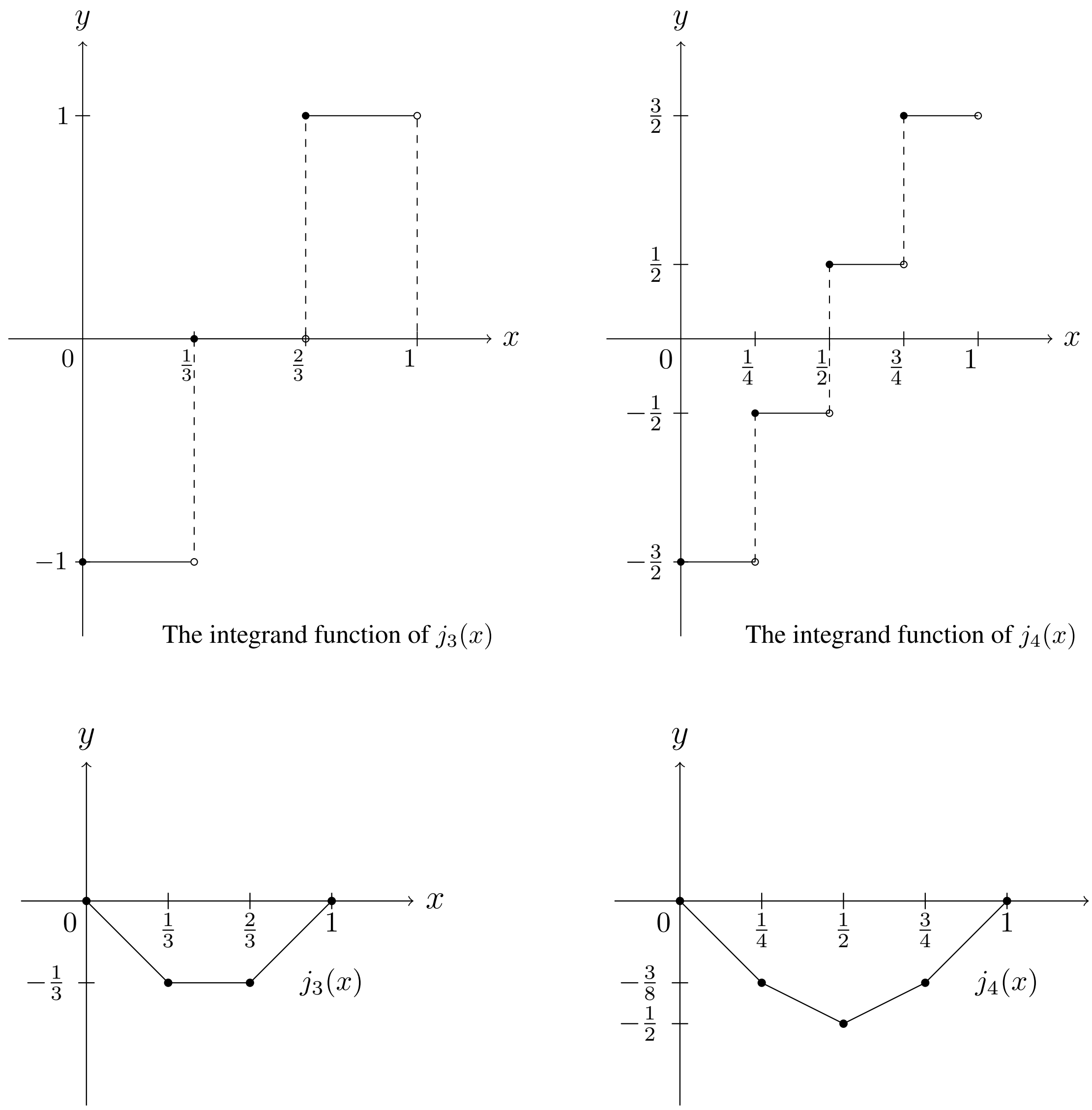

Figure 1. Integrand functions in j and the graphs of j for some small q

When a is a positive integer such that $0 \leq a \leq q$, we can evaluate j by evaluating the net area of its integrand function: $j\left(\frac{a}{q}\right) = \frac{(a-q)a}{2q}$. Hence, $j((q-1)x) = -g(x)$ and $h((q-1)x) = -f(x)$. Now we can evaluate h and get

$$\begin{aligned}
h\left(\frac{1+(q-1)x}{q}\right) &= \sum_{r=0}^{\infty} \frac{j\left(q^r \cdot \frac{1+(q-1)x}{q}\right)}{q^r} \\
&= j\left(\frac{1+(q-1)x}{q}\right) + \sum_{r=1}^{\infty} \frac{j\left(q^r \cdot \frac{1+(q-1)x}{q}\right)}{q^r} \\
&= j\left(\frac{1+(q-1)x}{q}\right) + \frac{1}{q}\sum_{r=0}^{\infty} \frac{j(q^r \cdot (1+(q-1)x)}{q^r} \\
&= j\left(\frac{1+(q-1)x}{q}\right) + \frac{1}{q}h(1+(q-1)x) \\
&= j\left(\frac{1+(q-1)x}{q}\right) + \frac{1}{q}h((q-1)x) \\
&= j\left(\frac{1+(q-1)x}{q}\right) - \frac{f(x)}{q}.
\end{aligned}$$

If we substitute this into (22), we have

$$nF_q\left(\log_q n\right) = n\left[\frac{q-1}{2}\left(1-\log_q(1+(q-1)x)\right) + \frac{q}{1+(q-1)x}\left(j\left(\frac{1+(q-1)x}{q}\right) - \frac{f(x)}{q}\right)\right].$$

After comparing this equation and the equation of $-E_q(n)$, we can cancel some terms and factors, which brings it down to proving the following equation:

$$\frac{(1+(q-1)x)(q-1)}{2} + qj\left(\frac{1+(q-1)x}{q}\right) = a_m(1-a_m+(q-1)x) + \frac{a_m(a_m-1)}{2}. \quad (23)$$

Since

$$\frac{a_m}{q} \leq \frac{1+(q-1)x}{q} < \frac{a_m+1}{q}, \quad (24)$$

we can evaluate $j\left(\frac{1+(q-1)x}{q}\right)$ and get

$$j\left(\frac{1+(q-1)x}{q}\right) = \frac{1}{q}\left(\left(\frac{1-q}{2}\right)a_m + \frac{(a_m-1)a_m}{2} + (1+(q-1)x-a_m)\left(a_m - \frac{q-1}{2}\right)\right). \quad (25)$$

Substituting this equation into the left hand side of (23), we can see that (23) is true. Therefore, we have proven $-E_q(n) = nF_q\left(\log_q n\right)$.

4 Conclusion

The error term of the sum of digital sum functions $A_q(n-1)$ has two formulas that were derived using different methods. However, they are the same quantity as we showed that one formula can be derived directly from the other.

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