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Some infinite series summations involving linear recurrence relations of order 2 and 3

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Dedicated to Professor Harald G. Niederreiter
on the occasion of his 80^{th} birthday!

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Abstract: This paper extends known results of second and third order recursive sequences through extensive formulations of properties of the roots of their characteristic equations, some

are old but most are new. They are applied to novel studies of $\sum_{n=0}^{\infty} \frac{a_{mn}}{10^{n+1}}, \quad m = 1, 2, 3$, including their convergence criteria, and applied to many standard sequences, as particular cases of a generic $\{a_n\}$. The detailed development of the algebra of the pertinent theorems, and their associated lemmas and corollaries, should open up new vistas for interested number theorists with the concluding results on series values.

Keywords: Arithmetic sequence, Balancing sequence, Fermat sequence, Fibonacci sequence, Geometric sequence, Jacobsthal sequence, Leonardo sequence, Lucas sequence, Mersenne sequence, Padovan sequence, Pell sequence, Perrin sequence.

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1 Introduction

Infinite series, built around sequences which generalize the Fibonacci sequences and the so-called Tribonacci sequences, are the focus of this paper. These are related in a fundamental sense to some of the work of Melham and Shannon three decades ago [12]. Table 1 contains a summary of some of the results from the paper, set out in a way for the interested reader to try to extend, as well as to navigate the details of the current paper.

$\{a_n\}$	$m=1$	$m=2$	$m=3$
$\{M_n\}$	$\frac{1}{72}$	$\frac{3}{54}$	$\frac{7}{18}$
$\{S_n\}$	$\frac{17}{72}$	$\frac{15}{54}$	$\frac{11}{18}$
$\{P_n\}$	$\frac{1}{79}$	$\frac{2}{41}$	#
$\{Q_n\}$	$\frac{18}{79}$	$\frac{14}{41}$	#
$\{J_n\}$	$\frac{1}{88}$	$\frac{1}{54}$	$\frac{3}{22}$
$\{\mathcal{J}_n\}$	$\frac{19}{88}$	$\frac{15}{54}$	$\frac{13}{22}$
$\{F_n\}$	$\frac{1}{89}$	$\frac{1}{71}$	$\frac{2}{59}$
$\{F_{n+1}\}$	$\frac{10}{89}$	$\frac{9}{71}$	$\frac{9}{59}$
$\{L_n\}$	$\frac{19}{89}$	$\frac{17}{71}$	$\frac{16}{59}$
$\{Le_n\}$	$\frac{91}{801}$	$\frac{91}{639}$	$\frac{103}{531}$
$\{\mathcal{L}_n\}$	$\frac{171}{801}$	$\frac{153}{639}$	$\frac{144}{531}$

Table 1. $\sum_{n=0}^{\infty} \frac{a_{mn}}{10^{n+1}}$, $m = 1, 2, 3$. # Does not satisfy the condition for convergence. The special sequences are defined in Table 2. Le_n and $\mathcal{L}_n$ are defined in Section 6.

The principal results of this paper are theorems which deal with the roots, both distinct and repeated, of the pertinent quadratic characteristic equations of the second order sequences, which lead directly to the infinite series with appropriate conditions for convergence. These include involvement of well-known and well-established results of Henry Gould, Rudi Lidl, Harald Niederreiter, Morgan Ward, and [12] in particular, as further links to the past.

2 Preliminaries

Definition 2.1. [13] *A number sequence $\{a_n\}$ is called a sequence of order 2 if it satisfies the recurrence relation of order 2:*

$$a_n = pa_{n-1} + qa_{n-2},\ n \geq 2, \tag{1}$$

for some constants $p, q \neq 0$ and initial conditions a_0, a_1.

Given the initial conditions a_0, a_1, and the recurrence relation (1), the entire sequence can be determined. Many common sequences, such as arithmetic sequences, geometric sequences, and Fibonacci sequences, are examples of second order linear recurrence sequence (refer to Table 2 below, see [6]).

Sequence of numbers $\{a_n\}$	a_0	a_1	p	q
Integers $0, 1, 2, 3, \ldots$	0	1	2	-1
Arithmetic sequence (common difference d)	a	$a+d$	2	-1
Geometric sequence (common ratio r)	a	r	$r+1$	$-r$
Fibonacci sequence F_n	0	1	1	1
Lucas sequence L_n	2	1	1	1
Fermat sequence of the first kind T_n	1	3	3	-2
Fermat sequence of the second kind S_n	2	3	3	-2
Pell sequence of the first kind P_n	1	2	2	1
Pell sequence of the second kind Q_n	2	2	2	1
Balancing sequence B_n	0	1	6	-1
Lucas-balancing sequence C_n	1	3	6	-1
Mersenne sequence M_n	0	1	3	-2
Jacobsthal sequence J_n	0	1	1	2
Jacobsthal–Lucas sequence $\mathcal{J}_n$	2	1	1	2

Table 2. Some common sequences.

Theorem 2.1. *Let $\{a_n\}$ be a second order linear recurrence sequence* (1)*. Suppose α and β are two roots of the characteristic equation $x^2 - px - q = 0$, then*

$$a_n = \begin{cases} \left(\frac{a_1 - \beta a_0}{\alpha - \beta}\right)\alpha^n - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta}\right)\beta^n, & \textit{if } \alpha \neq \beta; \\ na_1\alpha^{n-1} - (n-1)a_0\alpha^n, & \textit{if } \alpha = \beta. \end{cases} \tag{2}$$

The proof for Theorem 2.1 can be found in [4] and [13].

The purpose of this paper is to investigate the series in the form of $\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}$, with $r > 1$. One well-known series of this form, which piqued the curiosity of many and inspired this paper, is the Fibonacci sum $\sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} = \frac{1}{89} = \frac{1}{F_{11}}$, [11].

3 Main results

In our main theorem, we will use the following lemma.

Lemma 3.1. *Suppose* $|t| < 1$. *Then*

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}.$$

Theorem 3.1. [10] *Let* $\{a_n\}$ *satisfy the second order linear recurrence sequence defined in* (1). *Suppose* α *and* β *are two different roots to the characteristic equation* $x^2 - px - q = 0$, k *is a positive integer, and* $r > 1$ *satisfying*

$$k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|, |\beta|\})} \right\rceil. \tag{3}$$

Then

$$\sum_{n=0}^{\infty} \frac{a_{nk}}{r^{n+1}} = \frac{a_0 r - a_0 \cdot \frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} + a_1 \cdot \frac{\alpha^k-\beta^k}{\alpha-\beta}}{(r-\alpha^k)(r-\beta^k)}. \tag{4}$$

Proof. From Theorem 2.1 and Lemma 3.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a_{nk}}{r^{n+1}} &= \frac{1}{r} \sum_{n=0}^{\infty} \left[\left(\frac{a_1 - \beta a_0}{\alpha - \beta} \right) \frac{\alpha^{nk}}{r^n} - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta} \right) \frac{\beta^{nk}}{r^n} \right] \\
&= \frac{1}{r} \left[\left(\frac{a_1 - \beta a_0}{\alpha - \beta} \right) \sum_{n=0}^{\infty} \frac{\alpha^{nk}}{r^n} - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta} \right) \sum_{n=0}^{\infty} \frac{\beta^{nk}}{r^n} \right] \\
&= \frac{1}{r} \left[\left(\frac{a_1 - \beta a_0}{\alpha - \beta} \right) \frac{r}{r - \alpha^k} - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta} \right) \frac{r}{r - \beta^k} \right] \\
&= \frac{1}{\alpha - \beta} \left[\frac{(\alpha - \beta) a_0 r - a_0(\alpha^{k+1} - \beta^{k+1}) + a_1(\alpha^k - \beta^k)}{(r - \alpha^k)(r - \beta^k)} \right] \\
&= \frac{a_0 r - a_0 \cdot \frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} + a_1 \cdot \frac{\alpha^k-\beta^k}{\alpha-\beta}}{(r-\alpha^k)(r-\beta^k)}
\end{aligned}$$

Note that Lemma 3.1 can be used because (3) is equivalent to

$$\max\left\{ \frac{|\alpha|^k}{r}, \frac{|\beta|^k}{r} \right\} < 1,$$

which is a necessary criterion for the convergence of the present geometric series. □

If the characteristic equation $x^2 - px - q = 0$ has repeated roots, we have a similar result.

Lemma 3.2. *Suppose* $|t| < 1$. *Then*

$$\sum_{n=1}^{\infty} n t^n = \frac{t}{(1-t)^2}.$$

Proof. Differentiating the geometric series in Lemma 3.1 with respect to t, we have

$$\sum_{n=1}^{\infty} nt^{n-1} = \frac{1}{(1-t)^2}.$$

Hence,

$$\sum_{n=1}^{\infty} nt^{n} = \frac{t}{(1-t)^2}.$$

□

Theorem 3.2. *Let $\{a_n\}$ satisfy the second order linear recurrence sequence defined in* (1). *Suppose α is a repeated root to the characteristic equation $x^2 - px - q = 0$ and satisfies $k < \left\lceil \frac{\log r}{\log|\alpha|} \right\rceil$. Then*

$$\sum_{n=0}^{\infty} \frac{a_{nk}}{r^{n+1}} = \frac{a_0 r - (k+1)a_0\alpha^k + ka_1\alpha^{k-1}}{(r-\alpha^k)^2}, \tag{5}$$

where k is a positive integer and $r > 1$.

Proof. By Theorem 2.1 and Corollary 3.2, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a_{nk}}{r^{n+1}} &= \sum_{n=0}^{\infty} \frac{nka_1\alpha^{nk-1}}{r^{n+1}} - \sum_{n=0}^{\infty} \frac{(nk-1)a_0\alpha^{nk}}{r^{n+1}} \\
&= \frac{ka_1}{\alpha r}\sum_{n=1}^{\infty} n\left(\frac{\alpha^k}{r}\right)^n - \frac{ka_0}{r}\sum_{n=1}^{\infty} n\left(\frac{\alpha^k}{r}\right)^n + \frac{a_0}{r}\sum_{n=0}^{\infty}\left(\frac{\alpha^k}{r}\right)^n \\
&= \frac{ka_1}{\alpha r}\cdot\frac{\frac{\alpha^k}{r}}{\left(1-\frac{\alpha^k}{r}\right)^2} - \frac{ka_0}{r}\cdot\frac{\frac{\alpha^k}{r}}{\left(1-\frac{\alpha^k}{r}\right)^2} + \frac{a_0}{r}\cdot\frac{1}{1-\frac{\alpha^k}{r}}r \\
&= \frac{ka_1\alpha^{k-1}}{(r-\alpha^k)^2} - \frac{ka_0\alpha^k}{(r-\alpha^k)^2} + \frac{a_0}{r-\alpha^k} \\
&= \frac{a_0 r - (k+1)a_0\alpha^k + ka_1\alpha^{k-1}}{(r-\alpha^k)^2}
\end{aligned}$$

Note that the condition $k < \left\lceil \frac{\log r}{\log|\alpha|} \right\rceil$ if and only if $\frac{|\alpha|^k}{r} < 1$, which guarantees convergence of the geometric series seen in the above steps. We then impose the ceiling function since k is a positive integer. □

Remark 3.1. *In future proofs, the equivalence between $k < \left\lceil \frac{\log r}{\log|\alpha|} \right\rceil$ and $\frac{|\alpha|^k}{r} < 1$ will be exercised implicitly, trusting that the reader will recognize when it is used. Similarly, the equivalence between $k < \left\lceil \frac{\log r}{\log(\max\{|\alpha_1|,|\alpha_2|,\ldots,|\alpha_n|\})} \right\rceil$ and $\max\left\{\frac{|\alpha_1|^k}{r}, \frac{|\alpha_2|^k}{r}, \ldots, \frac{|\alpha_n|^k}{r}\right\} < 1$ will be used in the later sections.*

Remark 3.2. *Let α be fixed and $\beta \to \alpha$ in the right hand side of* (4). *Since*

$$\lim_{\beta\to\alpha} \frac{\alpha^k - \beta^k}{\alpha - \beta} = \lim_{\beta\to\alpha} \sum_{j=0}^{k-1} \alpha^j \beta^{k-j-1} = k\alpha^{k-1},$$

then the limit of the right hand side of (4) *is the same as* (5).

Example 3.1. *Let $a_n = n$, where $a_0 = 0$, $a_1 = 1$, and a_n satisfies*

$$a_n = 2a_{n-1} - a_{n-2},\ n \geq 2.$$

The characteristic equation $x^2 - 2x + 1 = 0$ has two repeated roots. By Theorem 3.2, we have

$$\sum_{n=1}^{\infty} \frac{nk}{r^{n+1}} = \frac{k}{(r-1)^2},\ r > 1.$$

For example, let $k = 1$ and $r = 2$, we obtain $\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1$ that leads to $\sum_{n=1}^{\infty} \frac{n}{2^n} = 2$.

Example 3.2. *Let $\{a_n\}$ be an arithmetic sequence with $a_0 = 0$ and a common difference d such that $a_n = a + nd$. Since $\{a_n\}$ satisfies*

$$a_n = 2a_{n-1} - a_{n-2},\ n \geq 2,$$

we have by Theorem 3.2 that,

$$\sum_{n=0}^{\infty} \frac{a + nkd}{r^{n+1}} = \frac{a(r-1) + kd}{(r-1)^2},\ r > 1.$$

If we let $r = 2$, we obtain

$$\sum_{n=1}^{\infty} \frac{a + nkd}{2^{n+1}} = a + kd.$$

When $k = d = 1$ and $a = 0$, we have the result in Example 3.1.

Remark 3.3. *Example 3.2 can be used to find a closed expression for*

$$\sum_{n=1}^{\infty} \frac{n^2}{r^{n+1}}.$$

We start with the calculation

$$\begin{aligned}
(r-1)\sum_{n=1}^{\infty} \frac{n^2}{r^{n+1}} &= \sum_{n=1}^{\infty} \left(\frac{n^2}{r^n} - \frac{n^2}{r^{n+1}} \right) \\
&= \frac{1}{r} + \sum_{n=1}^{\infty} \frac{(n+1)^2 - n^2}{r^{n+1}} \\
&= \frac{1}{r} + \sum_{n=1}^{\infty} \frac{2n+1}{r^{n+1}} \\
&= \frac{1}{r} + \frac{r+1}{(r-1)^2}
\end{aligned}$$

where the last step comes from applying Example 3.2 with $a = k = 1$ and $d = 2$. Then

$$\sum_{n=1}^{\infty} \frac{n^2}{r^{n+1}} = \frac{1}{r-1} \left(\frac{1}{r} + \frac{r+1}{(r-1)^2} \right) = \frac{2r^2 - r + 1}{r(r-1)^3}.$$

Remark 3.4. *Example 3.2 and Remark 3.3 can also be used to find a closed expression for*

$$\sum_{n=1}^{\infty} \frac{n^3}{r^{n+1}}.$$

A similar process to Remark 3.2 yields

$$\begin{aligned}
(r-1)\sum_{n=1}^{\infty} \frac{n^3}{r^{n+1}} &= \sum_{n=1}^{\infty}\left(\frac{n^3}{r^n} - \frac{n^3}{r^{n+1}}\right) \\
&= \frac{1}{r} + \sum_{n=1}^{\infty} \frac{(n+1)^3 - n^3}{r^{n+1}} \\
&= \frac{1}{r} + \sum_{n=1}^{\infty} \frac{3n^2+3n+1}{r^{n+1}} \\
&= \frac{1}{r} + 3\sum_{n=1}^{\infty} \frac{n^2}{r^{n+1}} + \sum_{n=1}^{\infty} \frac{3n+1}{r^{n+1}} \\
&= \frac{1}{r} + 3\cdot\frac{2r^2-r+1}{r(r-1)^3} + \frac{r+2}{(r-1)^3},
\end{aligned}$$

where the last step comes from applying Remark 3.2 and Example 3.2 with $a = k = 1$ and $d = 3$. Then

$$\sum_{n=1}^{\infty} \frac{n^3}{r^{n+1}} = \frac{2r^3+4r^2-2r+4}{r(r-1)^4}.$$

Remarks 3.3 and 3.4 culminate into the following theorem that generalizes inductively to integral powers.

Theorem 3.3. *Let k be a positive integer and $r > 1$. Then*

$$(r-1)^2\sum_{n=1}^{\infty} \frac{n^k}{r^n} = (r-1)\left(k\sum_{n=1}^{\infty} \frac{n^{k-1}}{r^n} + \binom{k}{2}\sum_{n=1}^{\infty} \frac{n^{k-2}}{r^n} + \cdots + k\sum_{n=1}^{\infty} \frac{n}{r^n}\right) + r.$$

In particular, if $r = 2$, then

$$\sum_{n=1}^{\infty} \frac{n^k}{2^n} = k\sum_{n=1}^{\infty} \frac{n^{k-1}}{2^n} + \binom{k}{2}\sum_{n=1}^{\infty} \frac{n^{k-2}}{2^n} + \cdots + k\sum_{n=1}^{\infty} \frac{n}{2^n} + 2$$

Proof. Let S be the series,

$$S = \sum_{n=1}^{\infty} \frac{n^k}{r^n} = \frac{1^k}{r} + \frac{2^k}{r^2} + \cdots .$$

Treating S as a power series $\sum n^k z^n$ where $z = \frac{1}{r}$, one can easily use the ratio test to find that the radius of convergence is 1. Thus, S converges for $|z| < 1$, or $r > 1$.

Then algebraic manipulation begins by noting that

$$rS - S = 1^k + \frac{2^k - 1^k}{r} + \frac{3^k - 2^k}{r^2} + \cdots$$

and so,

$$
\begin{aligned}
(r-1)S &= \sum_{n=1}^{\infty} \frac{(n+1)^k - n^k}{r^n} + 1 \\
&= \sum_{n=1}^{\infty} \frac{\binom{k}{1} n^{k-1} + \binom{k}{2} n^{k-2} + \cdots + \binom{k}{k-1} n + 1}{r^n} + 1 \\
&= k \sum_{n=1}^{\infty} \frac{n^{k-1}}{r^n} + \binom{k}{2} \sum_{n=1}^{\infty} \frac{n^{k-2}}{r^n} + \cdots + k \sum_{n=1}^{\infty} \frac{n}{r^n} + \sum_{n=1}^{\infty} \frac{1}{r^n} + 1 \\
&= k \sum_{n=1}^{\infty} \frac{n^{k-1}}{r^n} + \binom{k}{2} \sum_{n=1}^{\infty} \frac{n^{k-2}}{r^n} + \cdots + k \sum_{n=1}^{\infty} \frac{n}{r^n} + \frac{1}{r-1} + 1.
\end{aligned}
$$

Hence,

$$
(r-1)^2 S = (r-1)^2 \sum_{n=1}^{\infty} \frac{n^k}{r^n} = (r-1) \left(k \sum_{n=1}^{\infty} \frac{n^{k-1}}{r^n} + \binom{k}{2} \sum_{n=1}^{\infty} \frac{n^{k-2}}{r^n} + \cdots + k \sum_{n=1}^{\infty} \frac{n}{r^n} \right) + r.
$$

□

Example 3.3. *Making the following substitutions into Theorem 3.3, we calculate*

$$
\begin{aligned}
\mathbf{r = 2, k = 2}: &\quad \sum_{n=1}^{\infty} \frac{n^2}{2^n} = 2 \sum_{n=1}^{\infty} \frac{n}{r^n} + 2 = 6 \\
\mathbf{r = 2, k = 3}: &\quad \sum_{n=1}^{\infty} \frac{n^3}{2^n} = 3 \sum_{n=1}^{\infty} \frac{n^2}{2^n} + 3 \sum_{n=1}^{\infty} \frac{n}{r^n} + 2 = 26 \\
\mathbf{r = 2, k = 4}: &\quad \sum_{n=1}^{\infty} \frac{n^4}{2^n} = 4 \sum_{n=1}^{\infty} \frac{n^3}{2^n} + 6 \sum_{n=1}^{\infty} \frac{n^2}{2^n} + 4 \sum_{n=1}^{\infty} \frac{n}{2^n} + 2 = 150.
\end{aligned}
$$

Remark 3.5. *Let $\{a_n\}$ be a second order linear recurrence sequence satisfying* (1)*. If $a_1 = \alpha a_0 \neq 0$, where α is a root of the characteristic equation $x^2 - px - q = 0$, then substituting into the formula of Theorem 2.1, it can be deduced that (regardless of whether the characteristic equation has repeated roots) $a_n = a_0 \alpha^n$; therefore, $\{a_n\}$ forms a geometric sequence. If k is a positive integer, $\left\{\frac{a_{nk}}{r^{n+1}}\right\}$ also forms a geometric sequence with the common ratio $\frac{\alpha^k}{r}$. Hence, $\sum_{n=0}^{\infty} \frac{a_{nk}}{r^{n+1}}$ converges if and only if $\frac{|\alpha|^k}{r} < 1$.*

On the other hand, if it is known that $\{a_n\}$ is a geometric sequence satisfying (1) *with the common ratio d, substituting $n = 2$ in* (1) *yields*

$$
a_0 d^2 = p(a_0 d) + q a_0.
$$

Dividing both sides by a_0, we obtain $d^2 = pd + q$. Hence, d is the root of $x^2 - px - q = 0$.

When $k = 1$, the formula $\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}$ can be further simplified:

Corollary 3.1. *Let $\{a_n\}$ satisfy the second order linear recurrence sequence defined in* (1)*. Suppose α and β are two roots to the characteristic equation $x^2 - px - q = 0$ such that $|\alpha|, |\beta| < r$. Then*

$$
\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \frac{a_0(r-p) + a_1}{r^2 - pr - q}.
$$

Proof. In Theorem 3.1 and 3.2, regardless of whether $x^2 - px - q = 0$ has repeated roots, i.e., $\alpha = \beta$ or $\alpha \neq \beta$, we have

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \frac{a_0 r - a_0(\alpha + \beta) + a_1}{(r-\alpha)(r-\beta)}$$
$$= \frac{a_0 r - a_0(\alpha + \beta) + a_1}{r^2 - (\alpha+\beta)r + \alpha\beta}.$$

Furthermore, from the relationship between the roots and coefficients of the quadratic equation

$$\alpha + \beta = p,\ \alpha\beta = -q,$$

we have

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \frac{a_0(r-p) + a_1}{r^2 - pr - q}. \qquad \square$$

Remark 3.6. *Alternatively, we may prove Corollary 3.1 in the following way. This way, we do not need to use the roots of the characteristic equation to prove the formula $\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}$. However, we will still need the roots to prove the convergence of $\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}$.*

Proof. Let $a_{n+2} = pa_{n+1} + qa_n$. Then

$$\sum_{n=0}^{\infty} \frac{a_{n+2}}{r^{n+1}} = p\sum_{n=0}^{\infty} \frac{a_{n+1}}{r^{n+1}} + q\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}.$$

Since

$$\sum_{n=0}^{\infty} \frac{a_{n+2}}{r^{n+1}} = r^2\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - (ra_0 + a_1),$$
$$\sum_{n=0}^{\infty} \frac{a_{n+1}}{r^{n+1}} = r\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - a_0,$$

we have

$$r^2\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - (ra_0 + a_1) = pr\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - pa_0 + q\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}.$$

Then

$$(r^2 - pr - q)\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = (r-p)a_0 + a_1.$$

Hence,

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \frac{a_0(r-p) + a_1}{r^2 - pr - q}. \qquad \square$$

The next corollary focuses on the special case $p = q = 1$ and $r = 10$.

Corollary 3.2. *Let $\{a_n\}$ satisfy the second order linear recurrence sequence $a_n = a_{n-1} + a_{n-2}$, $n \geq 2$, with initial conditions a_0 and a_1. Then*

$$\sum_{n=0}^{\infty} \frac{a_n}{10^{n+1}} = \frac{9a_0 + a_1}{89}.$$

Proof. Applying Theorem 3.1, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a_n}{10^{n+1}} &= \frac{10a_0 - a_0 \cdot \frac{\phi^2-\psi^2}{\phi-\psi} + a_1 \cdot \frac{\phi-\psi}{\phi-\psi}}{(10-\phi)(10-\psi)} \\ &= \frac{10a_0 - a_0(\phi+\psi) + a_1}{100 - 10(\phi+\psi) + \phi\psi} \\ &= \frac{9a_0 + a_1}{100 - 10 - 1} \\ &= \frac{9a_0 + a_1}{89}.\end{aligned}$$

□

In particular, if $a_0 = 0$ and $a_1 = 1$, we have $\sum_{n=0}^{\infty} \frac{a_n}{10^{n+1}} = \sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} = \frac{1}{89}$ ([11]).

4 Lucas sequence of the first kind

Definition 4.1. [13, 17] *A second-order linear recursive sequence $\{u_n\}$ is called a Lucas sequence of the first kind if it satisfies*

$$u_n = pu_{n-1} + qu_{n-2}, \tag{6}$$

for some constants $p, q \neq 0$ and initial conditions $u_0 = 0$, $u_1 = 1$.

Theorem 4.1. *Let $\{u_n\}$ be the Lucas sequence of the first kind. Let α and β be two roots of the characteristic equation $x^2 - px - q = 0$ and satisfy*

$$k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|, |\beta|\})} \right\rceil,$$

where $r > 1$. Then

$$\sum_{n=1}^{\infty} \frac{u_{nk}}{r^{n+1}} = \frac{u_k}{r^2 - (\alpha^k + \beta^k)r + (-q)^k}.$$

Proof. Note that

$$k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|, |\beta|\})} \right\rceil \text{ if and only if } \max\left\{\left|\frac{\alpha^k}{r}\right|, \left|\frac{\beta^k}{r}\right|\right\} < 1,$$

which is a necessary criterion for the two theorems used in this proof.

When $\alpha \neq \beta$, we apply Theorem 3.1. Then for $u_0 = 0$ and $u_1 = 1$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{u_{nk}}{r^{n+1}} &= \frac{\frac{\alpha^k-\beta^k}{\alpha-\beta}}{(r-\alpha^k)(r-\beta^k)} \\ &= \frac{u_k}{(r-\alpha^k)(r-\beta^k)} = \frac{u_k}{r^2-(\alpha^k+\beta^k)r+(\alpha\beta)^k} \\ &= \frac{u_k}{r^2-(\alpha^k+\beta^k)r+(-q)^k}.\end{aligned}$$

On the other hand, if $\alpha = \beta$, we have by Theorem 3.2,

$$\sum_{n=1}^{\infty} \frac{u_{nk}}{r^{n+1}} = \frac{k\alpha^{k-1}}{(r-\alpha^k)^2} = \frac{u_k}{r^2-2r\alpha^k+\alpha^{2k}}.$$

□

Theorem 4.2. *Let $\{u_n\}$ be the Lucas sequence of the first kind. Then*

$$\sum_{n=0}^{\infty} \frac{u_{kn-1}}{r^{n+1}} = \frac{1}{q(\alpha-\beta)} \left[\frac{\alpha(r-\alpha^k)-\beta(r-\beta^k)}{(r-\alpha^k)(r-\beta^k)}\right], k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|,|\beta|\})} \right\rceil,$$

where α, β are roots to the characteristic equation $x^2 - px - q = 0$.

Proof. Let $w_n = u_{n-1}$, then $w_{kn} = u_{kn-1}$ with $w_0 = \frac{1}{q}$ and $w_1 = 0$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{u_{kn-1}}{r^n} = r\sum_{n=0}^{\infty} \frac{w_{kn}}{r^{n+1}} &= r\left[\frac{w_0 r - w_0\cdot\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} + w_1\cdot\frac{\alpha^k-\beta^k}{\alpha-\beta}}{(r-\alpha^k)(r-\beta^k)}\right] \\ &= r\left[\frac{\frac{r}{q} - \frac{1}{q}\cdot\frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta}}{(r-\alpha^k)(r-\beta^k)}\right] \\ &= \frac{r}{q(\alpha-\beta)}\left[\frac{r(\alpha-\beta)-(\alpha^{k+1}-\beta^{k+1})}{(r-\alpha^k)(r-\beta^k)}\right] \\ &= \frac{r}{q(\alpha-\beta)}\left[\frac{\alpha(r-\alpha^k)-\beta(r-\beta^k)}{(r-\alpha^k)(r-\beta^k)}\right].\end{aligned}$$

□

Theorem 4.3. *Let $\{u_n\}$ be the Lucas sequence of the first kind. Then*

$$\sum_{n=0}^{\infty} \frac{u_{kn+1}}{r^n} = \frac{1}{\alpha-\beta}\left(\frac{\alpha(r-\beta^k)-\beta(r-\alpha^k)}{(r-\alpha^k)(r-\beta^k)}\right), k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|,|\beta|\})} \right\rceil,$$

where α, β are roots to the characteristic equation $x^2 - px - q = 0$.

Proof. Let $w_n = u_{n+1}$ for $n \geq 0$, then $w_{kn} = u_{kn+1}$ for $n, k \geq 0$ with $w_0 = u_1 = 1$ and $w_1 = u_2 = pu_1 + qu_0 = p = \alpha + \beta$. Then using Theorem 3.1, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{u_{kn+1}}{r^{n+1}} = \sum_{n=0}^{\infty} \frac{w_{kn}}{r^{n+1}} &= \frac{r - \frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} + (\alpha+\beta)\frac{\alpha^k-\beta^k}{\alpha-\beta}}{(r-\alpha^k)(r-\beta^k)} \\ &= \frac{r(\alpha-\beta)-(\alpha^{k+1}-\beta^{k+1})+(\alpha+\beta)(\alpha^k-\beta^k)}{(\alpha-\beta)(r-\alpha^k)(r-\beta^k)} \\ &= \frac{1}{\alpha-\beta}\left(\frac{\alpha(r-\beta^k)-\beta(r-\alpha^k)}{(r-\alpha^k)(r-\beta^k)}\right).\end{aligned}$$

□

A generalization of Theorem 4.3 can be found in [12].

Example 4.1. [11] *Let* $\{F_n\}$ *be the Fibonacci sequence, i.e.,*

$$F_n = F_{n-1} + F_{n-2},$$

with $F_0 = 0$ *and* $F_1 = 1$. *The characteristic equation is* $x^2 - x - 1 = 0$. *The roots are* $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$. *By Theorem 4.1, we have*

$$\sum_{n=0}^{\infty} \frac{F_{nk}}{r^{n+1}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}\left(r - \left(\frac{1+\sqrt{5}}{2}\right)^k\right)\left(r - \left(\frac{1-\sqrt{5}}{2}\right)^k\right)}, \ k < \left\lceil \frac{\log r}{\log\left(\frac{1+\sqrt{5}}{2}\right)} \right\rceil. \tag{7}$$

Let $r = 10$, *then only* $k = 1, 2, 3, 4$ *satisfy the condition for convergence. We have*

$$\sum_{n=0}^{\infty} \frac{F_n}{10^{n+1}} = \frac{\sqrt{5}}{\sqrt{5}\left(10 - \frac{1+\sqrt{5}}{2}\right)\left(10 - \frac{1-\sqrt{5}}{2}\right)} = \frac{1}{89}, \tag{8}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n}}{10^{n+1}} = \frac{\sqrt{5}}{\sqrt{5}\left(10 - \left(\frac{1+\sqrt{5}}{2}\right)^2\right)\left(10 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right)} = \frac{1}{71}, \tag{9}$$

$$\sum_{n=0}^{\infty} \frac{F_{3n}}{10^{n+1}} = \frac{2\sqrt{5}}{\sqrt{5}\left(10 - \left(\frac{1+\sqrt{5}}{2}\right)^3\right)\left(10 - \left(\frac{1-\sqrt{5}}{2}\right)^3\right)} = \frac{2}{59}, \tag{10}$$

$$\sum_{n=0}^{\infty} \frac{F_{4n}}{10^{n+1}} = \frac{3\sqrt{5}}{\sqrt{5}\left(10 - \left(\frac{1+\sqrt{5}}{2}\right)^4\right)\left(10 - \left(\frac{1-\sqrt{5}}{2}\right)^4\right)} = \frac{3}{31}. \tag{11}$$

Corollary 4.1. *Let* $\{F_n\}$ *be the Fibonacci sequence. Then by Theorem 4.3,*

$$\sum_{n=0}^{\infty} \frac{F_{kn+1}}{r^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)}\right), \tag{12}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, *and* $k < \left\lceil \frac{\log r}{\log \phi} \right\rceil$. *Furthermore, if* $r = 10$ *in* (12), *then only* $k = 1, 2, 3, 4$ *satisfy the condition for convergence. Thus,*

$$\sum_{n=0}^{\infty} \frac{F_{n+1}}{10^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi) - \psi(r-\phi)}{(r-\phi)(r-\psi)}\right) = \frac{10}{89}, \tag{13}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n+1}}{10^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi^2) - \psi(r-\phi^2)}{(r-\phi^2)(r-\psi^2)}\right) = \frac{9}{71}, \tag{14}$$

$$\sum_{n=0}^{\infty} \frac{F_{3n+1}}{10^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi^3) - \psi(r-\phi^3)}{(r-\phi^3)(r-\psi^3)}\right) = \frac{9}{59}, \tag{15}$$

$$\sum_{n=0}^{\infty} \frac{F_{4n+1}}{10^{n+1}} = \frac{1}{\sqrt{5}}\left(\frac{\phi(r-\psi^4) - \psi(r-\phi^4)}{(r-\phi^4)(r-\psi^4)}\right) = \frac{8}{31}. \tag{16}$$

Example 4.2. [18] *Let* $a_0 = 0$, $a_1 = 1$, *and* $a_n = 4a_{n-1} - a_{n-2}$, $n \geq 2$, *i.e.,* $\{a_n\} = \{0, 1, 4, 15, 56, \ldots\}$. *The two roots of the characteristic equation* $x^2 - 4x + 1 = 0$ *are* $\alpha = 2 + \sqrt{3}$ *and* $\beta = 2 - \sqrt{3}$, *and satisfy* $\alpha + \beta = 4$ *and* $\alpha\beta = 1$. *Hence, from Theorem 4.1, we have*

$$\sum_{n=1}^{\infty} \frac{a_{nk}}{r^{n+1}} = \frac{a_k}{r^2 - (\alpha^k + \beta^k)r + 1}, \tag{17}$$

with the condition for convergence $k < \frac{\log r}{\log(2+\sqrt{3})}$, $r > 1 \in \mathbb{R}$. *With some calculations, we obtain*

$$\begin{aligned}
\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta = 14, \\
\alpha^3 + \beta^3 &= (\alpha^2 + \beta^2)(\alpha + \beta) - \alpha\beta(\alpha + \beta) = 52, \\
\alpha^4 + \beta^4 &= (\alpha^3 + \beta^3)(\alpha + \beta) - \alpha\beta(\alpha^2 + \beta^2) = 194.
\end{aligned}$$

In fact, using Girard–Waring formulas ([3], [9]), we can directly compute $\alpha^k + \beta^k$ *for larger values of* k:

$$\begin{aligned}
\alpha^k + \beta^k &= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-m-1)!k}{(k-2m)!m!}(-1)^m(\alpha + \beta)^{k-2m}(\alpha\beta)^m \\
&= \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k}{k-m}\binom{k-m}{m}(-1)^m 4^{k-2m}.
\end{aligned}$$

For $k = 1, 2, 3, 4$, *we can express* (17) *more clearly:*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{u_n}{r^{n+1}} &= \frac{1}{r^2 - 4r + 1}, \quad \textit{if } k = 1, \\
\sum_{n=1}^{\infty} \frac{u_{2n}}{r^{n+1}} &= \frac{1}{r^2 - 14r + 1}, \quad \textit{if } k = 2,
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{u_{3n}}{r^{n+1}} &= \frac{1}{r^2 - 52r + 1}, \quad \textit{if } k = 3, \\
\sum_{n=1}^{\infty} \frac{u_{4n}}{r^{n+1}} &= \frac{1}{r^2 - 194r + 1}, \quad \textit{if } k = 4.
\end{aligned}$$

Example 4.3. *Let* $\{B_n\}$ *be the sequence of balancing numbers satisfying*

$$B_n = 6B_{n-1} - B_{n-2}, \; n \geq 2,$$

with $B_0 = 0$ *and* $B_1 = 1$. *The characteristic equation for* (18) *is* $x^2 - 6x + 1 = 0$. *The roots are* $\alpha = 3 + 2\sqrt{2}$ *and* $\beta = 3 - 2\sqrt{2}$.

By Theorem 4.1, we have

$$\sum_{n=1}^{\infty} \frac{B_{nk}}{r^{n+1}} = \frac{B_k}{r^2 - r\left((3 + 2\sqrt{2})^k + (3 - 2\sqrt{2})^k\right) + 1}, \; k < \left\lceil \frac{\log r}{\log(3 + 2\sqrt{2})} \right\rceil. \tag{18}$$

Let $r = 10$, *then only* $k = 1$ *satisfies the condition for convergence. Thus,*

$$\sum_{n=1}^{\infty} \frac{B_n}{10^{n+1}} = \frac{1}{100 - 10(6) + 1} = \frac{1}{41}.$$

Example 4.4. *Let $\{P_n\}$ be the sequence of Pell numbers satisfying*

$$P_n = 2P_{n-1} + P_{n-2},\ n \geq 2,$$

with $P_0 = 0$ and $P_1 = 1$. The characteristic equation for (19) *is $x^2 - 2x - 1 = 0$. The roots are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. By Theorem 4.1, we have*

$$\sum_{n=1}^{\infty} \frac{P_{nk}}{r^{n+1}} = \frac{P_k}{r^2 - r\left((1+\sqrt{2})^k + (1-\sqrt{2})^k\right) + (-1)^k},\ k < \left[\frac{\log r}{\log(1+\sqrt{2})}\right]. \tag{19}$$

Let $r = 10$, then only $k = 1, 2$ satisfy the condition for convergence. Thus,

$$\sum_{n=1}^{\infty} \frac{P_n}{10^{n+1}} = \frac{1}{100 - 10(2) - 1} = \frac{1}{79}, \tag{20}$$

$$\sum_{n=1}^{\infty} \frac{P_{2n}}{10^{n+1}} = \frac{2}{100 - 10(6) + 1} = \frac{2}{41}. \tag{21}$$

By Theorem 4.2, we have

$$\sum_{n=0}^{\infty} \frac{P_{kn-1}}{r^{n+1}} = \frac{1}{2\sqrt{2}}\left[\frac{(1+\sqrt{2})(r-(1+\sqrt{2})^k) - (1-\sqrt{2})(r-(1-\sqrt{2})^k)}{(r-(1+\sqrt{2})^k)(r-(1-\sqrt{2})^k)}\right]. \tag{22}$$

Let $r = 10$, then for $k = 2$, we have

$$\sum_{n=0}^{\infty} \frac{P_{2n-1}}{10^{n+1}} = \frac{1}{2\sqrt{2}}\left[\frac{10\sqrt{2}}{41}\right] = \frac{5}{41}. \tag{23}$$

By Theorem 4.3, we have

$$\sum_{n=0}^{\infty} \frac{P_{kn+1}}{r^{n+1}} = \frac{1}{2\sqrt{2}}\left[\frac{(1+\sqrt{2})(r-(1-\sqrt{2})^k) - (1-\sqrt{2})(r-(1+\sqrt{2})^k)}{(r-(1+\sqrt{2})^k)(r-(1-\sqrt{2})^k)}\right]. \tag{24}$$

Let $r = 10$, then for $k = 2$, we have

$$\sum_{n=0}^{\infty} \frac{P_{2n+1}}{10^{n+1}} = \frac{1}{2\sqrt{2}}\left[\frac{18\sqrt{2}}{41}\right] = \frac{9}{41}. \tag{25}$$

Remark 4.1. *It is known that $B_n = \frac{P_{2n}}{2}$ ([14]). We have*

$$\sum_{n=1}^{\infty} \frac{\frac{P_{2n}}{2}}{10^{n+1}} = \sum_{n=1}^{\infty} \frac{B_n}{10^{n+1}} = \frac{1}{41}.$$

Example 4.5. *Let $\{M_n\}$ be the sequence of Mersenne numbers satisfying*

$$M_n = 3M_{n-1} - 2M_{n-2},\ n \geq 2,$$

with $M_0 = 0$ and $M_1 = 1$. The characteristic equation is $x^2 - 3x + 2 = 0$. The roots are $\alpha = 2$ and $\beta = 1$.

By Theorem 4.1, we have

$$\sum_{n=1}^{\infty} \frac{M_{nk}}{r^{n+1}} = \frac{M_k}{r^2 - r\left(2^k+1\right) + 2^k},\ k < \left\lceil \frac{\log r}{\log 2} \right\rceil. \tag{26}$$

Let $r = 10$, then only $k = 1, 2, 3$ satisfy the condition for convergence. Thus,

$$\sum_{n=1}^{\infty} \frac{M_n}{10^{n+1}} = \frac{1}{100 - 10(3) + 2} = \frac{1}{72}, \tag{27}$$

$$\sum_{n=1}^{\infty} \frac{M_{2n}}{10^{n+1}} = \frac{3}{100 - 10(5) + 4} = \frac{3}{54} = \frac{1}{18}, \tag{28}$$

$$\sum_{n=1}^{\infty} \frac{M_{3n}}{10^{n+1}} = \frac{7}{100 - 10(9) + 8} = \frac{7}{18}. \tag{29}$$

Example 4.6. *Let $\{J_n\}$ be the sequence of Jacobsthal numbers satisfying*

$$J_n = J_{n-1} + 2J_{n-2},\ n \geq 2,$$

with $J_0 = 0$ and $J_1 = 1$. The characteristic equation is $x^2 - x - 2 = 0$. The roots are $\alpha = 2$ and $\beta = -1$.

By Theorem 4.1,

$$\sum_{n=1}^{\infty} \frac{J_{nk}}{r^{n+1}} = \frac{J_k}{r^2 - r\left(2^k + (-1)^k\right) + (-2)^k},\ k < \left\lceil \frac{\log r}{\log 2} \right\rceil. \tag{30}$$

Let $r = 10$, then only $k = 1, 2, 3$ satisfy the condition for convergence. Thus,

$$\sum_{n=1}^{\infty} \frac{J_n}{10^{n+1}} = \frac{1}{100 - 10 - 2} = \frac{1}{88}, \tag{31}$$

$$\sum_{n=1}^{\infty} \frac{J_{2n}}{10^{n+1}} = \frac{1}{100 - 10(5) + 4} = \frac{1}{54}, \tag{32}$$

$$\sum_{n=1}^{\infty} \frac{J_{3n}}{10^{n+1}} = \frac{3}{100 - 10(7) - 8} = \frac{3}{22}. \tag{33}$$

5 Lucas sequence of the second kind

Definition 5.1. [13] *A second-order linear recursive sequence $\{v_n\}$ is called a Lucas sequence of the second kind if it satisfies*

$$v_n = pv_{n-1} + qv_{n-2}, \tag{34}$$

for some constants $p, q \neq 0$ and initial conditions $v_0 = 2$, $v_1 = p$.

Theorem 5.1. *Let $\{v_n\}$ be the Lucas sequence of the second kind. Let α and β be two roots of the characteristic equation $x^2 - px - q = 0$ and satisfy*

$$k < \left\lceil \frac{\log r}{\log(\max\{|\alpha|, |\beta|\})} \right\rceil.$$

Then

$$\sum_{n=0}^{\infty} \frac{v_{nk}}{r^{n+1}} = \frac{2r - (\alpha^k + \beta^k)}{r^2 - (\alpha^k + \beta^k)r + (-q)^k}. \tag{35}$$

Proof. By Theorem 3.1, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{v_{nk}}{r^{n+1}} &= \frac{v_0 r - v_0 \cdot \frac{\alpha^{k+1}-\beta^{k+1}}{\alpha-\beta} + v_1 \cdot \frac{\alpha^k-\beta^k}{\alpha-\beta}}{(r-\alpha^k)(r-\beta^k)} \\
&= \frac{1}{(\alpha-\beta)(r-\alpha^k)(r-\beta^k)} \left(2r(\alpha-\beta) - 2(\alpha^{k+1}-\beta^{k+1}) + p(\alpha^k-\beta^k)\right) \\
&= \frac{1}{(\alpha-\beta)(r-\alpha^k)(r-\beta^k)} \left(2r(\alpha-\beta) - (\alpha-\beta)(\alpha^k+\beta^k)\right) \\
&= \frac{2r-(\alpha^k+\beta^k)}{(r-\alpha^k)(r-\beta^k)} = \frac{2r-(\alpha^k+\beta^k)}{r^2-(\alpha^k+\beta^k)r+(-q)^k}.
\end{aligned}$$

□

Example 5.1. *Let $\{L_n\}$ be the sequence of Lucas numbers satisfying*

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2,$$

with $L_0 = 2$ and $L_1 = 1$. The characteristic equation is $x^2 - x - 1 = 0$. The roots are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

By Theorem 5.1, we have

$$\sum_{n=0}^{\infty} \frac{L_{nk}}{r^{n+1}} = \frac{2r - \left[\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k\right]}{r^2 - \left[\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k\right] r + (-1)^k},\ k < \left\lceil \frac{\log r}{\log(\frac{1+\sqrt{5}}{2})} \right\rceil. \tag{36}$$

Let $r = 10$, then only $k = 1, 2, 3, 4$ satisfy the condition for convergence. Thus,

$$\sum_{n=0}^{\infty} \frac{L_n}{10^{n+1}} = \frac{20-1}{100-10-1} = \frac{19}{89}, \tag{37}$$

$$\sum_{n=0}^{\infty} \frac{L_{2n}}{10^{n+1}} = \frac{20-3}{100-30+1} = \frac{17}{71}, \tag{38}$$

$$\sum_{n=0}^{\infty} \frac{L_{3n}}{10^{n+1}} = \frac{20-4}{100-40-1} = \frac{16}{59}, \tag{39}$$

$$\sum_{n=0}^{\infty} \frac{L_{4n}}{10^{n+1}} = \frac{20-7}{100-70+1} = \frac{13}{31}. \tag{40}$$

Example 5.2. *Let $\{S_n\}$ be the Fermat sequence of numbers of the second kind satisfying*

$$S_n = 3S_{n-1} - 2S_{n-2},\ n \geq 2,$$

with $S_0 = 2$ and $S_1 = 3$. The characteristic equation is $x^2 - 3x + 2 = 0$. The roots are $\alpha = 2$ and $\beta = 1$.

By Theorem 5.1, we have

$$\sum_{n=0}^{\infty} \frac{S_{nk}}{r^{n+1}} = \frac{2r - \left(2^k + 1\right)}{r^2 - (2^k + 1)\, r + 2^k},\ k < \left\lceil \frac{\log r}{\log 2} \right\rceil. \tag{41}$$

Let $r = 10$, then only $k = 1, 2, 3$ satisfy the condition for convergence. Thus,

$$\sum_{n=0}^{\infty} \frac{S_n}{10^{n+1}} = \frac{20 - 3}{100 - 30 + 2} = \frac{17}{72}, \tag{42}$$

$$\sum_{n=0}^{\infty} \frac{S_{2n}}{10^{n+1}} = \frac{20 - 5}{100 - 50 + 4} = \frac{15}{54}, \tag{43}$$

$$\sum_{n=0}^{\infty} \frac{S_{3n}}{10^{n+1}} = \frac{20 - 9}{100 - 90 + 8} = \frac{11}{18}. \tag{44}$$

Example 5.3. *Let $\{Q_n\}$ be the Pell sequence of numbers of the second kind satisfying*

$$Q_n = 2Q_{n-1} + Q_{n-2},\ n \geq 2,$$

with $Q_0 = 2$ and $Q_1 = 2$. The characteristic equation is $x^2 - 2x - 1 = 0$. The roots are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

By Theorem 5.1, we have

$$\sum_{n=0}^{\infty} \frac{Q_{nk}}{r^{n+1}} = \frac{2r - \left((1+\sqrt{2})^k + (1-\sqrt{2})^k\right)}{r^2 - \left((1+\sqrt{2})^k + (1-\sqrt{2})^k\right) r + (-1)^k},\ k < \left\lceil \frac{\log r}{\log(1+\sqrt{2})} \right\rceil. \tag{45}$$

Let $r = 10$, then only $k = 1, 2$ satisfy the condition for convergence. Thus,

$$\sum_{n=0}^{\infty} \frac{Q_n}{10^{n+1}} = \frac{20 - 2}{100 - 20 - 1} = \frac{18}{79}, \tag{46}$$

$$\sum_{n=0}^{\infty} \frac{Q_{2n}}{10^{n+1}} = \frac{20 - 6}{100 - 60 + 1} = \frac{14}{41}. \tag{47}$$

Example 5.4. *Let $\{\mathcal{J}_n\}$ be the Jacobsthal–Lucas sequence of numbers satisfying*

$$\mathcal{J}_n = \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2},\ n \geq 2,$$

with $\mathcal{J}_0 = 2$ and $\mathcal{J}_1 = 1$. The characteristic equation is $x^2 - x - 2 = 0$. The roots are $\alpha = 2$ and $\beta = -1$.

By Theorem 5.1, we have

$$\sum_{n=0}^{\infty} \frac{\mathcal{J}_{nk}}{r^{n+1}} = \frac{2r - \left(2^k + (-1)^k\right)}{r^2 - (2^k + (-1)^k)\, r + (-2)^k},\ k < \left\lceil \frac{\log r}{\log 2} \right\rceil. \tag{48}$$

Let $r = 10$, then only $k = 1, 2, 3$ satisfy the condition for convergence. Thus,

$$\sum_{n=0}^{\infty} \frac{\mathcal{J}_n}{10^{n+1}} = \frac{20 - 1}{100 - 10 - 2} = \frac{19}{88}, \tag{49}$$

$$\sum_{n=0}^{\infty} \frac{\mathcal{J}_{2n}}{10^{n+1}} = \frac{20 - 5}{100 - 50 + 4} = \frac{15}{54}, \tag{50}$$

$$\sum_{n=0}^{\infty} \frac{\mathcal{J}_{3n}}{10^{n+1}} = \frac{20 - 7}{100 - 70 - 8} = \frac{13}{22}. \tag{51}$$

6 Leonardo sequence and generalized Leonardo sequence

Definition 6.1. [2] *A second-order linear recursive sequence $\{Le_n\}$ is called the Leonardo sequence if it satisfies*

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \tag{52}$$

with $Le_0 = 1$ and $Le_1 = 1$.

Next, we will us the generalized Leonardo sequence $\{\mathcal{L}_{m,n}\}$ as follows:

Definition 6.2. [8] *The generalized Leonardo sequence $\{\mathcal{L}_{m,n}\}$, with a fixed positive integer m, is defined by*

$$\mathcal{L}_{m,n} = \mathcal{L}_{m,n-1} + \mathcal{L}_{m,n-2} + m,\ n \geq 2, \tag{53}$$

with the initial conditions $\mathcal{L}_{m,0} = \mathcal{L}_{m,1} = 1$.

Theorem 6.1. [8] *The closed formula for the generalized Leonardo sequence $\{\mathcal{L}_{m,n}\}$ is*

$$\mathcal{L}_{m,n} = (1+m)F_{n+1} - m. \tag{54}$$

Corollary 6.1. [2] *Let $\{Le_n\}$ be the classical Leonardo sequence be defined by $Le_n = Le_{n-1} + Le_{n-2} + 1,\ n \geq 2$ with initial conditions $Le_0 = Le_1 = 1$. Then*

$$Le_n = 2F_{n+1} - 1.$$

Theorem 6.2. *Let $\{\mathcal{L}_{m,n}\}$ be the generalized Leonardo sequence defined in Definition 6.2. Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ be two roots of the characteristic equation $x^2 - x - 1 = 0$ and satisfy*

$$k < \left\lceil \frac{\log r}{\log(\frac{1+\sqrt{5}}{2})} \right\rceil.$$

Then

$$\sum_{n=0}^{\infty} \frac{\mathcal{L}_{m,kn}}{r^{n+1}} = \frac{(1+m)}{\sqrt{5}} \left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)} \right) - \frac{m}{r-1}. \tag{55}$$

Proof. Using (54) and Theorem 4.3 yields

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\mathcal{L}_{m,kn}}{r^{n+1}} &= \sum_{n=0}^{\infty} \left(\frac{(1+m)F_{kn+1}}{r^{n+1}} - \frac{m}{r^{n+1}} \right) \\
&= (1+m) \sum_{n=0}^{\infty} \frac{F_{kn+1}}{r^{n+1}} - \frac{m}{r} \sum_{n=0}^{\infty} \left(\frac{1}{r} \right)^n \\
&= \frac{(1+m)}{\sqrt{5}} \left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)} \right) - \frac{m}{r-1}.
\end{aligned}$$

□

Corollary 6.2. *Let $m = 1$ in Theorem 6.2, i.e., $\mathcal{L}_{1,kn} = Le_{kn}$. Then*

$$\sum_{n=0}^{\infty} \frac{Le_{kn}}{r^{n+1}} = \frac{2}{\sqrt{5}} \left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)} \right) - \frac{1}{r-1}, \tag{56}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, and $k < \left\lceil \frac{\log r}{\log \phi} \right\rceil$.

Proof. Let $m = 1$ in (55), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\mathcal{L}_{1,kn}}{r^{n+1}} &= \sum_{n=0}^{\infty} \frac{Le_{kn}}{r^{n+1}} \\ &= \frac{2}{\sqrt{5}} \left(\frac{\phi(r-\psi^k) - \psi(r-\phi^k)}{(r-\phi^k)(r-\psi^k)} \right) - \frac{1}{r-1},\end{aligned}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$, and $k < \left\lceil \frac{\log r}{\log \phi} \right\rceil$. $\square$

Example 6.1. *Let $r = 10$ in (56), then only $k = 1, 2, 3, 4$ satisfy the condition for convergence. Thus,*

$$\sum_{n=0}^{\infty} \frac{Le_{n}}{10^{n+1}} = \frac{2}{\sqrt{5}} \left(\frac{\phi(10-\psi) - \psi(10-\phi)}{(10-\phi)(10-\psi)} \right) - \frac{1}{10-1} = \frac{91}{801}, \tag{57}$$

$$\sum_{n=0}^{\infty} \frac{Le_{2n}}{10^{n+1}} = \frac{2}{\sqrt{5}} \left(\frac{\phi(10-\psi^2) - \psi(10-\phi^2)}{(10-\phi^2)(10-\psi^2)} \right) - \frac{1}{10-1} = \frac{91}{639}, \tag{58}$$

$$\sum_{n=0}^{\infty} \frac{Le_{3n}}{10^{n+1}} = \frac{2}{\sqrt{5}} \left(\frac{\phi(10-\psi^3) - \psi(10-\phi^3)}{(10-\phi^3)(10-\psi^3)} \right) - \frac{1}{10-1} = \frac{103}{531}, \tag{59}$$

$$\sum_{n=0}^{\infty} \frac{Le_{4n}}{10^{n+1}} = \frac{2}{\sqrt{5}} \left(\frac{\phi(10-\psi^4) - \psi(10-\phi^4)}{(10-\phi^4)(10-\psi^4)} \right) - \frac{1}{10-1} = \frac{113}{279}. \tag{60}$$

Next, we will consider a version of the Leonardo-like sequence $\{C_n(a,b,m)\}$ defined by

$$C_n(a,b,m) = C_{n-1}(a,b,m) + C_{n-2}(a,b,m) + m, \tag{61}$$

with $C_0(a,b,m) = b - a - m$, $C_1(a,b,m) = a$, and m is a constant ([1]). The generalized Leonardo sequence arises as a special case of C_n :

$$\mathcal{L}_{m,n} = C_n(1, 2+m, m).$$

Lemma 6.1. [1] *Consider the Leonardo-like sequence $\{C_n(a,b,m)\}$ as defined in (61). Then*

$$C_n(a,b,m) = aF_{n-2} + bF_{n-1} + m(F_n - 1).$$

The proof of this lemma can be found in [1].

Theorem 6.3. *Let $\{C_n(a,b,m)\}$ be the Leonardo-like sequence. Then*

$$\sum_{n=0}^{\infty} \frac{C_{kn}(a,b,m)}{r^{n+1}} = \frac{(r-\phi^k)((b-a)\phi - m - a) - (r-\psi^k)((b-a)\psi - m - a)}{\sqrt{5}(r-\phi^k)(r-\psi^k)} - \frac{m}{r-1}.$$

Proof. Using Lemma 6.1, we have

$$\sum_{n=0}^{\infty}\frac{C_{kn}(a,b,m)}{r^{n+1}}=a\sum_{n=0}^{\infty}\frac{F_{kn-2}}{r^{n+1}}+b\sum_{n=0}^{\infty}\frac{F_{kn-1}}{r^{n+1}}+m\sum_{n=0}^{\infty}\frac{F_{kn}}{r^{n+1}}-m\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}.$$

We will deal with the series involving the Fibonacci sequence separately. Denote $\phi=\frac{1+\sqrt{5}}{2}$ and $\psi=\frac{1-\sqrt{5}}{2}$. Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{F_{kn-2}}{r^{n+1}}&=\sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left(\frac{\phi^{kn-2}-\psi^{kn-2}}{r^{n+1}}\right)\\&=\frac{1}{r\phi^2\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{\phi^k}{r}\right)^n-\frac{1}{r\psi^2\sqrt{5}}\sum_{n=0}^{\infty}\left(\frac{\psi^k}{r}\right)^n\\&=\frac{1}{r\phi^2\sqrt{5}}\left(\frac{1}{1-\frac{\phi^k}{r}}\right)-\frac{1}{r\psi^2\sqrt{5}}\left(\frac{1}{1-\frac{\psi^k}{r}}\right)\\&=\frac{1}{\sqrt{5}}\left(\frac{\phi^{-2}(r-\psi^k)-\psi^{-2}(r-\phi^k)}{(r-\phi^k)(r-\psi^k)}\right).\end{aligned}$$

Next, by Theorem 4.2, we have

$$\sum_{n=0}^{\infty}\frac{F_{kn-1}}{r^{n+1}}=\frac{1}{\sqrt{5}}\left(\frac{\phi(r-\phi^k)-\psi(r-\psi^k)}{(r-\phi^k)(r-\psi^k)}\right).$$

Next, by Theorem 3.1, we have

$$\sum_{n=0}^{\infty}\frac{F_{kn}}{r^{n+1}}=\frac{1}{\sqrt{5}}\left(\frac{\phi^k-\psi^k}{(r-\phi^k)(r-\psi^k)}\right).$$

Putting the above together, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{C_{kn}(a,b,m)}{r^{n+1}}&=a\sum_{n=0}^{\infty}\frac{F_{kn-2}}{r^{n+1}}+b\sum_{n=0}^{\infty}\frac{F_{kn-1}}{r^{n+1}}+m\sum_{n=0}^{\infty}\frac{F_{kn}}{r^{n+1}}-m\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\\&=\frac{a}{\sqrt{5}}\left(\frac{\phi^{-2}(r-\psi^k)-\psi^{-2}(r-\phi^k)}{(r-\phi^k)(r-\psi^k)}\right)+\frac{b}{\sqrt{5}}\left(\frac{\phi(r-\phi^k)-\psi(r-\psi^k)}{(r-\phi^k)(r-\psi^k)}\right)\\&\quad+\frac{m}{\sqrt{5}}\left(\frac{\phi^k-\psi^k}{(r-\phi^k)(r-\psi^k)}\right)-\frac{m}{r-1}\\&=\frac{(r-\phi^k)(-a\psi^{-2}+b\phi-m)+(r-\psi_k)(a\phi^{-2}-b\psi+m)}{\sqrt{5}(r-\phi^k)(r-\psi^k)}-\frac{m}{r-1}\\&=\frac{(r-\phi^k)((b-a)\phi-m-a)-(r-\psi^k)((b-a)\psi-m-a)}{\sqrt{5}(r-\phi^k)(r-\psi^k)}-\frac{m}{r-1}.\ \square\end{aligned}$$

We can generalize further by letting $\{w_n(w_0,w_1,p,q,t,j)\}$ be a sequence of order 2 satisfying the non-homogeneous linear relation in the following form, [16]:

$$w_n=pw_{n-1}+qw_{n-2}+(p+q-1)(tn+j),\quad n\geq 2,\ t,j\in\mathbb{Z},S \tag{62}$$

where w_0,w_1,p,q are given constants such that $p+q\neq 1$.

Lemma 6.2. [16] *Let $\{w_n(w_0, w_1, p, q, t, j)\}$ be the sequence as defined in (62). Then*

$$w_n = w_n(w_0, w_1, p, q, 0, 0) + \left(j - \tfrac{t(p+2q)}{1-p-q}\right)(w_n(1, 1, p, q, 0, 0) - 1) + t(w_n(0, 1, p, q, 0, 0) - n).$$

Theorem 6.4. *Let $\{w_n(w_0, w_1, p, q, t, j)\}$ be the sequence as defined in (62). Then*

$$\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} = \frac{(r-p)w_0 + w_1 + (p+q-1)\left(\frac{j}{r-1} + \frac{t}{(r-1)^2}\right)}{r^2 - pr - q}, \tag{63}$$

with $r > 1$.

Proof. Let S be the desired series, i.e.,

$$S = \sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}}.$$

Consider the straightforward calculations

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{w_{n+1}}{r^{n+1}} &= r\sum_{n=1}^{\infty} \frac{w_n}{r^{n+1}} = r\left(\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} - \frac{w_0}{r}\right) = r\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} - w_0, \\
\sum_{n=0}^{\infty} \frac{w_{n+2}}{r^{n+1}} &= r^2\sum_{n=2}^{\infty} \frac{w_n}{r^{n+1}} = r^2\left(\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} - \frac{w_1}{r^2} - \frac{w_0}{r}\right) = r^2\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} - w_1 - rw_0.
\end{aligned} \tag{64}$$

From (62), it is clear that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{w_{n+2}}{r^{n+1}} &= \sum_{n=0}^{\infty} \frac{pw_{n+1} + qw_n + (p+q-1)(tn+j)}{r^{n+1}} \\
&= p\sum_{n=0}^{\infty} \frac{w_{n+1}}{r^{n+1}} + q\sum_{n=0}^{\infty} \frac{w_n}{r^{n+1}} + t(p+q-1)\sum_{n=0}^{\infty} \frac{n}{r^{n+1}} + j(p+q-1)\sum_{n=0}^{\infty} \frac{1}{r^{n+1}}.
\end{aligned}$$

Then by (64), we get

$$\begin{aligned}
qS &= \sum_{n=0}^{\infty} \frac{w_{n+2}}{r^{n+1}} - p\sum_{n=0}^{\infty} \frac{w_{n+1}}{r^{n+1}} - \frac{t(p+q-1)}{r}\sum_{n=0}^{\infty} n\left(\frac{1}{r}\right)^n - \frac{j(p+q-1)}{r}\sum_{n=0}^{\infty} \left(\frac{1}{r}\right)^n \\
&= (Sr^2 - w_0 r - w_1) - p(Sr - w_0) - \frac{t(p+q-1)}{r} \cdot \frac{\frac{1}{r}}{\left(1 - \frac{1}{r}\right)^2} - \frac{j(p+q-1)}{r} \cdot \frac{1}{1 - \frac{1}{r}} \\
&= S(r^2 - pr) - w_0 r - pw_0 - w_1 - \frac{t(p+q-1)}{(r-1)^2} - \frac{j(p+q-1)}{r-1}.
\end{aligned}$$

Solving this equation for S yields the desired equality. $\square$

7 Linear recurrence sequences of order three

We now analyze the natural corresponding results for sequences of order three. First and foremost, we derive a closed form expression for third order linear recurrence sequences. A matrix representation for the closed form can be found in [5].

Theorem 7.1. *Let $\{a_n\}$ be a sequence satisfying the third order recurrence relation*

$$a_n = pa_{n-1} + qa_{n-2} + ta_{n-3}, \quad n \geq 3, \tag{65}$$

for some constants $p, q, t \neq 0$ and initial conditions $a_0, a_1,$ and a_2. Let α, β, γ be roots of the characteristic equation $x^3 - px^2 - qx - t = 0$. Then we have the following third order analogue of Theorem 2.1:

$$a_n = \begin{cases} \frac{a_1(\beta+\gamma)-a_0\beta\gamma-a_2}{(\alpha-\beta)(\gamma-\alpha)}\alpha^n + \frac{a_1(\alpha+\gamma)-a_0\alpha\gamma-a_2}{(\alpha-\beta)(\beta-\gamma)}\beta^n + \frac{a_1(\alpha+\beta)-a_0\alpha\beta-a_2}{(\beta-\gamma)(\gamma-\alpha)}\gamma^n, & \text{if } \alpha \neq \beta \neq \gamma, \\ \frac{2a_1-2a_0\gamma}{(\gamma-\alpha)^2}\alpha^{n+1} + \frac{a_0\gamma^2-a_2}{(\gamma-\alpha)^2}\alpha^n + \frac{a_1(\gamma+\alpha)-a_0\alpha\gamma-a_2}{\gamma-\alpha}n\alpha^{n-1} + \frac{a_0\alpha^2-2a_1\alpha+a_2}{(\gamma-\alpha)^2}\gamma^n, & \text{if } \alpha = \beta \neq \gamma, \\ \frac{1}{2}\left[a_0(n-1)(n-2)\alpha^n - 2a_1n(n-2)\alpha^{n-1} + a_2n(n-1)\alpha^{n-2}\right], & \text{if } \alpha = \beta = \gamma. \end{cases} \tag{66}$$

Proof. First, suppose that α, β, γ are all distinct. By Vieta's formula, we have

$$\begin{aligned} p &= \alpha + \beta + \gamma \\ q &= -(\alpha\beta + \beta\gamma + \gamma\alpha) \\ r &= \alpha\beta\gamma. \end{aligned}$$

Substituting these into (65) yields

$$a_n = (\alpha + \beta + \gamma)a_{n-1} - (\alpha\beta + \beta\gamma + \gamma\alpha)a_{n-2} + \alpha\beta\gamma a_{n-3}.$$

This is equivalent to

$$a_n - (\alpha + \beta)a_{n-1} + \alpha\beta a_{n-2} = \gamma(a_{n-1} - (\alpha + \beta)a_{n-2} + \alpha\beta a_{n-3}),$$

which implies that the sequence $\{a_n - (\alpha + \beta)a_{n-1} + \alpha\beta a_{n-2}\}$ is geometric, with γ being its common ratio. Thus,

$$a_n - (\alpha + \beta)a_{n-1} + \alpha\beta a_{n-2} = (a_2 - (\alpha + \beta)a_1 + \alpha\beta a_0)\gamma^{n-2},$$

which implies

$$\frac{a_n}{\gamma^n} = \frac{\alpha + \beta}{\gamma} \cdot \frac{a_{n-1}}{\gamma^{n-1}} - \frac{\alpha\beta}{\gamma^2} \cdot \frac{a_{n-2}}{\gamma^{n-2}} + \frac{a_2 - (\alpha + \beta)a_1 + \alpha\beta a_0}{\gamma^2}.$$

We can make the substitution $A_n := a_n/\gamma^n$ to yield the second-order nonhomogeneous recurrence relation

$$A_n = \frac{\alpha + \beta}{\gamma}A_{n-1} - \frac{\alpha\beta}{\gamma^2}A_{n-2} + \frac{a_2 - (\alpha + \beta)a_1 + \alpha\beta a_0}{\gamma^2}, \tag{67}$$

where $A_0 = a_0$ and $A_1 = a_1/\gamma$.

Solving the corresponding characteristic equation

$$x^2 - \frac{\alpha + \beta}{\gamma}x + \frac{\alpha\beta}{\gamma^2} = \frac{a_2 - (\alpha + \beta)a_1 + \alpha\beta a_0}{\gamma^2},$$

we get the roots $x = \frac{\alpha}{\gamma}, \frac{\beta}{\gamma}$. This implies that the closed form for $\{a_n\}$ is of the form

$$A_n = c_1 \left(\frac{\alpha}{\gamma}\right)^n + c_2 \left(\frac{\beta}{\gamma}\right)^n + C, \tag{68}$$

with undetermined coefficients c_1, c_2, and C.

With C being a particular solution of A_n, it satisfies (67), and so

$$C = \frac{\alpha+\beta}{\gamma} \cdot C - \frac{\alpha\beta}{\gamma^2} \cdot C + \frac{a_2 - (\alpha+\beta)a_1 + \alpha\beta a_0}{\gamma^2},$$

which implies

$$C = \frac{a_1(\alpha+\beta) - a_0\alpha\beta - a_2}{(\beta-\gamma)(\gamma-\alpha)}.$$

Evaluating (67) at $n = 0$ and $n = 1$ yields

$$\begin{aligned} a_0 = A_0 &= c_1 + c_2 + C, \\ \frac{a_1}{\gamma} = A_1 &= c_1\frac{\alpha}{\gamma} + c_2\frac{\beta}{\gamma} + C, \end{aligned}$$

respectively. This system of two equations with two unknowns c_1 and c_2 can be solved to get

$$\begin{aligned} c_1 &= \frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{(\alpha-\beta)(\gamma-\alpha)}, \\ c_2 &= \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{(\alpha-\beta)(\beta-\gamma)}. \end{aligned}$$

Thus, multiplying both sides of (67) by γ^n gifts us with

$$a_n = \frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{(\alpha-\beta)(\gamma-\alpha)}\alpha^n + \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{(\alpha-\beta)(\beta-\gamma)}\beta^n + \frac{a_1(\alpha+\beta) - a_0\alpha\beta - a_2}{(\beta-\gamma)(\gamma-\alpha)}\gamma^n. \tag{69}$$

Now, supposing $\alpha = \beta \neq \gamma$, we can simply let β approach α in (69). Starting with the third term of the right-hand side of (69), we get

$$\lim_{\beta\to\alpha} \frac{a_1(\alpha+\beta) - a_0\alpha\beta - a_2}{(\beta-\gamma)(\gamma-\alpha)}\gamma^n = \frac{a_0\alpha^2 - 2a_1\alpha + a_2}{(\gamma-\alpha)^2}\gamma^n. \tag{70}$$

Treating the first two terms of the the right-hand side of (69) carefully,

$$\begin{aligned} &\lim_{\beta\to\alpha}\left[\frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{(\alpha-\beta)(\gamma-\alpha)}\alpha^n + \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{(\alpha-\beta)(\beta-\gamma)}\beta^n\right] \\ &= \lim_{\beta\to\alpha}\left[\frac{a_1(\beta^2-\gamma^2) - a_0\beta\gamma(\beta-\gamma) - a_2(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}\alpha^n + \frac{a_1(\gamma^2-\alpha^2) - a_0\alpha\gamma(\gamma-\alpha) - a_2(\gamma-\alpha)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}\beta^n\right], \end{aligned}$$

we employ L'Hôpital rule to continue to

$$\begin{aligned} &= \lim_{\beta\to\alpha}\left[\frac{2a_1\beta - 2a_0\beta\gamma + a_0\gamma^2 - a_2}{(\alpha-2\beta+\gamma)(\gamma-\alpha)}\alpha^n + \frac{na_1(\gamma^2-\alpha^2) - a_0\alpha\gamma(\gamma-\alpha) - a_2(\gamma-\alpha)}{(\alpha-2\beta+\gamma)(\gamma-\alpha)}\beta^{n-1}\right] \\ &= \frac{2a_1 - 2a_0}{(\gamma-\alpha)^2}\alpha^{n+1} + \frac{a_0\gamma^2 - a_2}{(\gamma-\alpha)^2}\alpha^n + \frac{n\alpha^{n-1}\left(a_1(\gamma+\alpha) - a_0\alpha\gamma - a_2\right)}{\gamma-\alpha}. \end{aligned} \tag{71}$$

Combining (70) and (71), we obtain the formula for a_n when $\alpha = \beta \neq \gamma$:

$$a_n = \frac{2a_1 - 2a_0\gamma}{(\gamma-\alpha)^2}\alpha^{n+1} + \frac{a_0\gamma^2 - a_2}{(\gamma-\alpha)^2}\alpha^n + \frac{a_1(\gamma+\alpha) - a_0\alpha\gamma - a_2}{\gamma-\alpha}n\alpha^{n-1} + \frac{a_0\alpha^2 - 2a_1\alpha + a_2}{(\gamma-\alpha)^2}\gamma^n \tag{72}$$

Finally, to get the case $\alpha = \beta = \gamma$, we let γ approach α in (72). Indeed,

$$\begin{aligned} a_n = \lim_{\gamma\to\alpha} \frac{1}{(\gamma-\alpha)^2}\Big[&(2a_1 - 2a_0\gamma)\alpha^{n+1} + (a_0\gamma^2 - a_2)\alpha^n \\ &+ n\alpha^{n-1}(a_1\gamma + a_1\alpha - a_0\alpha\gamma - a_2)(\gamma-\alpha) + (a_0\alpha^2 - 2a_1\alpha + a_2)\gamma^n\Big] \end{aligned}$$

Applying L'Hôpital rule once, we have

$$\begin{aligned} a_n = \lim_{\gamma\to\alpha} \frac{1}{2(\gamma-\alpha)}\Big[&-2a_0\alpha^{n+1} + 2a_0\gamma\alpha^n + n\gamma^{n-1}(a_0\alpha^2 - 2a_1\alpha + a_2) \\ &+ n\alpha^{n-1}(2a_1\gamma - 2a_0\alpha\gamma - a_2 + a_0\alpha^2)\Big]. \end{aligned}$$

Applying L'Hôpital rule again and rearranging terms, we have the result for $\alpha = \beta = \gamma$:

$$a_n = \frac{1}{2}\Big[a_0(n-1)(n-2)\alpha^n - 2a_1n(n-2)\alpha^{n-1} + a_2n(n-1)\alpha^{n-2}\Big]. \qquad \square$$

Next, we consider criteria for convergence of the usual series. To do so, we introduce the following lemma.

Lemma 7.1. *Suppose* $|t| < 1$. *Then*

$$\sum_{n=0}^{\infty} n^2t^n = \frac{t^2+t}{(1-t)^3}.$$

Proof. Differentiating the series in Lemma 3.2 with respect to t, we have

$$\sum_{n=1}^{\infty} n^2t^{n-1} = \frac{1-t^2}{(1-t)^4} = \frac{t+1}{(1-t)^3}.$$

Multiplying by t yields the desired result. $\square$

Lemma 7.1 serves as the final stepping stone needed for the following result on convergence.

Lemma 7.2. *Let* $\{a_n\}$ *be a sequence as defined in Theorem 7.1. Let* α, β, γ *be the roots of its characteristic equation that may or may not be distinct. Then the series*

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}$$

converges if $r > \max\{|\alpha|, |\beta|, |\gamma|\}$.

Proof. Considering the first branch of (66), i.e., when α, β, γ are distinct, we get

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \sum_{n=0}^{\infty} \frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{(\alpha-\beta)(\gamma-\alpha)} \frac{\alpha^n}{r^{n+1}} + \sum_{n=0}^{\infty} \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{(\alpha-\beta)(\beta-\gamma)} \frac{\beta^n}{r^{n+1}}$$
$$+ \frac{a_1(\alpha+\beta) - a_0\alpha\beta - a_2}{(\beta-\gamma)(\gamma-\alpha)} \frac{\gamma^n}{r^{n+1}}$$
$$= \frac{a_1(\beta+\gamma) - a_0\beta\gamma - a_2}{r(\alpha-\beta)(\gamma-\alpha)} \sum_{n=0}^{\infty} \left(\frac{\alpha}{r}\right)^n + \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{r(\alpha-\beta)(\beta-\gamma)} \sum_{n=0}^{\infty} \left(\frac{\beta}{r}\right)^n$$
$$+ \frac{a_1(\alpha+\gamma) - a_0\alpha\gamma - a_2}{r(\alpha-\beta)(\beta-\gamma)} \sum_{n=0}^{\infty} \left(\frac{\gamma}{r}\right)^n,$$

where the geometric series in the above line converge when $\frac{|\alpha|}{r}, \frac{|\beta|}{r}, \frac{|\gamma|}{r} < 1$. Or, equivalently, when $r > \max\{|\alpha|, |\beta|, |\gamma|\}$.

As for the second branch of (66), another direct computation foretells

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \sum_{n=0}^{\infty} \frac{2a_1 - 2a_0\gamma}{(\gamma-\alpha)^2} \frac{\alpha^{n+1}}{r^{n+1}} + \sum_{n=0}^{\infty} \frac{a_0\gamma^2 - a_2}{(\gamma-\alpha)^2} \frac{\alpha^n}{r^{n+1}} + \sum_{n=0}^{\infty} \frac{a_1(\gamma+\alpha) - a_0\alpha\gamma - a_2}{\gamma-\alpha} \cdot n \frac{\alpha^{n-1}}{r^{n+1}}$$
$$+ \sum_{n=0}^{\infty} \frac{a_0\alpha^2 - 2a_1\alpha + a_2}{(\gamma-\alpha)^2} \frac{\gamma^n}{r^{n+1}}$$
$$= \frac{2a_1 - 2a_0\gamma}{(\gamma-\alpha)^2} \cdot \frac{\alpha}{r} \sum_{n=0}^{\infty} \left(\frac{\alpha}{r}\right)^n + \frac{a_0\gamma^2 - a_2}{r(\gamma-\alpha)^2} \sum_{n=0}^{\infty} \left(\frac{\alpha}{r}\right)^n$$
$$+ \frac{a_1(\gamma+\alpha) - a_0\alpha\gamma - a_2}{\alpha r(\gamma-\alpha)} \sum_{n=0}^{\infty} n \left(\frac{\alpha}{r}\right)^n + \frac{a_0\alpha^2 - 2a_1\alpha + a_2}{r(\gamma-\alpha)^2} \sum_{n=0}^{\infty} \left(\frac{\gamma}{r}\right)^n$$

where the same condition elicits convergence of the above series.

Finally, the last branch results in

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \frac{1}{2}\left[\sum_{n=0}^{\infty} a_0(n-1)(n-2)\frac{\alpha^n}{r^{n+1}} - \sum_{n=0}^{\infty} 2a_1 n(n-2)\frac{\alpha^{n-1}}{r^{n+1}} + \sum_{n=0}^{\infty} a_2 n(n-1)\frac{\alpha^{n-2}}{r^{n+1}}\right]$$
$$= \frac{a_0}{2r}\left[\sum_{n=0}^{\infty} n^2\left(\frac{\alpha}{r}\right)^n - 3\sum_{n=0}^{\infty} n\left(\frac{\alpha}{r}\right)^n + 2\sum_{n=0}^{\infty}\left(\frac{\alpha}{r}\right)^n\right] - \frac{a_1}{\alpha r}\left[\sum_{n=0}^{\infty} n^2\left(\frac{\alpha}{r}\right)^n - 2\sum_{n=0}^{\infty} n\left(\frac{\alpha}{r}\right)^n\right]$$
$$+ \frac{a_2}{2\alpha^2 r}\left[\sum_{n=0}^{\infty} n^2\left(\frac{\alpha}{r}\right)^n - \sum_{n=0}^{\infty} n\left(\frac{\alpha}{r}\right)^n\right],$$

where a special appearance of Lemma 7.1 guarantees convergence with the same condition. □

Behold, the main theorem of this section.

Theorem 7.2. *Let* $\{a_n\}$ *be a sequence satisfying the recurrence relation in (65) and let* $r > \max\{|\alpha|, |\beta|, |\gamma|, 1\}$. *Then*

$$\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} = \frac{a_0 r^2 + (a_1 - pa_0)r + (a_2 - pa_1 - qa_0)}{r^3 - pr^2 - qr - t}.$$

Proof. Denote the series to be evaluated by S, i.e.,

$$S = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}.$$

We first note that

$$\sum_{n=0}^{\infty} \frac{a_{n+3}}{r^{n+1}} = \sum_{n=0}^{\infty} \frac{pa_{n+2} + qa_{n+1} + ta_n}{r^{n+1}} = p\sum_{n=0}^{\infty} \frac{a_{n+2}}{r^{n+1}} + q\sum_{n=0}^{\infty} \frac{a_{n+1}}{r^{n+1}} + t\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}}. \tag{73}$$

Since we know from Lemma 7.2 that S converges, we can use the usual process to yield the equalities:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{a_{n+1}}{r^{n+1}} &= r\sum_{n=1}^{\infty} \frac{a_n}{r^{n+1}} = r\left(\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - \frac{a_0}{r}\right) = Sr - a_0, \\
\sum_{n=0}^{\infty} \frac{a_{n+2}}{r^{n+1}} &= r^2\sum_{n=2}^{\infty} \frac{a_n}{r^{n+1}} = r^2\left(\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - \frac{a_0}{r} - \frac{a_1}{r^2}\right) = Sr^2 - a_0 r - a_1, \\
\sum_{n=0}^{\infty} \frac{a_{n+3}}{r^{n+1}} &= r^3\sum_{n=3}^{\infty} \frac{a_n}{r^{n+1}} = r^3\left(\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} - \frac{a_0}{r} - \frac{a_1}{r^2} - \frac{a_2}{r^3}\right) = Sr^3 - a_0 r^2 - a_1 r - a_2.
\end{aligned}$$

Combining these in (73), we get

$$\begin{aligned}
tS &= (Sr^3 - a_0 r^2 - a_1 r - a_2) - p(Sr^2 - a_0 r - a_1) - q(Sr - a_0) \\
&= S(r^3 - pr^2 - qr) - a_0 r^2 + (pa_0 - a_1)r + (qa_0 + pa_1 - a_2).
\end{aligned}$$

Upon solving for S, the desired result arises. □

Example 7.1. *The Padovan sequence $\{p_n\}$ is defined by the recurrence relation*

$$p_n = p_{n-2} + p_{n-3}, \quad n \geq 3,$$

with $p_0 = p_1 = p_2 = 1$. The real root to the characteristic equation $x^3 - x - 1 = 0$ is $\alpha \approx 1.3247$ (Plastic ratio, [15]). Hence, for any $r > 1.3247$, we have

$$\sum_{n=0}^{\infty} \frac{p_n}{r^{n+1}} = \frac{r^2 + r}{r^3 - r - 1}, \tag{74}$$

and in particular,

$$\sum_{n=0}^{\infty} \frac{p_n}{10^{n+1}} = \frac{110}{989} \quad \textit{and} \quad \sum_{n=0}^{\infty} \frac{p_n}{2^{n+1}} = \frac{24}{5}. \tag{75}$$

Example 7.2. *The Perrin sequence $\{q_n\}$ is defined by the recurrence relation*

$$q_n = q_{n-2} + q_{n-3}, \quad n \geq 3,$$

with the initial conditions $q_0 = 3, q_1 = 0, q_2 = 2$. For $r > 1.3247$, we have

$$\sum_{n=0}^{\infty} \frac{q_n}{r^{n+1}} = \frac{3r^2 - 1}{(r-\alpha)(r-\beta)(r-\gamma)} = \frac{3r^2 - 1}{r^3 - r - 1}.$$

Example 7.3. *A third-order linear recursive sequence $\{\mathcal{T}_n\}$ is called a Tribonacci sequence if it satisfies*

$$\mathcal{T}_n = \mathcal{T}_{n-1} + \mathcal{T}_{n-2} + \mathcal{T}_{n-3},\ n \geq 3,$$

with $\mathcal{T}_0 = 0,\ \mathcal{T}_1 = 1,\ \mathcal{T}_2 = 1$. The real root to the characteristic equation $x^3 - x^2 - x - 1 = 0$ is $\alpha \approx 1.8393$ (Tribonacci constant, [7]*). Hence, for any $r > 1.8393$, we have*

$$\sum_{n=0}^{\infty} \frac{\mathcal{T}_n}{r^{n+1}} = \frac{r}{r^3 - r^2 - 2},$$

and particularly,

$$\sum_{n=0}^{\infty} \frac{\mathcal{T}_n}{10^{n+1}} = \frac{5}{449} \quad \textit{and} \quad \sum_{n=0}^{\infty} \frac{\mathcal{T}_n}{2^{n+1}} = 1. \tag{76}$$

The series for the Tribonacci sequence was also discussed by [7].

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References

[1] Bicknell-Johnson, M., & Bergum, G. E. (1988). The generalized Fibonacci numbers $\{C_n\}$, $C_n = C_{n-1} + C_{n-2} + k$. In: Philippou, A. N., Horadam, A. F., Bergum, G. E. (Eds.). *Applications of Fibonacci Numbers: Proceedings of The Second International Conference on Fibonacci Numbers and Their Applications*, Springer, Dordrecht, pp. 193–205.

[2] Catarino, P. M. M. C., & Borges, A. (2019). On Leonardo numbers. *Acta Mathematica Universitatis Comenianae*, 89(1), 75–86.

[3] Gould, H. W. (1999). The Girard–Waring power sum formulas for symmetric functions, and Fibonacci sequences. *The Fibonacci Quarterly*, 37, 135–140.

[4] He, T.-X., & Shiue, P. J.-S. (2009). On sequences of numbers and polynomials defined by linear recurrence relations of order 2. *International Journal of Mathematics and Mathematical Sciences*, 2009, Article ID 709386.

[5] He, T.-X., Liao, H.-C., & Shiue, P. J.-S. (2018). Matrix representation of recursive sequences of order 3 and its applications. *Journal of Mathematical Research with Applications*, 38(3), 221–235.

[6] Horadam, A. F. (1967). Special properties of the sequence $w_n(a, b; p, q)$. *The Fibonacci Quarterly*, 5(5), 424–434.

[7] Hudson, R. H. (1987). Convergence of Tribonacci decimal expansions. *The Fibonacci Quarterly*, 25, 163–170.

[8] Kuhapatanakul, K., & Chobsorn, J. (2022). On the generalized Leonardo numbers. *Integers*, 22, Article ID A48.

[9] Lidl, R., & Niederreiter, H. (1997). *Finite Fields*, 20. Cambridge University Press.

[10] Liu, Y.-C., & Shiue, P. J.-S. (2022). Convergent series of a second-order linear recursive sequence. *Math Media*, 44(3), 31–41 (In Chinese).

[11] Long, C. T. (1981). The decimal expansion of $1/89$ and related results. *The Fibonacci Quarterly*, 19(1), 53–56.

[12] Melham, R. S, & Shannon, A. G. (1995). Some infinite series summations using power-series evaluated at a matrix. *The Fibonacci Quarterly*, 33(1), 13–20.

[13] Niven, I., Zuckerman, H. S., & Montgomery, H. L. (1991). *An Introduction to the Theory of Numbers*. John Wiley & Sons.

[14] Panda, G. K., & Ray, P. K. (2011). Some links of balancing and cobalancing numbers with Pell and associated Pell numbers. *Bulletin of the Institute of Mathematics, Academia Sinica (New Series)*, 6(1), 41–72.

[15] Shannon, A. G., Anderson, P. G., & Horadam, A. F. (2006). Properties of Cordonnier, Perrin and van der Laan numbers. *International Journal of Mathematical Education in Science and Technology*, 37(7), 825–831.

[16] Shannon, A. G., Shiue, P. J.-S., & Huang, S. C. (2023). Notes on generalized and extended Leonardo numbers. *Notes on Number Theory and Discrete Mathematics*, 29(4), 752–773.

[17] Sloane, N. J. A. (2008). *The On-line Encyclopedia of Integer Sequences*. Available online at: `https://oeis.org`.

[18] Ward, M. (1948). Memoir on elliptic divisibility sequences. *American Journal of Mathematics*, 70(1), 31–74.