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The generalized order (k,t)-Mersenne sequences in groups

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Abstract: The purpose of this paper is to determine the algebraic properties of finite groups via a Mersenne-like sequence. Firstly, we introduce the generalized order (k,t)-Mersenne number sequences and study the periods of these sequences modulo m. Then, we get some interesting structural results. Furthermore, we expand the generalized order (k,t)-Mersenne number sequences to groups and we give the definition of the generalized order (k,t)-Mersenne sequences, $MQ_k^t(G,X)$, in the j-generator groups and also, investigate these sequences in the non-Abelian finite groups in detail. At last, we obtain the periods of the generalized order (k,t)-Mersenne sequences in some special groups as applications of the results produced.

Keywords: Period, Mersenne number, The generalized order (k,t)-Mersenne number sequences p-group.
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1 Introduction

In mathematics and other sciences, sequences play a crucial role. There are numerous scientific applications for sequences in fields such as coding and encryption (see [5, 7, 15, 17]). Fibonacci and Pell are two of the most important sequences. The sequences have been studied in many works for example [1, 3, 6, 9, 10, 16, 18]. Another one of the sequence is Mersenne.

A number of the form $M_n = 2^n - 1,\ n \geq 2$ is said to be a Mersenne number [11]. In 2013, T. Koshy and Z. Gao investigated some divisibility properties of Catalan numbers with Mersenne numbers as their subscripts (see [12]).

In 2016, studied some properties Mersenne, Jacobsthal and Jacobsthal–Lucas sequence and obtained some results with matrices involving Mersenne numbers such as the generating matrix (see [2]). T. Goy [8], calculated determinats the Toeplitz–Hessenberg matrices whose entries are Mersenne numbers. In [13], definded the generalized Mersenne number as follows

Definition 1.1. *For an integer $k \geq 3$, the generalized Mersenne number, denoted by $\{M(k,n)\}_{n=0}^{\infty}$, is defined by*

$$M(k, n) = kM(k, n-1) - (k-1)M(k, n-2), \quad n \geq 0,$$

and we seed the sequence with $M(k, 0) = 0$ and $M(k, 1) = 1$.

Then, studied generalized Mersenne numbers, their properties, matrix generators and some combinatorial interpretations.

Definition 1.2. *After a certain point, a sequence is periodic if it is only composed of repeating subsequences. Period is the number of elements in shortest repeating subsequence. For example, we cosider the sequence $a_1, a_2, a_3, a_4, a_2, a_3, a_4, \ldots$ is periodic after the initial element a and has period 3. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a_1, a_2, a_3, a_4, a_5, a_1, a_2, a_3, a_4, a_5, \ldots$ is simply periodic with period 5.*

For $m, u, l \in \mathbb{N}$, we consider the finitely presented group H_m and $H_{(u,l,m)}$ as follows:

$$H_m = \langle a, b \mid a^{m^2} = b^m = 1, b^{-1}ab = a^{1+m} \rangle,\ m \geq 2.$$
$$H_{(u,l,m)} = \langle a, b, c \mid a^u = b^l = c^m = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle.$$

Lemma 1.1. *Every element of H_m can be written uniquely in the form $b^u a^w$, where $0 \leq u \leq m-1$ and $0 \leq w \leq m^2 - 1$. Also, $|H_m| = m^2$ (see [4]).*

Lemma 1.2. [14] *Element $H(u, l, m)$ of the Heisenberg group can be written uniquely in the form $a^i b^j c^k$ where $1 \leq i \leq u$, $1 \leq j \leq l$ and $1 \leq k \leq m$.*

In this paper, we introduce the generalized order (k, t)-Mersenne number sequences and define them on groups. Then, as a result of our analysis, we show that these sequences are simply periodic, and we study them in some finite groups.

It is the purpose of Section 2 to define the generalized order (k, t)-Mersenne number sequences and to discuss some results related to them. Section 3 introduces the generalized order (k, t)-Mersenne sequences in a finite group and discusses perodic sequences.

2 The generalized order (k, t)-Mersenne number sequences

In this section, we introduce the generalized order (k, t)-Mersenne number sequences and get some properties of them that we use later.

Definition 2.1. *For $k,\ t \geq 3$, the generalized order (k, t)-Mersenne number sequences, $\{M_n(k, t)\}_{n=0}^{\infty}$, defined as follows*

$$M_n(k,t) = kM_{n-1}(k,t) - (k-1)M_{n-2}(k,t) + M_{n-3}(k,t) + \cdots + M_{n-t}(k,t), \quad n \geq t, \quad (1)$$

with initial conditions $M_0(k,t) = M_1(k,t) = \cdots = M_{t-2}(k,t) = 0$ and $M_{t-1}(k,t) = 1$.

Example 2.1. *For $t = 3$ and $k = 3$, we have $M_n(3,3) = 3M_{n-1}(3,3) - 2M_{n-2}(3,3) + M_{n-3}(3,3)$. So that, $\{M_n(3,3)\}_{n=0}^{\infty} = \{0,0,1,3,7,16,37,\ldots\}$.*

For $k = 4$ and $t = 3$, we have $M_n(4,3) = 4M_{n-1}(4,3) - 3M_{n-2}(4,3) + M_{n-3}(4,3)$. So that, $\{M_n(4,3)\}_{n=0}^{\infty} = \{0,0,1,4,13,41,129,\ldots\}$.

In Table 1, we calculate $M_n(3,t)$, for $0 \leq n \leq 8$ and $3 \leq t \leq 8$.

Table 1. $M_n(3,t)$, for $0 \leq n \leq 8$ and $3 \leq t \leq 8$

n	$M_0(3,t)$	$M_1(3,t)$	$M_2(3,t)$	$M_3(3,t)$	$M_4(3,t)$	$M_5(3,t)$	$M_6(3,t)$	$M_7(3,t)$	$M_8(3,t)$
$t=3$	0	0	1	3	7	16	37	86	200
$t=4$	0	0	0	1	3	7	16	38	92
$t=5$	0	0	0	0	1	3	7	16	38
$t=6$	0	0	0	0	0	1	3	7	16
$t=7$	0	0	0	0	0	0	1	3	7
$t=8$	0	0	0	0	0	0	0	1	3

The generalized order (k,t)-Mersenne number sequences modulo α, $\{M_n^{\alpha}(k,t)\} = \{M_0^{\alpha}(k,t), M_1^{\alpha}(k,t), \ldots, M_i^{\alpha}(k,t), \ldots\}$ where $M_i^{\alpha}(k,t) = M_i(k,t) \pmod{\alpha}$.

Theorem 2.1. *For $k, t \geq 3$, the sequence $\{M_n^{\alpha}(k,t)\}$ is simply periodic.*

Proof. Suppose that $X_t = \{(x_1, x_2, \cdots, x_t) \mid x_i \in \mathbb{N}\ and\ 1 \leq x_t \leq \alpha\}$. So that we have $|X_t| = \alpha^t$. Since there are α^t distinct t-tuples of elements of Z_{α}, at least one of the t-tuples appears twice in the sequence $\{M_n^{\alpha}(k,t)\}$. Then the subsequence follows this t-tuple. Thus, it is obvious that the sequence $\{M_n^{\alpha}(k,t)\}$ is periodic.
Hence, it is cearly for $w \geq 0$, there exist $w \geq v$ such that

$$M_w^{\alpha}(k,t) \equiv M_v^{\alpha}(k,t),\ M_{w+1}^{\alpha}(k,t) \equiv M_{v+1}^{\alpha}(k,t), \ldots, M_{w+t}^{\alpha}(k,t) \equiv M_{v+t}^{\alpha}(k,t).$$

By definition of the generalized order (k,t)-Mersenne number sequences, we have

$$M_n(k,t) = kM_{n-1}(k,t) - (k-1)M_{n-2}(k,t) + M_{n-3}(k,t) + \cdots + M_{n-t}(k,t),$$

Thus we can easily derive that

$$M_{w-v}^{\alpha}(k,t) \equiv M_0^{\alpha}(k,t),\ M_{w-v+1}^{\alpha}(k,t) \equiv M_1^{\alpha}(k,t), \ldots,\ M_{w-v+t}^{\alpha}(k,t) \equiv M_t^{\alpha}(k,t),$$

which indicates that the generalized order (k,t)-Mersenne number sequences is simply periodic. □

We use $hM_r(k,t)$ to denoted the minimal period of the generalized order (k,t)-Mersenne number sequences modulo r. In Table 2, we calculate $hM_r(k,3)$, for $2 \le r \le 10$ and $3 \le k \le 8$.

Table 2. $hM_r(k,3)$, for $2 \le r \le 10$ and $3 \le k \le 8$

r	$hM_r(3,3)$	$hM_r(4,3)$	$hM_r(5,3)$	$hM_r(6,3)$	$hM_r(7,3)$	$hM_r(8,3)$
2	7	7	7	7	7	7
3	13	8	13	13	8	13
4	14	14	14	14	14	14
5	8	31	24	31	12	8
6	91	56	91	91	56	91
7	16	38	16	21	16	57
8	28	28	28	28	28	28
9	39	24	39	39	24	39
10	56	217	168	217	84	56

From the recurrence relation (2.1), we have

$$\begin{bmatrix} M_n(k,t) \\ M_{n-1}(k,t) \\ \vdots \\ M_{n-t+2}(k,t) \\ M_{n-t+1}(k,t) \end{bmatrix} = \begin{bmatrix} k & -(k-1) & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} M_{n-1}(k,t) \\ M_{n-2}(k,t) \\ \vdots \\ M_{n-t+1}(k,t) \\ M_{n-t}(k,t) \end{bmatrix}.$$

The generalized order (k,t)-Mersenne number sequences have the following companion matrix

$$M_t(k) = \begin{bmatrix} k & -(k-1) & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{t\times t},$$

and is called the generalized order (k,t)-Mersenne matrix.

Lemma 2.1. *For $k=3$, $t=3$ and $n \ge t$, we have*

$$(M_3(3))^n = \begin{bmatrix} M_{n+2}(3,3) & -(2M_{n+1}(3,3) - M_n(3,3)) & M_{n+1}(3,3) \\ M_{n+1}(3,3) & -(2M_n(3,3) - M_{n-1}(3,3)) & M_n(3,3) \\ M_n(3,3) & -(2M_{n-1}(3,3) - M_{n-2}(3,3)) & M_{n-1}(3,3) \end{bmatrix}.$$

Proof. We use induction method on n. The result is clear if $n=3$. We have

$$(M_3(3))^3 = \begin{bmatrix} 16 & -11 & 7 \\ 7 & -5 & 3 \\ 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} M_5(3,3) & -(2M_4(3,3) - M_3(3,3)) & M_4(3,3) \\ M_4(3,3) & -(2M_3(3,3) - M_2(3,3)) & M_4(3,3) \\ M_3(3,3) & -(2M_2(3,3) - M_1(3,3)) & M_2(3,3) \end{bmatrix}.$$

Assume that Lemma holds for n such that $3 \leq n \leq s$. Let us show that it holds for $n = s+1$.

$$(M_3(3))^{s+1} = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} M_{s+2}(3,3) & -(2M_{s+1}(3,3) - M_s(3,3)) & M_{s+1}(3,3) \\ M_{s+1}(3,3) & -(2M_s(3,3) - M_{s-1}(3,3)) & M_s(3,3) \\ M_s(3,3) & -(2M_{s-1}(3,3) - M_{s-2}(3,3)) & M_{s-1}(3,3) \end{bmatrix}$$

$$= \begin{bmatrix} M_{s+3}(3,3) & -(2M_{s+2}(3,3) - M_{s+1}(3,3)) & M_{s+2}(3,3) \\ M_{s+2}(3,3) & -(2M_{s+1}(3,3) - M_s(3,3)) & M_{s+1}(3,3) \\ M_{s+1}(3,3) & -(2M_s(3,3) - M_{s-1}(3,3)) & M_s(3,3) \end{bmatrix}.$$

Lemma is proved. □

Corollary 2.1. *For $k \geq 4$, $t = 3$ and $n \geq t$, we have*

$$(M_3(k))^n = \begin{bmatrix} M_{n+2}(k,3) & -((k-1)M_{n+1}(k,3) - M_n(k,3)) & M_{n+1}(k,3) \\ M_{n+1}(k,3) & -((k-1)M_n(k,3) - M_{n-1}(k,3)) & M_n(k,3) \\ M_n(k,3) & -((k-1)M_{n-1}(k,3) - M_{n-2}(k,3)) & M_{n-1}(k,3) \end{bmatrix}.$$

Example 2.2. *If $k = 4$ and $t = 3$, we have*

$$(M_3(4))^3 = \begin{bmatrix} 41 & -35 & 13 \\ 13 & -11 & 4 \\ 4 & -3 & 1 \end{bmatrix}.$$

In general by induction on n, for $k \geq 3$, $t \geq 4$ and $n \geq t$, we have

$$(M_t(k))^n \begin{bmatrix} M_{n+t-1}(k,t) & -((k-1)M_{n+t-2}(k,t) - (M_{n+t-3}(k,t) + M_{n+t-4}(k,t) + \ldots + M_n(k,t))) & & M_{n+t-2}(k,t) \\ M_{n+t-2}(k,t) & -((k-1)M_{n+t-3}(k,t) - (M_{n+t-4}(k,t) + M_{n+t-5}(k,t) + \ldots + M_{n-1}(k,t))) & & M_{n+t-3}(k,t) \\ \vdots & \vdots & M_k^* & \vdots \\ M_{n+1}(k,t) & -((k-1)M_n(k,t) - (M_{n-1}(k,t) + M_{n-2}(k,t) + \ldots + M_{n-t}(k,t))) & & M_n(k,t) \\ M_n(k,t) & -((k-1)M_{n-1}(k,t) - (M_{n-2}(k,t) + M_{n-3}(k,t) + \ldots + M_{n-t-1}(k,t))) & & M_{n-1}(k,t) \end{bmatrix},$$

$$M_k^* = \begin{bmatrix} \sum_{i=0}^{t-3} M_{n+t-(2+i)}(k,t) & \sum_{i=0}^{t-4} M_{n+t-(2+i)}(k,t) & \ldots & \sum_{i=0}^{1} M_{n+t-(2+i)}(k,t) \\ \sum_{i=0}^{t-3} M_{n+t-(2+i-1)}(k,t) & \sum_{i=0}^{t-4} M_{n+t-(2+i-1)}(k,t) & \ldots & \sum_{i=0}^{1} M_{n+t-(2+i-1)}(k,t) \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{i=0}^{t-3} M_{n-i}(k,t) & \sum_{i=0}^{t-4} M_{n-i}(k,t) & \ldots & \sum_{i=0}^{1} M_{n-i}(k,t) \\ \sum_{i=0}^{t-3} M_{n-(i+1)}(k,t) & \sum_{i=0}^{t-4} M_{n-(i+1)}(k,t) & \ldots & \sum_{i=0}^{1} M_{n-(i+1)}(k,t) \end{bmatrix},$$

where M_k^* is a $(t-3) \times (t-3)$ matrix.

Lemma 2.2. *Let $g_{M_n(k,t)}$ be the generating function of the generalized order (k,t)-Mersenne number sequences. Then,*

$$g_{M_n(k,t)} = \frac{x^{t-1}}{1 - kx + (k-1)x^2 - \cdots - x^t}. \tag{2}$$

Proof. Let $g_{M_n(k,t)}$ be the generating function of the generating function of the generalized order (k,t)-Mersenne number sequences. We have

$$\begin{aligned}
g_{M_n(k,t)} &= \sum_{n=1}^{\infty} M_n(k,t)x^n \\
&= M_1(k,t)x + M_2(k,t)x^2 + \cdots + M_{t-1}(k,t)x^{t-1} + \sum_{n=t}^{\infty} M_n(k,t)x^n \\
&= x^{t-1} + \sum_{n=t}^{\infty} kM_{n-1}(k,t) - (k-1)M_{n-2}(k,t) + M_{n-3}(k,t) + \cdots + M_{n-t}(k,t)x^n \\
&= x^{t-1} + k\sum_{n=t}^{\infty} M_{n-1}(k,t)x^n - (k-1)\sum_{n=t}^{\infty} M_{n-2}(k,t)x^n + \cdots + \sum_{n=t}^{\infty} M_{n-t}(k,t)x^n \\
&= x^{t-1} + kx\sum_{n=1}^{\infty} M_n(k,t)x^n - (k-1)x^2\sum_{n=1}^{\infty} M_n(k,t)x^n + \cdots + x^t\sum_{n=1}^{\infty} M_n(k,t)x^n \\
&= x^{t-1} + kxg_{M_n(k,t)} - (k-1)x^2 g_{M_n(k,t)} + \cdots + x^t g_{M_n(k,t)}.
\end{aligned}$$

Thus,

$$g_{M_n(k,t)} = \frac{x^{t-1}}{1-kx+(k-1)x^2-\cdots-x^t}. \qquad \square$$

Lemma 2.3. *The generating function of the generalized order (k,t)-Mersenne number sequences has the following exponential representation*

$$g_{M_n(k,t)} = x^{t-1}\exp\sum_{i=1}^{\infty}\frac{x^i}{i}(k-(k-1)x+\cdots+x^{t-1})^i,$$

where $t \geq 3$.

Proof. Using (2.2), we have

$$\ln\frac{g_{M_n(k,t)}}{x^{t-1}} = -\ln(1-kx+(k-1)x^2-\cdots-x^t).$$

$$\begin{aligned}
&-\ln(1-kx+(k+1)x^2-\cdots-x^t) = -[-x(k-(k-1)+\cdots+x^{t-1}) \\
&-\frac{1}{2}x^2(k-(k-1)+\cdots+x^{t-1})^2-\cdots-\frac{1}{i}x^i(k-(k-1)+\cdots+x^{t-1})^i-\cdots],
\end{aligned}$$

result has been achieved. $\square$

Lemma 2.4. *For integers s, n and $m \geq 2$, we have*

(i) $M_{hM_m(k,t)+n}(k,t) \equiv M_n(k,t) \pmod{m}$,

(ii) $M_{s\times(hM_m(k,t))+n}(k,t) \equiv M_n(k,t) \pmod{m}$.

Proof. (i) The result follows by using the definition of the period of the (k,t)-Mersenne number sequences modulo m.

(ii) We have

$$M_{s\times(hM_m(k,t))+n}(k,t) \equiv M_{(hM_m(k,t))+(s-1)\times(hM_m(k,t))+n}(k,t) \equiv M_{(s-1)(hM_m(k,t))+n}(k,t) \equiv \cdots \equiv M_n(k,t) \pmod{m}.$$

The result is obtained. □

Lemma 2.5. *Let n be an integer and $m \geq 2$ is a positive number. If*

$$\begin{cases} M_n(k,t) \equiv 0 & \pmod{m}, \\ M_{n+1}(k,t) \equiv 0 & \pmod{m}, \\ \vdots \\ M_{n+t-2}(k,t) \equiv 0 & \pmod{m}, \\ M_{n+t-1}(k,t) \equiv 1 & \pmod{m}, \\ M_{n+t}(k,t) \equiv k & \pmod{m}, \end{cases}$$

then $hM_m(k,t) \mid n$.

Proof. There exists $0 \leq i \leq hM_m(k,t)$ such that $n = t \times (hM_m(k,t)) + i$. Also, $\forall\, 0 \leq j \leq t$, $M_{n+j}(t,k) \equiv M_{i+j}(t,k) \pmod{m}$, then we have the following equations

$$\begin{cases} M_i(k,t) \equiv 0 & \pmod{m}, \\ M_{i+1}(k,t) \equiv 0 & \pmod{m}, \\ \vdots \\ M_{i+t-2}(k,t) \equiv 0 & \pmod{m}, \\ M_{i+t-1}(k,t) \equiv 1 & \pmod{m}, \\ M_{i+t}(k,t) \equiv k & \pmod{m}. \end{cases}$$

So that i is a period of (k,t)-Mersenne number sequences modulo m, i.e., $hM_m(k,t) \mid i$. Since $0 \leq i < hM_m(k,t)$, we have $i = 0$. Therefore, we get the results. □

3 The generalized order (k,t)-Mersenne sequences in finite groups

In this section, we define the generalized order (k,t)-Mersenne sequences in finite groups and prove that generalized order (k,t)-Mersenne sequences in finite groups are simply periodic. Also, we study these sequences on H_m and $H_{(u,l,m)}$. First, we define the generalized order (k,t)-Mersenne sequence in a finite group as follows.

Definition 3.1. *The generalized order (k,t)-Mersenne sequence in a finite group is a sequence of group elements $x_0, x_1, \ldots, x_n, \ldots$ for which, given an initail (seed) set in $X = \{a_1, a_2, \ldots, a_j\}$, each element is defined by*

$$x_n = \begin{cases} a_n, & \text{for } n \leq j, \\ x_0 x_1 \cdots x_{n-3}(x_{n-2})^{-(k-1)} x_{n-1}^k, & \text{for } j < n \leq t, \\ x_{n-t} \cdots x_{n-3}(x_{n-2})^{-(k-1)} x_{n-1}^k, & \text{for } n > t. \end{cases} \tag{3}$$

The element of the generalized order (k,t)-Mersenne sequences in group are denoted by $Q_k^t(G,X)$ and its period is denoted by $MQ_k^t(G,X)$.

Theorem 3.1. *The generalized order (k,t)-Mersenne sequences in group is simply periodic.*

Proof. Let G be an $i-$generator group and let $(a_0, a_1, \dots, a_{i-1})$ be a generating $i-$tuple for G. If $|G| = m$, then there are m^i distinict $i-$tuple of elements of G. Thus, at least one of the $i-$tuple appers twice in $Q_k^t(G,X)$. Because of the repetition, the sequence $Q_k^t(G,X)$ is periodic. Now, we show simply periodic. It is clear that there are r and s in $\mathbb{N}$, with $s > r$, such that $x_{s+1} = x_{r+1},\ x_{s+2} = x_{r+2}, \dots, x_{s+t} = x_{r+t}$. From Definition 3.1, we write $x_{s-1} = x_{r-1},\ x_{s-2} = x_{r-2}, \dots, x_{s-r} = a_0 = x_0 = x_{r-s}$. Thus, $Q_k^t(G,X)$ is simply periodic. □

For $m \in \mathbb{N}$, we consider the finitely presented group H_m as follows:

$$H_m = \langle a, b | a^{m^2} = b^m = 1, b^{-1}ab = a^{1+m}\rangle,\ m \geq 2.$$

Now, we obtain sequence $T_n(k)$ as follows:

$$\begin{aligned}
&T_0(k) = 1,\ T_1(k) = 0,\ T_2(k) = -(k-1) - mk(k-1),\\
&T_n(k) = T_{n-3}(k) - (k-1)T_{n-2}(k) + kT_{n-1}(k) - m((T_{n-3}(k) - T_{n-2}(k))M_{n-1}(k,3)\\
&\qquad + \cdots + (T_{n-3}(k) - (k-1)T_{n-2}(k))M_{n-1}(k,3) - (T_{n-3}(k) - (k-1)T_{n-2}(k)\\
&\qquad + T_{n-1}(k))M_n(k,3) - \cdots - (T_{n-3}(k) - (k-1)T_{n-2}(k) + (k-1)T_{n-1}(k))M_n(k,3),\\
&\qquad n \geq 3.
\end{aligned}$$

Lemma 3.1. *Every element of $Q_k^3(H_m, X)$ may be persented by $x_n = b^{M_{n+1}(k,3)}a^{T_n(k)}$, $k \geq 3$.*

Proof. For $n = 2$ and $n = 3$, we have $x_2 = a^{-(k-1)}(b)^k = b^k a^{-(k-1)-mk(k-1)}$ and $x_3 = ab^{-(k-1)}(b^k a^{-(k-1)-mk(k-1)})^k = b^{-(k-1)}a^{1-m(k-1)}(b^k a^{-(k-1)-mk(k-1)})^k$ $= b^{-(k-1)+k^2}a^{1-(k-1)-m(1-k^3-k^2-2k)}$.

Now, by induction on n, we have

$$\begin{aligned}
x_n &= x_{n-3}(x_{n-2})^{-(k-1)}(x_{n-1})^k\\
&= b^{M_{n-2}(k,3)}a^{T_{n-3}(k)}(b^{M_{n-1}(k,3)}a^{T_{n-2}(k)})^{-(k-1)}(b^{M_n(k,3)}a^{T_{n-1}(k)})^k\\
&= b^{M_{n-2}(k,3)}a^{T_{n-3}(k)}(b^{M_{n-1}(k,3)}a^{T_{n-2}(k)})^{-1}(a^{T_{n-3}(k)}b^{M_{n-1}(k,3)}a^{T_{n-2}(k)})^{-(k-2)}(b^{M_n(k,3)}a^{T_{n-1}(k)})^k\\
&= b^{M_{n-2}(k,3)}a^{T_{n-3}(k)}a^{-T_{n-2}(k)}b^{-M_{n-1}(k,3)}(b^{M_{n-1}(k,3)}a^{T_{n-2}(k)})^{-(k-2)}(b^{M_n(k,3)}a^{T_{n-1}(k)})^k\\
&= b^{M_{n-2}(k,3)-M_{n-1}(k,3)}a^{T_{n-3}(k)-T_{n-2}(k)-m(T_{n-3}(k)-T_{n-2}(k))M_{n-1}(k,3)}(b^{M_{n-1}(k,3)}a^{T_{n-2}(k)})^{-(k-2)}\\
&\quad (b^{M_n(k,3)}a^{T_{n-1}(k)})^k\\
&= \cdots\\
&= b^{M_{n-2}(k,3)-(k-1)M_{n-1}(k,3)}\\
&\quad a^{T_{n-3}(k)-(k-1)T_{n-2}(k)-m((T_{n-3}(k)-T_{n-2}(k))M_{n-1}(k,3)+\cdots+(T_{n-3}(k)-(k-1)T_{n-2}(k))M_{n-1}(k,3)}\\
&\quad b^{M_n(k,3)}a^{T_{n-1}(k)}(b^{M_n(k,3)}a^{T_{n-1}(k)})^{k-1}\\
&= b^{M_{n-2}(k,3)-(k-1)M_{n-1}(k,3)+M_n(k,3)}a^{T_{n-3}(k)-(k-1)T_{n-2}(k)}\\
&\quad a^{-m((T_{n-3}(k)-T_{n-2}(k))M_{n-1}(k,3)+\cdots+(T_{n-3}(k)-(k-1)T_{n-2}(k))M_{n-1}(k,3)-(T_{n-3}(k)-(k-1)T_{n-2}(k)+T_{n-1}(k))M_n(k,3)}\\
&\quad b^{M_n(k,3)}a^{T_{n-1}(k)}(b^{M_n(k,3)}a^{T_{n-1}(k)})^{k-2}\\
&= \cdots\\
&= b^{M_{n+1}(k,3)}a^{T_n(k)}.
\end{aligned}$$

Therefore, the assertion holds. □

Lemma 3.2. *If $MQ_k^3(H_m, X) = s$, then s is the least integer such that all of the equations*

$$\begin{cases} M_{s+1}(k,t) \equiv 0 & \pmod{m}, \\ M_{s+2}(k,t) \equiv 1 & \pmod{m}, \\ M_{s+3}(k,t) \equiv k & \pmod{m}, \\ T_s(k) \equiv 1 & \pmod{m^2}, \\ T_{s+1}(k) \equiv 0 & \pmod{m^2}, \\ T_{s+2}(k) \equiv -(k-1) - m(k^2-k) & \pmod{m^2}, \end{cases}$$

holds. Moreover, $hM_m(k,3)$ divides $MQ_k^3(H_m, X)$.

Proof. By Lemma 3.1, we obtain $x_n = b^{M_{n+1}(k,3)}a^{T_n(k)}$. Since $x_s = a$, $x_{s+1} = b$ and $x_{s+2} = b^k a^{-(k-1)-mk(k-1)}$, by Lemma 1.1 we have

$$\begin{cases} M_{s+1}(k,t) \equiv 0 & \pmod{m}, \\ M_{s+2}(k,t) \equiv 1 & \pmod{m}, \\ M_{s+3}(k,t) \equiv k & \pmod{m}, \\ T_s(k) \equiv 1 & \pmod{m^2}, \\ T_{s+1}(k) \equiv 0 & \pmod{m^2}, \\ T_{s+2}(k) \equiv -(k-1) - m(k^2-k) & \pmod{m^2}. \end{cases}$$

So, Lemma 2.5 proved that $hM_m(k,3) \mid MQ_k^3(H_m, X)$. □

Here, we consider Heisenberg group $H_{(u,l,m)} = \langle a, b, c \mid a^u = b^l = c^m = 1, [a,b] = c, [a,c] = [b,c] = 1\rangle$ and define sequences $g_n(3,4)$ and $S_n(3,4)$ as follows:

$$\begin{aligned} &g_0(3,4) = 0,\ g_1(3,4) = 1,\ g_2(3,4) = 0,\ g_3(3,4) = -2, \\ &g_n(3,4) = g_{n-4}(3,4) + g_{n-3}(3,4) - 2g_{n-2}(3,4) + 3g_{n-1}(3,4), n \geq 4. \\ &S_0(3,4) = 0,\ S_1(3,4) = 0,\ S_2(3,4) = 1,\ S_3(3,4) = 3, \\ &S_n(3,4) = S_{n-4}(3,4) + S_{n-3}(3,4) - 2S_{n-2}(3,4) + 3S_{n-1}(3,4) - g_{n-4}(3,4)M_{n-3}(3,4) \\ &\qquad + (g_{n-4}(3,4) + g_{n-3}(3,4) - g_{n-2}(3,4))M_{n-2}(3,4) + (g_{n-4}(3,4) + g_{n-3}(3,4) \\ &\qquad - 2g_{n-2}(3,4))M_{n-2}(3,4) - (g_{n-4}(3,4) + g_{n-3}(3,4) - 2g_{n-2}(3,4))M_{n-1}(3,4) \\ &\qquad - (g_{n-4}(3,4) + g_{n-3}(3,4) - 2g_{n-2}(3,4) + g_{n-1}(3,4))M_{n-1}(3,4) \\ &\qquad - (g_{n-4}(3,4) + g_{n-3}(3,4) - 2g_{n-2}(3,4) + 2g_{n-1}(3,4))M_{n-1}(3,4),\ n \geq 4. \end{aligned}$$

Lemma 3.3. *Every element of $Q_3^4(H_{(u,l,m)}, X)$ may be presented by $x_n = a^{M_n(3,4)}b^{g_n(3,4)}c^{S_n(3,4)}$, $n \geq 3$.*

Proof. For $n = 3$, $n = 4$ and $n = 5$, we have $x_3 = a(b)^{-2}c^3$, $x_4 = abc^{-2}(ab^{-2}c^3)^3 = a^4b^{-5}c^{11}$ and $x_5 = bc(ab^{-2}c^3)^{-2}(a^4b^{-5}c^{11})^3 = a^7b^{-10}c^{37}$. Now, by induction on n, we have:

$$
\begin{aligned}
x_n &= x_{n-4}x_{n-3}(x_{n-2})^{-2}(x_{n-1})^3 \\
&= a^{M_{n-4}(3,4)}b^{g_{n-4}(3,4)}c^{S_{n-4}(3,4)}a^{M_{n-3}(3,4)}b^{g_{n-3}(3,4)}c^{S_{n-3}(3,4)}(a^{M_{n-2}(3,4)}b^{g_{n-2}(3,4)}c^{S_{n-2}(3,4)})^{-2} \\
&\quad (a^{M_{n-1}(3,4)}b^{g_{n-1}(3,4)}c^{S_{n-1}(3,4)})^3 \\
&= a^{M_{n-4}(3,4)+M_{n-3}(3,4)}b^{g_{n-4}(3,4)+g_{n-3}(3,4)}c^{S_{n-3}(3,4)-g_{n-4}(3,4)M_{n-3}(3,4)}(a^{M_{n-2}(3,4)}b^{g_{n-2}(3,4)}c^{S_{n-2}(3,4)})^{-2} \\
&\quad (a^{M_{n-1}(3,4)}b^{g_{n-1}(3,4)}c^{S_{n-1}(3,4)})^3 \\
&= a^{M_{n-4}(3,4)+M_{n-3}(3,4)}b^{g_{n-4}(3,4)+g_{n-3}(3,4)}c^{S_{n-4}(3,4)+S_{n-3}(3,4)-g_{n-4}(3,4)M_{n-3}(3,4)} \\
&(a^{M_{n-2}(3,4)}b^{g_{n-2}(3,4)}c^{S_{n-2}(3,4)})^{-1}(a^{M_{n-2}(3,4)}b^{g_{n-2}(3,4)}c^{S_{n-2}(3,4)})^{-1}(a^{M_{n-1}(3,4)}b^{g_{n-1}(3,4)}c^{S_{n-1}(3,4)})^3 \\
&= a^{M_{n-4}(3,4)+M_{n-3}(3,4)-M_{n-2}(3,4)}b^{g_{n-4}(3,4)+g_{n-3}(3,4)-g_{n-2}(3,4)} \\
&\quad c^{S_{n-4}(3,4)+S_{n-3}(3,4)-g_{n-4}(3,4)M_{n-3}(3,4)+(g_{n-4}(3,4)+g_{n-3}(3,4)-g_{n-2}(3,4))M_{n-2}(3,4)} \\
&\quad (a^{M_{n-2}(3,4)}b^{g_{n-2}(3,4)}c^{S_{n-2}(3,4)})^{-1}(a^{M_{n-1}(3,4)}b^{g_{n-1}(3,4)}c^{S_{n-1}(3,4)})^3 \\
&= \cdots \\
&= a^{M_{n-4}(3,4)+M_{n-3}(3,4)-2M_{n-2}(3,4)+3M_{n-1}(3,4)}b^{g_{n-4}(3,4)+g_{n-3}(3,4)-2g_{n-2}(3,4)+3g_{n-1}(3,4)} \\
&\quad c^{S_{n-4}(3,4)+S_{n-3}(3,4)-2S_{n-2}(3,4)+3S_{n-1}(3,4)-g_{n-4}(3,4)M_{n-3}(3,4)+(g_{n-4}(3,4)+g_{n-3}(3,4)-g_{n-2}(3,4))M_{n-2}(3,4)} \\
&\quad c^{+(g_{n-4}(3,4)+g_{n-3}(3,4)-2g_{n-2}(3,4))M_{n-2}(3,4)-(g_{n-4}(3,4)+g_{n-3}(3,4)-2g_{n-2}(3,4))M_{n-1}(3,4)-(g_{n-4}(3,4)+g_{n-3}(3,4)} \\
&\quad c^{-2g_{n-2}(3,4)+g_{n-1}(3,4))M_{n-1}(3,4)-(g_{n-4}(3,4)+g_{n-3}(3,4)-2g_{n-2}(3,4)+2g_{n-1}(3,4))M_{n-1}(3,4)} \\
&= a^{M_n(3,4)}b^{g_n(3,4)}c^{S_n(3,4)},
\end{aligned}
$$

Lemma is proved. □

Theorem 3.2. *For $u \geq 1$, we have $hM_u(3,4) \mid MQ_3^4(H_{(u,l,m)}, X)$.*

Proof. By Lemma 3.3,we obtain $x_n = a^{M_n(3,4)}b^{g_n(3,4)}c^{S_n(3,4)}$. Suppose that $MQ_3^4(H_{(u,l,m)}, X) = i$. Since $x_i = a$, $x_{i+1} = b$, $x_{i+2} = c$ and $x_{i+3} = ab^{-2}c^3$, by Lemma 1.2, we have

$$
\begin{cases}
M_i(3,4) \equiv 1 & (\text{mod } u), \\
M_{i+1}(3,4) \equiv 0 & (\text{mod } u), \\
M_{i+2}(3,4) \equiv 0 & (\text{mod } u), \\
M_{i+3}(3,4) \equiv 1 & (\text{mod } u), \\
g_i(3,4) \equiv 0 & (\text{mod } l), \\
g_{i+1}(3,4) \equiv 1 & (\text{mod } l), \\
g_{i+2}(3,4) \equiv 0 & (\text{mod } l), \\
g_{i+3}(3,4) \equiv -2 & (\text{mod } l), \\
S_i(3,4) \equiv 0 & (\text{mod } m), \\
S_{i+1}(3,4) \equiv 0 & (\text{mod } m), \\
S_{i+2}(3,4) \equiv 1 & (\text{mod } m), \\
S_{i+3}(3,4) \equiv 3 & (\text{mod } m).
\end{cases}
$$

So, Lemma 2.5 proved that $hM_u(3,4) \mid MQ_3^4(H_{(u,l,m)}, X)$. □

Now, we define $g_n(k,4)$ and $S_n(k,4)$ as follows

$$g_0(k,4)=0,\ g_1(k,4)=1,\ g_2(k,4)=0,\ g_3(k,4)=-2,$$
$$g_n(k,4)=g_{n-4}(k,4)+g_{n-3}(k,4)-(k-1)g_{n-2}(k,4)+kg_{n-1}(k,4),\ n\geq 4.$$
$$S_0(k,4)=0,\ S_1(k,4)=0,\ S_2(k,4)=1,\ S_3(k,4)=3,$$
$$\begin{aligned}S_n(k,4)=&\ S_{n-4}(k,4)+S_{n-3}(k,4)-(k-1)S_{n-2}(k,4)+kS_{n-1}(k,4)-g_{n-4}(k,4)M_{n-3}(k,4)\\&+(g_{n-4}(k,4)+g_{n-3}(k,4)-g_{n-2}(k,4))M_{n-2}(3,4)+(g_{n-4}(k,4)+g_{n-3}(k,4)\\&-2g_{n-2}(k,4))M_{n-2}(k,4)-(g_{n-4}(k,4)+g_{n-3}(k,4)-2g_{n-2}(k,4))M_{n-1}(3,4)+\cdots\\&+(g_{n-4}(k,4)+g_{n-3}(k,4)-(k-1)g_{n-2}(k,4)+g_{n-1}(k,4))M_{n-1}(k,4)-\cdots\\&-(g_{n-4}(k,4)+g_{n-3}(k,4)-(k-1)g_{n-2}(k,4)+(k-1)g_{n-1}(k,4))M_{n-1}(k,4),\end{aligned}$$

$n\geq 4$.

Lemma 3.4. *Every element of $Q_k^4(H_{(u,l,m)},X)$ may be presented by $x_n=a^{M_n(k,4)}b^{g_n(k,4)}c^{S_n(k,4)}$, $n\geq 3$.*

Proof. Similar to Lemma 3.3, the proof follows. □

Similarly, Theorem 3.2 can be applied to the proof of the following Lemma.

Lemma 3.5. *For $k\geq 4$ and $u\geq 1$, we have $hM_u(k,4)\mid MQ_k^4(H_{(u,l,m)},X)$.*

In conclusion, we have two open questions.

Open question 1. *Prove or disprove:*

i. $MQ_k^3(H_m,X)=hM_m(k,3)$.

ii. $MQ_k^4(H_{(u,l,m)},X)=hM_u(k,4)$.

Open question 2. *Is it possible to prove that each finite group G divides the minimal period of the generalized order (k,t)-Mersenne number sequence?*

References

[1] Bennett, M. A., Patel, V., & Siksek, S. (2019). Shifted powers in Lucas–Lehmer sequences. *Research in Number Theory*, 5(1), Article ID 15.

[2] Catarino, P., Campos, H., & Vasco, P. (2016). On the Mersenne sequence. *Annales Mathematicae et Informaticae*, 46, 37–53.

[3] Deveci, Ö., & Shannon, A. G. (2018). The quaternion-Pell sequence. *Communications in Algebra*, 46(12), 5403–5409.

[4] Doostie, H. & Hashemi, M. (2006). Fibonacci lengths involving the Wall number $k(n)$. *Journal of Applied Mathematics and Computing*, 20(1–2), 171–180.

[5] Esmaeili, M., Moosavi, M., & Gulliver, T. A. (2017). A new class of Fibonacci sequence based error correcting codes. *Cryptography and Communications*, 9(3), 379–396.

[6] Falcon, S. (2013). On the generating matrices of the k-Fibonacci numbers. *Proyecciones*, 32(4), 347–357.

[7] Gautam, R. (2018). Balancing numbers and application. *Journal of Advanced College of Engineering and Management*, 4, 137–143.

[8] Goy, T. (2018). On new identities for Mersenne numbers. *Applied Mathematics E-Notes*, 18, 100–105.

[9] Hashemi, M., & Mehraban, E. (2021). The generalized order k-Pell sequences in some special groups of nilpotency class 2. *Communications in Algebra*, 50(4), 1768–1784.

[10] Hashemi, M., & Mehraban, E. (2023). Fibonacci length and the generalized order k-Pell sequences of the 2-generator p-groups of nilpotency class 2. *Journal of Algebra and Its Applications*, 22(3), Article ID 2350061.

[11] Jaroma, J. H., & Reddy, K. N. (2007). Classical and alternative approaches to the Mersenne and Fermat numbers. *The American Mathematical Monthly*, 114(8), 677–687.

[12] Koshy, T., & Gao, Z. (2013). Catalan numbers with Mersenne subscripts. *Mathematical Scientist*, 38(2), 86–91.

[13] Ochalik, P., & Wloch, A. (2018). On generalized Mersenne numbers, their interpretations and matrix generators. *Annales Universitatis Mariae Curie-Skłodowska. Section A*, 1, 69–76.

[14] Osipov, D. V. (2015). The discrete Heisenberg group and its automorphism group. *Mathematical Notes*, 98(1–2), 185–188.

[15] Prasad, K., & Mahato, H. (2022). Cryptography using generalized Fibonacci matrices with Affine-Hill cipher. *Journal of Discrete Mathematical Sciences and Cryptography*, 25(8), 2341–2352.

[16] Shannon, A. G., Erdağ, Ö., & Deveci, Ö. (2021). On the connections between Pell numbers and Fibonacci p-numbers. *Notes on Number Theory and Discrete Mathematics*, 21(1), 148–160.

[17] Stakhov, A. P. (2006). Fibonacci matrices, a generalization of the Cassini formula and new coding theory. *Chaos, Solitions & Fractals*, 30(1), 56–66.

[18] Taşci, D., Tuğlu, N., & Asci, M. (2011). On Fibo–Pascal matrix involving k-Fibonacci and k-Pell matrices. *Arabian Journal for Science and Engineering*, 36, 1031–1037.