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On some identities for the $\mathcal{DGC}$ Leonardo sequence

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Abstract: In this study, we examine the Leonardo sequence with dual-generalized complex ($\mathcal{DGC}$) coefficients for $\mathfrak{p} \in \mathbb{R}$. Firstly, we express some summation formulas related to the $\mathcal{DGC}$ Fibonacci, $\mathcal{DGC}$ Lucas, and $\mathcal{DGC}$ Leonardo sequences. Secondly, we present some order-2 characteristic relations, involving d'Ocagne's, Catalan's, Cassini's, and Tagiuri's identities. The essential point of the paper is that one can reduce the calculations of the $\mathcal{DGC}$ Leonardo sequence by considering $\mathfrak{p}$. This generalization gives the dual-complex Leonardo sequence for $\mathfrak{p} = -1$, hyper-dual Leonardo sequence for $\mathfrak{p} = 0$, and dual-hyperbolic Leonardo sequence for $\mathfrak{p} = 1$.
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1 Basic notations and literature review

Integer sequences and their applications are widely used in technology, nature, biology, theoretical physics, and chemistry. Among the other sequences, the Fibonacci and Lucas sequences have played an important role in number theory. One can see some basic notations and results related to Fibonacci and Lucas numbers in [4, 11, 18, 24, 42]. The Fibonacci sequence is defined by the following linear recurrence relation

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2 \tag{1}$$

with the initial values $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has been generalized in many ways, some by preserving the initial values, and others by preserving the recurrence relation.The Lucas sequence is defined recursively by:

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2 \tag{2}$$

with the initial values $L_0 = 2$ and $L_1 = 1$. As it seen from equations (1) and (2), the Fibonacci and Lucas sequences satisfy the same recurrence relation with different initial values. The Binet's formulas of the Fibonacci and Lucas sequences are as follows, respectively:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{3}$$

and

$$L_n = \alpha^n + \beta^n, \tag{4}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are roots of the characteristic equation $\lambda^2 - \lambda - 1 = 0$, [24]. These two sequences have more in common than their recursive structure.

The Leonardo sequence is another integer sequence which is related to the Fibonacci sequence and also to the Lucas sequence. The Leonardo sequence is given by the following non-homogeneous recurrence relation:

$$Le_n = Le_{n-1} + Le_{n-2} + 1,\ n \geq 2 \tag{5}$$

with the initial values $Le_0 = Le_1 = 1$. The homogeneous recurrence relation of the Leonardo sequence is

$$Le_{n+1} = 2Le_n - Le_{n-2},\ n \geq 2 \tag{6}$$

with the initial values $Le_0 = Le_1 = 1$ and $Le_2 = 3$, [5]. The Binet formula of the Leonardo sequence is

$$Le_n = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1.$$

There are many well-known and established relations between the Fibonacci, Lucas, and Leonardo sequences. For a positive integer n, the fundamental relations between them are as follows [5]:

$$\begin{cases} Le_n = & 2F_{n+1} - 1, \\ Le_n = & 2\left(\dfrac{L_n + L_{n+2}}{5}\right) - 1, \\ Le_{n+3} = & \dfrac{L_{n+1} + L_{n+7}}{5} - 1, \\ Le_n = & L_{n+2} - F_{n+2} - 1, \end{cases} \tag{7}$$

and the summation formulas are (see [5, 7]):

$$\begin{cases} \sum_{j=0}^{n} F_j &= F_n + F_{n+1} - 1, \\ \sum_{j=0}^{n} L_j &= L_n + L_{n+1} - 1, \\ \sum_{j=0}^{n} Le_j &= Le_{n+2} - (n+2). \end{cases} \tag{8}$$

Characteristic properties and some generalizations of the Leonardo sequence have been studied by researchers. In 2019, the Leonardo sequence was examined in detail by P. Catarino and A. Borges in [5]. A. G. Shannon studied the generalization of the Leonardo sequence, [33]. P. Catarino and A. Borges defined incomplete Leonardo numbers and analyzed their recurrence relations, some properties, and the generating functions in [6]. The generating matrices and the matrix form of the Leonardo sequence were investigated in [43]. Y. Alp and E. G. Köçer introduced the hybrid Leonardo sequence in [2]. The relations among Fibonacci, Lucas, and Leonardo numbers were given in [3]. In 2021, the relations between the hybrid Leonardo sequence and the hybrid Fibonacci sequence, Catalan's, Cassini's, and d'Ocagne's identities were presented in [26]. The elliptical biquaternion Leonardo sequence was discussed in [27], and the hyperbolic Leonardo sequence was examined in [44]. In 2022, the hybrid quaternion Leonardo sequence and their identities were presented in [28]. The real and complex generalizations of the Leonardo sequence were introduced in [34]. A. Karataş discussed the complex Leonardo sequence and their special identities in [22]. S. Ö. Karakuş, S. K. Nurkan and M. Tosun characterised the hyper-dual Leonardo sequence in [21]. M. Shattuck provided combinatorial proofs of some important identities satisfied by the generalized Leonardo numbers in [35]. G.Y. Şentürk presented a brief study on the Leonardo sequence with dual-generalized complex coefficients in [40]. In 2023, S. Kaya Nurkan and I. A. Güven combined the Leonardo sequence and dual quaternions, [30]. Y. Soykan defined the modified p-Leonardo, p-Leonardo-Lucas, and p-Leonardo sequences as special cases of the generalized Leonardo sequence in [38]. H. Özimamoğlu introduced the hybrid q-Leonardo sequence by using q-integers in [31]. E. Tan and H. H. Leung investigated the Leonardo p-sequence, and incomplete Leonardo p-sequence in [41]. Z. İşbilir, M. Akyiğit and M. Tosun investigated the Pauli–Leonardo quaternion sequence in [19]. A. Karakaş defined the concept of dual Leonardo numbers in [23].

In the literature, one can see that the various hypercomplex number systems are used as components in generalizing sequences (see [2, 21–23, 26–28, 30, 31, 34, 40, 44] for some hypercomplex Leonardo numbers). The set of generalized complex numbers is given by:

$$\mathbb{C}_{\mathfrak{p}} := \left\{ z = a + bJ : \ a, b \in \mathbb{R}, \ J^2 = \mathfrak{p}, \ \mathfrak{p} \in \mathbb{R}, J \notin \mathbb{R} \right\}$$

and examined in [17, 20]. The set of generalized complex numbers forms an associative and commutative algebra of dimension 2 over $\mathbb{R}$ and includes other well-known 2-dimensional algebras as special cases. It is analogous to the set of:

- complex numbers $\mathbb{C}$ with elements $z = a + bi$, $i^2 = -1$ for $\mathfrak{p} = -1$ (see [45]),

- dual numbers $\mathbb{D}$ with elements $z = a + b\varepsilon$, $\varepsilon^2 = 0, \varepsilon \neq 0$ for $\mathfrak{p} = 0$ (see [32, 39, 46]),
- hyperbolic (perplex, double, split complex, binary) numbers $\mathbb{H}$ with elements $z = a + bj$, $j^2 = 1, j \neq 1$ for $\mathfrak{p} = 1$ (see [14, 37, 46]).

The set $\mathbb{C}_\mathfrak{p}$ is a vector space over $\mathbb{R}$. All these special number systems have led to the construction of some 4-dimensional number systems. By utilizing the generalized complex and dual numbers, the dual-generalized complex ($\mathcal{DGC}$) numbers are introduced in [15]. The set of $\mathcal{DGC}$ numbers was introduced as:

$$\mathbb{DC}_\mathfrak{p} := \left\{ \tilde{a} = z_1 + z_2\varepsilon : \ z_1, z_2 \in \mathbb{C}_\mathfrak{p}, \ \varepsilon^2 = 0, \, \varepsilon \neq 0, \varepsilon \notin \mathbb{R} \right\}.$$

and discussed in [15]. Any $\mathcal{DGC}$ number is of the form $\tilde{a} = z_1 + z_2\varepsilon = a_1 + a_2J + a_3\varepsilon + a_4J\varepsilon$. It gives the set of:

- dual-complex numbers with elements $\tilde{a} = a_1 + a_2 i + a_3\varepsilon + a_4 i\varepsilon$ for $\mathfrak{p} = -1$ (see [8,9,25,29]),
- hyper-dual numbers with elements $\tilde{a} = a_1 + a_2\epsilon + a_3\varepsilon + a_4\epsilon\varepsilon$ for $\mathfrak{p} = 0$ (see [10, 12, 13]),
- dual-hyperbolic numbers with elements $\tilde{a} = a_1 + a_2 j + a_3\varepsilon + a_4 j\varepsilon$ for $\mathfrak{p} = 1$ (see [1, 25]).

The base elements $\{1, J, \varepsilon, J\varepsilon\}$ satisfy the following multiplication rules (see [15]):

$$J^2 = \mathfrak{p}, \quad (J\varepsilon)^2 = 0, \quad J\varepsilon = \varepsilon J. \tag{9}$$

For $\tilde{a}_1 = z_{11} + z_{12}\varepsilon, \tilde{a}_2 = z_{21} + z_{22}\varepsilon \in \mathbb{DC}_\mathfrak{p}$ and $\lambda \in \mathbb{R}$, the basic algebraic operations can be given as follows:

- the equality: $\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow z_{11} = z_{21}, z_{12} = z_{22}$,
- the addition (and hence subtraction): $\tilde{a}_1 \pm \tilde{a}_2 = (z_{11} \pm z_{21}) + (z_{12} \pm z_{22})\,\varepsilon$,
- the scalar multiplication: $\lambda\tilde{a}_1 = \lambda z_{11} + \lambda z_{12}\varepsilon$,
- the multiplication: $\tilde{a}_1\tilde{a}_2 = (z_{11}z_{21}) + (z_{11}z_{22} + z_{12}z_{21})\,\varepsilon$.

The set of $\mathcal{DGC}$ numbers forms an associative and commutative ring with unity and a vector space of dimension 4 over $\mathbb{R}$, [15].

According to the above statements, we now give the $\mathcal{DGC}$ Fibonacci and $\mathcal{DGC}$ Lucas sequences. The n-th $\mathcal{DGC}$ Fibonacci number is of the form

$$\tilde{\mathcal{F}}_n = F_n + F_{n+1}J + F_{n+2}\varepsilon + F_{n+3}J\varepsilon, \tag{10}$$

and satisfies the recurrence relation $\tilde{\mathcal{F}}_n = \tilde{\mathcal{F}}_{n-1} + \tilde{\mathcal{F}}_{n-2}, \ n \geq 2$. The Binet's formula of the $\mathcal{DGC}$ Fibonacci sequence is

$$\tilde{\mathcal{F}}_n = \frac{\tilde{\alpha}\alpha^n - \tilde{\beta}\beta^n}{\alpha - \beta}, \tag{11}$$

where $\tilde{\alpha} = 1 + \alpha J + \alpha^2\varepsilon + \alpha^3 J\varepsilon$ and $\tilde{\beta} = 1 + \beta J + \beta^2\varepsilon + \beta^3 J\varepsilon$. Similarly, the n-th $\mathcal{DGC}$ Lucas number is of the form

$$\tilde{\mathcal{L}}_n = L_n + L_{n+1}J + L_{n+2}\varepsilon + L_{n+3}J\varepsilon, \tag{12}$$

and satisfies the reccurence relation $\tilde{\mathcal{L}}_n = \tilde{\mathcal{L}}_{n-1} + \tilde{\mathcal{L}}_{n-2}$ $n \geq 2$. The Binet's formula of the $\mathcal{DGC}$ Lucas sequence is

$$\widetilde{\mathcal{L}}_n = \tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n,$$

where $\tilde{\alpha} = 1 + \alpha J + \alpha^2\varepsilon + \alpha^3 J\varepsilon$ and $\tilde{\beta} = 1 + \beta J + \beta^2\varepsilon + \beta^3 J\varepsilon$, [16].

In this study, inspired by the theory of the hypercomplex sequences, we examined the concept of the $\mathcal{DGC}$ Leonardo sequence for $\mathfrak{p} \in \mathbb{R}$. We gave the characteristic formulas, involving d'Ocagne's, Catalan's, Cassini's, and Tagiuri's identities for the $\mathcal{DGC}$ Leonardo sequence.

2 Preliminaries

In this section, we follow [40] in presenting the basic notions of the $\mathcal{DGC}$ Leonardo sequence.

The n-th $\mathcal{DGC}$ Leonardo number is of the form

$$\tilde{\mathcal{L}}e_n = Le_n + Le_{n+1}J + Le_{n+2}\varepsilon + Le_{n+3}J\varepsilon. \tag{13}$$

The $\mathcal{DGC}$ Leonardo numbers satisfy the following second-order non-homogeneous relation

$$\tilde{\mathcal{L}}e_n = \tilde{\mathcal{L}}e_{n-1} + \tilde{\mathcal{L}}e_{n-2} + \tilde{1},\ n \geq 2, \tag{14}$$

where $\tilde{1} = 1 + J + \varepsilon + J\varepsilon$ with the initial values $\tilde{\mathcal{L}}e_0 = 1 + J + 3\varepsilon + 5J\varepsilon$, $\tilde{\mathcal{L}}e_1 = 1 + 3J + 5\varepsilon + 9J\varepsilon$. The homogeneous recurrence relation is

$$\tilde{\mathcal{L}}e_{n+1} = 2\tilde{\mathcal{L}}e_n - \tilde{\mathcal{L}}e_{n-2}$$

with the initial values $\tilde{\mathcal{L}}e_0 = 1 + J + 3\varepsilon + 5J\varepsilon$, $\tilde{\mathcal{L}}e_1 = 1 + 3J + 5\varepsilon + 9J\varepsilon$ and $\tilde{\mathcal{L}}e_2 = 3 + 5J + 9\varepsilon + 15J\varepsilon$. See basic notations in Table 1.

Table 1. Nomenclature of the sequences

The sequence	**The general term (n-th term)**
The Fibonacci sequence	F_n
The Lucas sequence	L_n
The Leonardo sequence	Le_n
The $\mathcal{DGC}$ Fibonacci sequence	$\tilde{\mathcal{F}}_n$
The $\mathcal{DGC}$ Lucas sequence	$\tilde{\mathcal{L}}_n$
The $\mathcal{DGC}$ Leonardo sequence	$\tilde{\mathcal{L}}e_n$

In the following Table 2, the several terms of the Fibonacci (see A000045 in [36]), Lucas (see A000032 in [36]), Leonardo (see A001595 in [36]), $\mathcal{DGC}$ Fibonacci, $\mathcal{DGC}$ Lucas, and $\mathcal{DGC}$ Leonardo sequences are given, respectively.

Table 2. Several terms for the Fibonacci, Lucas, Leonardo, $\mathcal{DGC}$ Fibonacci, $\mathcal{DGC}$ Lucas, and $\mathcal{DGC}$ Leonardo sequences

n	0	1	2	3
F_n	0	1	1	2
L_n	2	1	3	4
Le_n	1	1	3	5
$\tilde{\mathcal{F}}_n$	$J+\varepsilon+2J\varepsilon$	$1+J+2\varepsilon+3J\varepsilon$	$1+2J+3\varepsilon+5J\varepsilon$	$2+3J+5\varepsilon+8J\varepsilon$
$\tilde{\mathcal{L}}_n$	$2+J+3\varepsilon+4J\varepsilon$	$1+3J+4\varepsilon+7J\varepsilon$	$3+4J+7\varepsilon+11J\varepsilon$	$4+7J+11\varepsilon+18J\varepsilon$
$\tilde{\mathcal{L}}e_n$	$1+J+3\varepsilon+5J\varepsilon$	$1+3J+5\varepsilon+9J\varepsilon$	$3+5J+9\varepsilon+15J\varepsilon$	$5+9J+15\varepsilon+25J\varepsilon$

Theorem 2.1. *For a positive integer n, the fundamental relations between the $\mathcal{DGC}$ Fibonacci, Lucas, and Leonardo numbers are:*

$$\begin{cases} \tilde{\mathcal{L}}e_n = & 2\tilde{\mathcal{F}}_{n+1} - \tilde{1}, \\ \tilde{\mathcal{L}}e_n = & 2\left(\dfrac{\tilde{\mathcal{L}}_n + \tilde{\mathcal{L}}_{n+2}}{5}\right) - \tilde{1}, \\ \tilde{\mathcal{L}}e_{n+3} = & \dfrac{\tilde{\mathcal{L}}_{n+1} + \tilde{\mathcal{L}}_{n+7}}{5} - \tilde{1}, \\ \tilde{\mathcal{L}}e_n = & \tilde{\mathcal{L}}_{n+2} - \tilde{\mathcal{F}}_{n+2} - \tilde{1}. \end{cases} \tag{15}$$

Theorem 2.2. *For a positive integer n, the summation formulas are:*

1. $\sum_{j=0}^{n} \tilde{\mathcal{L}}e_{2j} = \tilde{\mathcal{L}}e_{2n+1} - \tilde{\mathcal{N}}_n - (J + J\varepsilon)$,

2. $\sum_{j=0}^{n} \tilde{\mathcal{L}}e_{2j+1} = \tilde{\mathcal{L}}e_{2n+2} - \tilde{\mathcal{N}}_{n+2} + (J - J\varepsilon)$,

where $\tilde{\mathcal{N}}_n = n + (n+1)J + (n+2)\varepsilon + (n+3)J\varepsilon$.

Theorem 2.3. *For a positive integer n, the Binet's formula for the $\mathcal{DGC}$ Leonardo sequence is*

$$\tilde{\mathcal{L}}e_n = 2\left(\frac{\tilde{\alpha}\alpha^{n+1} - \tilde{\beta}\beta^{n+1}}{\alpha - \beta}\right) - \tilde{1}, \tag{16}$$

where $\tilde{\alpha} = 1 + \alpha J + \alpha^2\varepsilon + \alpha^3 J\varepsilon$ and $\tilde{\beta} = 1 + \beta J + \beta^2\varepsilon + \beta^3 J\varepsilon$.

3 Some new identities for the $\mathcal{DGC}$ Leonardo Sequence

In the following theorems, we touch only a new aspect to the $\mathcal{DGC}$ Leonardo sequence.

Theorem 3.1. *For a positive integer n, the summation formulas related to the $\mathcal{DGC}$ Fibonacci, $\mathcal{DGC}$ Lucas, and $\mathcal{DGC}$ Leonardo sequences are as follows:*

1. $\sum_{j=0}^{n} \tilde{\mathcal{F}}_j = \tilde{\mathcal{F}}_{n+2} - \tilde{\mathcal{F}}_1$,

2. $\sum_{j=0}^{n} \tilde{\mathcal{L}}_j = \tilde{\mathcal{L}}_{n+2} - \tilde{\mathcal{L}}_1$,

3. $\sum_{j=0}^{n} \tilde{\mathcal{L}}e_j = \tilde{\mathcal{L}}e_{n+2} - \tilde{\mathcal{N}}_{n+2} - (J + 2\varepsilon + 5J\varepsilon)$,

4. $\sum_{j=0}^{n} \left(\tilde{\mathcal{L}}e_j + \tilde{\mathcal{F}}_j\right) = \tilde{\mathcal{L}}e_{n+2} + \tilde{\mathcal{F}}_{n+2} - \tilde{\mathcal{N}}_{n+3} - (J + 3\varepsilon + 7J\varepsilon)$,

5. $\sum_{j=0}^{n} \left(\tilde{\mathcal{L}}e_j + \tilde{\mathcal{F}}_j\right) = \tilde{\mathcal{F}}_{n+5} - \tilde{\mathcal{N}}_{n+4} - (J + 3\varepsilon + 7J\varepsilon)$,

6. $\sum_{j=0}^{n} \left(\tilde{\mathcal{L}}e_j + \tilde{\mathcal{L}}_j\right) = \tilde{\mathcal{L}}e_{n+2} + \tilde{\mathcal{L}}_{n+2} - \tilde{\mathcal{N}}_{n+3} - (3J + 5\varepsilon + 11J\varepsilon)$,

7. $\sum_{j=0}^{n} \left(\tilde{\mathcal{L}}e_j + \tilde{\mathcal{L}}_j\right) = \dfrac{7\tilde{\mathcal{L}}_{n+2} + 2\tilde{\mathcal{L}}_{n+4}}{5} - \tilde{\mathcal{N}}_{n+4} - (3J + 5\varepsilon + 11J\varepsilon)$,

where $\tilde{\mathcal{N}}_n = n + (n+1)J + (n+2)\varepsilon + (n+3)J\varepsilon$.

Proof. 3. According to Equation (13), we have:

$$
\begin{aligned}
\sum_{j=0}^{n} \tilde{\mathcal{L}}e_j = {} & \tilde{\mathcal{L}}e_0 + \tilde{\mathcal{L}}e_1 + \tilde{\mathcal{L}}e_2 + \cdots + \tilde{\mathcal{L}}e_n \\
= {} & (Le_0 + Le_1 J + Le_2\varepsilon + Le_3 J\varepsilon) + (Le_1 + Le_2 J + Le_3\varepsilon + Le_4 J\varepsilon) + \cdots \\
& + (Le_n + Le_{n+1} J + Le_{n+2}\varepsilon + Le_{n+3} J\varepsilon) \\
= {} & (Le_0 + Le_1 + \cdots + Le_n) + (Le_1 + Le_2 + \cdots + Le_{n+1})\, J \\
& + (Le_2 + Le_3 + \cdots + Le_{n+2})\,\varepsilon + (Le_3 + Le_4 + \cdots + Le_{n+3})\, J\varepsilon.
\end{aligned}
$$

Then, from Equation (8), we get:

$$
\begin{aligned}
\sum_{j=0}^{n} \tilde{\mathcal{L}}e_j = {} & \sum_{j=0}^{n} Le_j + \left(\sum_{j=0}^{n} Le_{j+1}\right) J + \left(\sum_{j=0}^{n} Le_{j+2}\right) \varepsilon + \left(\sum_{j=0}^{n} Le_{j+3}\right) J\varepsilon \\
= {} & Le_{n+2} - (n+2) + \left(\sum_{j=0}^{n+1} Le_j - Le_0\right) J - \left(\sum_{j=0}^{n+2} Le_j - Le_0 - Le_1\right) \varepsilon \\
& + \left(\sum_{j=0}^{n+3} Le_j - Le_0 - Le_1 - Le_2\right) J\varepsilon \\
= {} & Le_{n+2} + Le_{n+3} J + Le_{n+4}\varepsilon + Le_{n+5} J\varepsilon \\
& - ((n+2) + (n+4)J + (n+6)\varepsilon + (n+10)J\varepsilon) \\
= {} & Le_{n+2} + Le_{n+3} J + Le_{n+4}\varepsilon + Le_{n+5} J\varepsilon \\
& - ((n+2) + (n+3)J + (n+4)\varepsilon + (n+5)J\varepsilon) - (J + 2\varepsilon + 5J\varepsilon)) \\
= {} & \tilde{\mathcal{L}}e_{n+2} - \tilde{\mathcal{N}}_{n+2} - (J + 2\varepsilon + 5J\varepsilon).
\end{aligned}
$$

4. Using Theorem 3.1, items 1 and 3, we obtain:

$$\begin{aligned}\sum_{j=0}^{n}\left(\tilde{\mathcal{L}}e_j+\tilde{\mathcal{F}}_j\right) &= \sum_{j=0}^{n}\tilde{\mathcal{F}}_j+\sum_{j=0}^{n}\tilde{\mathcal{L}}e_j\\ &= \tilde{\mathcal{F}}_{n+2}-\tilde{\mathcal{F}}_1+\tilde{\mathcal{L}}e_{n+2}-\tilde{\mathcal{N}}_{n+2}-(J+2\varepsilon+5J\varepsilon)\\ &= \tilde{\mathcal{F}}_{n+2}-(1+J+2\varepsilon+3J\varepsilon)+\tilde{\mathcal{L}}e_{n+2}-\tilde{\mathcal{N}}_{n+2}-(J+2\varepsilon+5J\varepsilon)\\ &= \tilde{\mathcal{F}}_{n+2}+\tilde{\mathcal{L}}e_{n+2}-\tilde{\mathcal{N}}_{n+3}-(J+3\varepsilon+7J\varepsilon).\end{aligned}$$

5. From Equation (15), we get:

$$\begin{aligned}\sum_{j=0}^{n}\left(\tilde{\mathcal{L}}e_j+\tilde{\mathcal{F}}_j\right) &= \tilde{\mathcal{F}}_{n+2}-(1+J+2\varepsilon+3J\varepsilon)+\tilde{\mathcal{L}}e_{n+2}-\tilde{\mathcal{N}}_{n+2}-(J+2\varepsilon+5J\varepsilon)\\ &= \tilde{\mathcal{F}}_{n+2}-(\varepsilon+2J\varepsilon)-\tilde{\mathcal{N}}_{n+3}+2\tilde{\mathcal{F}}_{n+3}-\tilde{1}-(J+2\varepsilon+5J\varepsilon)\\ &= \tilde{\mathcal{F}}_{n+5}-\tilde{\mathcal{N}}_{n+3}-(1+2J+4\varepsilon+8J\varepsilon)\\ &= \tilde{\mathcal{F}}_{n+5}-\tilde{\mathcal{N}}_{n+4}-(J+3\varepsilon+7J\varepsilon).\end{aligned}$$

Similarly, the proofs of the other items are simple calculations, so we can omit them. □

Theorem 3.2. *For a positive integer n, the following order-2 summation formulas can be given:*

1. $\sum_{j=0}^{n}\left(\tilde{\mathcal{F}}_j\right)^2=\tilde{\mathcal{F}}_n\tilde{\mathcal{F}}_{n+1}-(J+(1+\mathfrak{p})\varepsilon+3J\varepsilon)$,

2. $\sum_{j=0}^{n}\left(\tilde{\mathcal{L}}_j\right)^2=\tilde{\mathcal{L}}_n\tilde{\mathcal{L}}_{n+1}-((-2+2\mathfrak{p})+3J+(11\mathfrak{p}-1)\varepsilon+9J\varepsilon)$,

3. $\sum_{j=0}^{n}\left(\tilde{\mathcal{L}}e_j\right)^2=\tilde{\mathcal{L}}e_n\tilde{\mathcal{L}}e_{n+1}+(\tilde{\mathcal{N}}_{n+3}-\tilde{\mathcal{L}}e_{n+2})\tilde{1}+(-\mathfrak{p}-J+2(1-2\mathfrak{p})\varepsilon+2J\varepsilon)$,

where $\tilde{\mathcal{N}}_n=n+(n+1)J+(n+2)\varepsilon+(n+3)J\varepsilon$.

Proof. 3. Using Equation (14), we have the following equations:

$$\begin{cases}(\tilde{\mathcal{L}}e_0)^2= & \tilde{\mathcal{L}}e_0\tilde{\mathcal{L}}e_0\\ (\tilde{\mathcal{L}}e_1)^2= & \tilde{\mathcal{L}}e_1\tilde{\mathcal{L}}e_1=\tilde{\mathcal{L}}e_1(\tilde{\mathcal{L}}e_2-\tilde{\mathcal{L}}e_0-\tilde{1})=\tilde{\mathcal{L}}e_1\tilde{\mathcal{L}}e_2-\tilde{\mathcal{L}}e_1\tilde{\mathcal{L}}e_0-\tilde{\mathcal{L}}e_1\tilde{1}\\ (\tilde{\mathcal{L}}e_2)^2= & \tilde{\mathcal{L}}e_2\tilde{\mathcal{L}}e_2=\tilde{\mathcal{L}}e_2(\tilde{\mathcal{L}}e_3-\tilde{\mathcal{L}}e_1-\tilde{1})=\tilde{\mathcal{L}}e_2\tilde{\mathcal{L}}e_3-\tilde{\mathcal{L}}e_2\tilde{\mathcal{L}}e_1-\tilde{\mathcal{L}}e_2\tilde{1}\\ (\tilde{\mathcal{L}}e_3)^2= & \tilde{\mathcal{L}}e_3\tilde{\mathcal{L}}e_3=\tilde{\mathcal{L}}e_3(\tilde{\mathcal{L}}e_4-\tilde{\mathcal{L}}e_2-\tilde{1})=\tilde{\mathcal{L}}e_3\tilde{\mathcal{L}}e_4-\tilde{\mathcal{L}}e_3\tilde{\mathcal{L}}e_2-\tilde{\mathcal{L}}e_3\tilde{1}\\ \quad\vdots & \\ (\tilde{\mathcal{L}}e_n)^2= & \tilde{\mathcal{L}}e_n(\tilde{\mathcal{L}}e_{n+1}-\tilde{\mathcal{L}}e_{n-1}-\tilde{1})=\tilde{\mathcal{L}}e_n\tilde{\mathcal{L}}e_{n+1}-\tilde{\mathcal{L}}e_n\tilde{\mathcal{L}}e_{n-1}-\tilde{\mathcal{L}}e_n\tilde{1}.\end{cases}$$

We thus get:

$$\sum_{j=0}^{n}\left(\tilde{\mathcal{L}}e_j\right)^2=\tilde{\mathcal{L}}e_n\tilde{\mathcal{L}}e_{n+1}-\tilde{\mathcal{L}}e_0(\tilde{\mathcal{L}}e_1-\tilde{\mathcal{L}}e_0)-(\tilde{\mathcal{L}}e_1+\tilde{\mathcal{L}}e_2+\cdots+\tilde{\mathcal{L}}e_n)\tilde{1}.$$

By using $\tilde{\mathcal{L}}e_0(\tilde{\mathcal{L}}e_1 - \tilde{\mathcal{L}}e_0) = 2\mathfrak{p} + 2J + (2+14\mathfrak{p})\varepsilon + 12J\varepsilon$ and Theorem 3.1, item 3, we have

$$\sum_{j=0}^{n} \tilde{\mathcal{L}}e_j - \tilde{\mathcal{L}}e_0 = \tilde{\mathcal{L}}e_{n+2} - \tilde{\mathcal{N}}_{n+2} - (1 + 2J + 5\varepsilon + 10J\varepsilon).$$

We thus get:

$$\begin{aligned}\sum_{j=0}^{n}\left(\tilde{\mathcal{L}}e_j\right)^2 &= \tilde{\mathcal{L}}e_n\tilde{\mathcal{L}}e_{n+1} - (2\mathfrak{p} + 2J + (2+14\mathfrak{p})\varepsilon + 12J\varepsilon)\\ &\quad - (\tilde{\mathcal{L}}e_{n+2} - \tilde{\mathcal{N}}_{n+3} - (J + 4\varepsilon + 9J\varepsilon))\tilde{1}\\ &= \tilde{\mathcal{L}}e_n\tilde{\mathcal{L}}e_{n+1} + (\tilde{\mathcal{N}}_{n+3} - \tilde{\mathcal{L}}e_{n+2})\tilde{1} + (-\mathfrak{p} - J + 2(1-2\mathfrak{p})\varepsilon + 2J\varepsilon).\end{aligned}$$

A similar proof can be used to verify the other items. □

3.1 Main results

In this subsection, we formulate the order-2 formulas for the $\mathcal{DGC}$ Leonardo sequence.

Theorem 3.3. *For n and m positive integers, with $m \geqslant n$, the following identities are true:*

1. $\tilde{\mathcal{L}}e_{n+1}^2 - \tilde{\mathcal{L}}e_n^2 = 4\tilde{\mathcal{F}}_n(\tilde{\mathcal{F}}_{n+3} - \tilde{1})$,
2. $\tilde{\mathcal{L}}e_{m+n}^2 - \tilde{\mathcal{L}}e_{m-n}^2 = 4F_{2n}\tilde{\mathcal{F}}_{m+1}\tilde{\mathcal{L}}_{m+1} - 2(\tilde{\mathcal{L}}e_{m+n} - \tilde{\mathcal{L}}e_{m-n})\tilde{1}$,
3. $\tilde{\mathcal{L}}e_{n+1}\tilde{\mathcal{F}}_{n+1} - \tilde{\mathcal{L}}e_n\tilde{\mathcal{F}}_n = \tilde{\mathcal{L}}e_n\tilde{\mathcal{F}}_{n-1} + 2\tilde{\mathcal{F}}_{n+1}\tilde{\mathcal{F}}_n$.

Proof. The main idea of the proof is to take the relation $\tilde{\mathcal{L}}e_n = 2\tilde{\mathcal{F}}_{n+1} - \tilde{1}$ (see Theorem 2.1).

1. Applying the $\mathcal{DGC}$ Fibonacci recurrence relation, we can assert that:

$$\begin{aligned}\tilde{\mathcal{L}}e_{n+1}^2 - \tilde{\mathcal{L}}e_n^2 &= (2\tilde{\mathcal{F}}_{n+2} - \tilde{1})^2 - (2\tilde{\mathcal{F}}_{n+1} - \tilde{1})^2\\ &= 4((\tilde{\mathcal{F}}_{n+2} - \tilde{\mathcal{F}}_{n+1})(\tilde{\mathcal{F}}_{n+2} + \tilde{\mathcal{F}}_{n+1}) - \tilde{\mathcal{F}}_n\tilde{1})\\ &= 4\tilde{\mathcal{F}}_n(\tilde{\mathcal{F}}_{n+3} - \tilde{1}).\end{aligned}$$

2. By using the relations

$$\widetilde{\mathcal{F}}_{n+r} + \widetilde{\mathcal{F}}_{n-r} = \begin{cases} L_r\widetilde{\mathcal{F}}_n, & r = 2k\\ F_r\widetilde{\mathcal{L}}_n, & r = 2k+1,\end{cases}$$

and

$$\widetilde{\mathcal{F}}_{n+r} - \widetilde{\mathcal{F}}_{n-r} = \begin{cases} F_r\widetilde{\mathcal{L}}_n, & r = 2k\\ L_r\widetilde{\mathcal{F}}_n, & r = 2k+1,\end{cases}$$

(see Theorem 2, items 2 and 3, in [16]), we see that

$$\begin{aligned}\tilde{\mathcal{L}}e_{m+n}^2 - \tilde{\mathcal{L}}e_{m-n}^2 &= (2\tilde{\mathcal{F}}_{m+n+1} - \tilde{1})^2 - (2\tilde{\mathcal{F}}_{m-n+1} - \tilde{1})^2\\ &= 4(\tilde{\mathcal{F}}_{m+n+1} - \tilde{\mathcal{F}}_{m-n+1})(\tilde{\mathcal{F}}_{m+n+1} + \tilde{\mathcal{F}}_{m-n+1}) - 2(\tilde{\mathcal{L}}e_{m+n} - \tilde{\mathcal{L}}e_{m-n})\tilde{1}\\ &= 4F_nL_n\tilde{\mathcal{F}}_{m+1}\tilde{\mathcal{L}}_{m+1} - 2(\tilde{\mathcal{L}}e_{m+n} - \tilde{\mathcal{L}}e_{m-n})\tilde{1}.\end{aligned}$$

We conclude from $F_nL_n = F_{2n}$ (see [24]) that

$$\tilde{\mathcal{L}}e_{m+n}^2 - \tilde{\mathcal{L}}e_{m-n}^2 = 4F_{2n}\tilde{\mathcal{F}}_{m+1}\tilde{\mathcal{L}}_{m+1} - 2(\tilde{\mathcal{L}}e_{m+n} - \tilde{\mathcal{L}}e_{m-n})\tilde{1}.$$

3. From the $\mathcal{DGC}$ Fibonacci recurrence relation, we have

$$\begin{aligned}\tilde{\mathcal{L}}e_{n+1}\tilde{\mathcal{F}}_{n+1} - \tilde{\mathcal{L}}e_n\tilde{\mathcal{F}}_n &= (2\tilde{\mathcal{F}}_{n+2} - \tilde{1})\tilde{\mathcal{F}}_{n+1} - (2\tilde{\mathcal{F}}_{n+1} - \tilde{1})\tilde{\mathcal{F}}_n \\ &= 2\tilde{\mathcal{F}}_{n+1}(\tilde{\mathcal{F}}_{n+2} - \tilde{\mathcal{F}}_n) - \tilde{1}(\tilde{\mathcal{F}}_{n+1} - \tilde{\mathcal{F}}_n) \\ &= 2\tilde{\mathcal{F}}_{n+1}(\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_{n-1}) - \tilde{1}\tilde{\mathcal{F}}_{n-1} \\ &= \tilde{\mathcal{L}}e_n\tilde{\mathcal{F}}_{n-1} + 2\tilde{\mathcal{F}}_{n+1}\tilde{\mathcal{F}}_n. \qquad \square\end{aligned}$$

Theorem 3.4. *For positive integers n and r, with $n \geq r$, the general Catalan's identity for the $\mathcal{DGC}$ Leonardo sequence is as follows:*

$$\begin{aligned}(\tilde{\mathcal{L}}e_n)^2 - \tilde{\mathcal{L}}e_{n-r}\tilde{\mathcal{L}}e_{n+r} = \; & 4(-1)^{n-r+1}F_r^2[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon] \\ & +\tilde{1}(\tilde{\mathcal{L}}e_{n-r} + \tilde{\mathcal{L}}e_{n+r} - 2\tilde{\mathcal{L}}e_n).\end{aligned}$$

Proof. We give two different proofs of the theorem.

Proof 1: By using the Binet's formulas in equations (3) and (16) for the Fibonacci and $\mathcal{DGC}$ Leonardo sequences, respectively, we deduce that:

$$\begin{aligned}(\tilde{\mathcal{L}}e_n)^2 - \tilde{\mathcal{L}}e_{n-r}\tilde{\mathcal{L}}e_{n+r} = \; & \left(\frac{2\tilde{\alpha}\alpha^{n+1} - 2\tilde{\beta}\beta^{n+1}}{\alpha-\beta} - \tilde{1}\right)\left(\frac{2\tilde{\alpha}\alpha^{n+1} - 2\tilde{\beta}\beta^{n+1}}{\alpha-\beta} - \tilde{1}\right) \\ & - \left(\frac{2\tilde{\alpha}\alpha^{n-r+1} - 2\tilde{\beta}\beta^{n-r+1}}{\alpha-\beta} - \tilde{1}\right)\left(\frac{2\tilde{\alpha}\alpha^{n+r+1} - 2\tilde{\beta}\beta^{n+r+1}}{\alpha-\beta} - \tilde{1}\right) \\ = \; & \frac{\left(4\tilde{\alpha}\tilde{\beta}(\alpha^{n-r+1}\beta^{n-r+1})(-2\alpha^r\beta^r + \beta^{2r} + \alpha^{2r}\right)}{(\alpha-\beta)^2} \\ & +\tilde{1}\left((\tilde{\mathcal{L}}e_{n-r} + \tilde{1}) + (\tilde{\mathcal{L}}e_{n+r} + \tilde{1}) - 2(\tilde{\mathcal{L}}e_n + \tilde{1})\right) \\ = \; & 4\left(\tilde{\alpha}\tilde{\beta}(\alpha\beta)^{n-r+1}F_r^2\right) + \tilde{1}\left(\tilde{\mathcal{L}}e_{n-r} + \tilde{\mathcal{L}}e_{n+r} - 2\tilde{\mathcal{L}}e_n\right).\end{aligned}$$

We conclude from α and β that $\alpha\beta = -1$, hence that

$$\tilde{\alpha}\tilde{\beta} = (1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon,$$

and finally that

$$\begin{aligned}(\tilde{\mathcal{L}}e_n)^2 - \tilde{\mathcal{L}}e_{n-r}\tilde{\mathcal{L}}e_{n+r} = \; & 4\left(\tilde{\alpha}\tilde{\beta}(\alpha\beta)^{n-r+1}F_r^2\right) + \tilde{1}\left(\tilde{\mathcal{L}}e_{n-r} + \tilde{\mathcal{L}}e_{n+r} - 2\tilde{\mathcal{L}}e_n\right) \\ = \; & (4(-1)^{n-r+1}F_r^2[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]) \\ & +\tilde{1}\left(\tilde{\mathcal{L}}e_{n-r} + \tilde{\mathcal{L}}e_{n+r} - 2\tilde{\mathcal{L}}e_n\right).\end{aligned}$$

Proof 2: From Theorem 2.1, it follows that

$$\begin{aligned}(\tilde{\mathcal{L}}e_n)^2 - \tilde{\mathcal{L}}e_{n-r}\tilde{\mathcal{L}}e_{n+r} = \; & (2\tilde{\mathcal{F}}_{n+1} - \tilde{1})^2 - (2\tilde{\mathcal{F}}_{n-r+1} - \tilde{1})(2\tilde{\mathcal{F}}_{n+r+1} - \tilde{1}) \\ = \; & 4(\tilde{\mathcal{F}}_{n+1}^2 - \tilde{\mathcal{F}}_{n-r+1}\tilde{\mathcal{F}}_{n+r+1}) + \tilde{1}(2\tilde{\mathcal{F}}_{n-r+1} + 2\tilde{\mathcal{F}}_{n+r+1} - 4\tilde{\mathcal{F}}_{n+1}).\end{aligned}$$

The equality

$$\tilde{\mathcal{F}}_m\tilde{\mathcal{F}}_n - \tilde{\mathcal{F}}_{m+r}\tilde{\mathcal{F}}_{n-r} = (-1)^{n-r}F_{m-n+r}F_r[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon], \tag{17}$$

(see Theorem 2, item 4 in [16] with $m \to n+1$ and $n \to n+1$), implies that

$$\begin{aligned}(\tilde{\mathcal{L}}e_n)^2 - \tilde{\mathcal{L}}e_{n-r}\tilde{\mathcal{L}}e_{n+r} = \; & 4((-1)^{n-r+1}F_r^2[(1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon]) \\ & + \tilde{1}(\tilde{\mathcal{L}}e_{n-r} + \tilde{\mathcal{L}}e_{n+r} - 2\tilde{\mathcal{L}}e_n). \qquad \square\end{aligned}$$

In what follows, $\tilde{\alpha}\tilde{\beta} = (1-\mathfrak{p}) + J + 3(1-\mathfrak{p})\varepsilon + 3J\varepsilon$.

Theorem 3.5. *For a positive integer n, the general Cassini's identity (sometimes called Simson's identity) for the $\mathcal{DGC}$ Leonardo sequence is as below:*

$$(\tilde{\mathcal{L}e}_n)^2 - \tilde{\mathcal{L}e}_{n-1}\tilde{\mathcal{L}e}_{n+1} = 4(-1)^n\tilde{\alpha}\tilde{\beta} + \tilde{1}(\tilde{\mathcal{L}e}_{n-1} + \tilde{\mathcal{L}e}_{n+1} - 2\tilde{\mathcal{L}e}_n).$$

Proof. Taking $r \to 1$ in the previous theorem, we obtain the Cassini's identity. □

Theorem 3.6. *For positive integers n and m, with $m \geq n$ the general d'Ocagne's identity for the $\mathcal{DGC}$ Leonardo sequence is as below:*

$$\tilde{\mathcal{L}e}_m\tilde{\mathcal{L}e}_{n+1} - \tilde{\mathcal{L}e}_{m+1}\tilde{\mathcal{L}e}_n = 4(-1)^{n+1}\tilde{\alpha}\tilde{\beta}F_{m-n} + \tilde{1}(\tilde{\mathcal{L}e}_{m-1} - \tilde{\mathcal{L}e}_{n-1}).$$

Proof. We give two different proofs of the theorem.

Proof 1: Substituting the Binet's formula in Equation (16) into left-hand side, then writing Equation (14), we see that:

$$\begin{aligned}\tilde{\mathcal{L}e}_m\tilde{\mathcal{L}e}_{n+1} - \tilde{\mathcal{L}e}_{m+1}\tilde{\mathcal{L}e}_n = \;& 4\tilde{\alpha}\tilde{\beta}\left(\frac{(\alpha-\beta)(\alpha^{m+1}\beta^{n+1}) - (\alpha-\beta)(\alpha^{n+1}\beta^{m+1})}{(\alpha-\beta)^2}\right)\\ & + \tilde{1}(\tilde{\mathcal{L}e}_{m-1} - \tilde{\mathcal{L}e}_{n-1})\\ =\;& 4\tilde{\alpha}\tilde{\beta}(\alpha\beta)^{n+1}\left(\frac{\alpha^{m-n} - \beta^{m-n}}{\alpha-\beta}\right) + \tilde{1}(\tilde{\mathcal{L}e}_{m-1} - \tilde{\mathcal{L}e}_{n-1}).\end{aligned}$$

By considering $\alpha\beta = -1$ and the Binet's formula in (3) for the Fibonacci sequence, the proof is straightforward.

Proof 2: Applying $\tilde{\mathcal{L}e}_n = 2\tilde{\mathcal{F}}_{n+1} - \tilde{1}$ (see Theorem 2.1) to the left-hand side gives

$$\begin{aligned}\tilde{\mathcal{L}e}_m\tilde{\mathcal{L}e}_{n+1} - \tilde{\mathcal{L}e}_{m+1}\tilde{\mathcal{L}e}_n =\;& (2\tilde{\mathcal{F}}_{m+1} - \tilde{1})(2\tilde{\mathcal{F}}_{n+2} - \tilde{1}) - (2\tilde{\mathcal{F}}_{m+2} - \tilde{1})(2\tilde{\mathcal{F}}_{n+1} - \tilde{1})\\ =\;& 4(\tilde{\mathcal{F}}_{m+1}\tilde{\mathcal{F}}_{n+2} - \tilde{\mathcal{F}}_{m+2}\tilde{\mathcal{F}}_{n+1})\\ & + 2(\tilde{\mathcal{F}}_{m+2} + \tilde{\mathcal{F}}_{n+1} - \tilde{\mathcal{F}}_{m+1} - \tilde{\mathcal{F}}_{n+2})\tilde{1}.\end{aligned}$$

From Equation (17) (see Theorem 2, item 4 in [16] with $m \to m+1$, $n \to n+2$ and $r \to 1$), and the recurrence relation of the $\mathcal{DGC}$ Fibonacci sequence, we obtain that:

$$\begin{aligned}\tilde{\mathcal{L}e}_m\tilde{\mathcal{L}e}_{n+1} - \tilde{\mathcal{L}e}_{m+1}\tilde{\mathcal{L}e}_n =\;& 4(-1)^{n+1}\tilde{\alpha}\tilde{\beta}F_{m-n} + \tilde{1}(2(\tilde{\mathcal{F}}_{m+2} - \tilde{\mathcal{F}}_{m+1}) + 2(\tilde{\mathcal{F}}_{n+1} - \tilde{\mathcal{F}}_{n+2}))\\ =\;& 4(-1)^{n+1}\tilde{\alpha}\tilde{\beta}F_{m-n} + \tilde{1}((2\tilde{\mathcal{F}}_m - \tilde{1}) - (2\tilde{\mathcal{F}}_n - \tilde{1}))\\ =\;& 4(-1)^{n+1}\tilde{\alpha}\tilde{\beta}F_{m-n} + \tilde{1}(\tilde{\mathcal{L}e}_{m-1} - \tilde{\mathcal{L}e}_{n-1}).\end{aligned}$$

This completes the proof. □

Theorem 3.7. *For positive integers n, m, r and s, with $r \geqslant s$, the special case of the Tagiuri identity for the $\mathcal{DGC}$ Leonardo sequence is as below:*

$$\begin{aligned}\tilde{\mathcal{L}e}_{n+r}\tilde{\mathcal{L}e}_{n+s} - \tilde{\mathcal{L}e}_n\tilde{\mathcal{L}e}_{n+r+s} =\;& \frac{4}{5}\tilde{\alpha}\tilde{\beta}(-1)^{n+1}(L_{r+s} - (-1)^sL_{r-s})\\ & + \tilde{1}(\tilde{\mathcal{L}e}_n + \tilde{\mathcal{L}e}_{n+r+s} - \tilde{\mathcal{L}e}_{n+r} - \tilde{\mathcal{L}e}_{n+s}).\end{aligned}$$

Proof. We first write the Binet's formula for the $\mathcal{DGC}$ Leonardo sequence in Equation (16) into the left-hand side and rearrange, then we see that:

$$\begin{aligned}\tilde{\mathcal{L}e}_{n+r}\tilde{\mathcal{L}e}_{n+s}-\tilde{\mathcal{L}e}_{n}\tilde{\mathcal{L}e}_{n+r+s}= &\ \frac{4}{5}\tilde{\alpha}\tilde{\beta}(\alpha^{n+1}\beta^{n+1})\left(\alpha^{r+s}+\beta^{r+s}-(\alpha^{s}\beta^{s})(\alpha^{r-s}+\beta^{r-s})\right)\\ &+\tilde{1}(\tilde{\mathcal{L}e}_{n}+\tilde{\mathcal{L}e}_{n+r+s}-\tilde{\mathcal{L}e}_{n+r}-\tilde{\mathcal{L}e}_{n+s}).\end{aligned}$$

By using $\alpha\beta=-1$ and the Binet's formula in Equation (4) for the Lucas sequence, the proof is completed. $\square$

Theorem 3.8. *For positive integers n and m, with $n \geqslant m$, the following identities hold:*

1. $\tilde{\mathcal{L}e}_{n+m}-(-1)^{m}\tilde{\mathcal{L}e}_{n-m}=\tilde{\mathcal{L}}_{n+1}(Le_{m-1}+1)+\tilde{1}((-1)^{m}-1)$,
2. $\tilde{\mathcal{L}e}_{n+m}+(-1)^{m}\tilde{\mathcal{L}e}_{n-m}=2\tilde{\mathcal{F}}_{n+1}L_{m}-\tilde{1}(1+(-1)^{m})$.

Proof. 1. The main idea of the proof is to use Equation (13) with identity

$$Le_{n+m}-(-1)^{m}Le_{n-m}=L_{n+1}(Le_{m-1}+1)-1+(-1)^{m}$$

(see [3]). Hence, we find:

$$\begin{aligned}\tilde{\mathcal{L}e}_{n+m}-(-1)^{m}\tilde{\mathcal{L}e}_{n-m}= &\ (L_{n+1}(Le_{m-1}+1)-1+(-1)^{m})\\ &+(L_{n+2}(Le_{m-1}+1)-1+(-1)^{m})J\\ &+(L_{n+3}(Le_{m-1}+1)-1+(-1)^{m})\varepsilon\\ &+(L_{n+4}(Le_{m-1}+1)-1+(-1)^{m})J\varepsilon\\ = &\ \tilde{\mathcal{L}}_{n+1}(Le_{m-1}+1)+\tilde{1}((-1)^{m}-1).\end{aligned}$$

2. Similarly, applying Equation (13) and Theorem 2.1 with the identity $Le_{n+m}+(-1)^{m}Le_{n-m} = L_{m}(Le_{n}+1)-1-(-1)^{m}$ (see [3]), the proof is clear. $\square$

Theorem 3.9. *For positive integers k, m, s and t, with $k \geq m$, $s \geq t$ and $k+m=s+t$, the following is true:*

$$\tilde{\mathcal{L}e}_{k}\tilde{\mathcal{L}e}_{m}-\tilde{\mathcal{L}e}_{s}\tilde{\mathcal{L}e}_{t}=\frac{4}{5}\tilde{\alpha}\tilde{\beta}((-1)^{m}L_{k-m}-(-1)^{t}L_{s-t})-\tilde{1}(\tilde{\mathcal{L}e}_{k}+\tilde{\mathcal{L}e}_{m}-\tilde{\mathcal{L}e}_{s}-\tilde{\mathcal{L}e}_{t}).$$

Proof. We begin by writing the Binet's formula for the $\mathcal{DGC}$ Leonardo sequence in Equation (16) into left-hand side. This gives

$$\begin{aligned}\tilde{\mathcal{L}e}_{k}\tilde{\mathcal{L}e}_{m}-\tilde{\mathcal{L}e}_{s}\tilde{\mathcal{L}e}_{t}= &\ 4\tilde{\alpha}\tilde{\beta}\left(\frac{\alpha^{s+1}\beta^{t+1}+\alpha^{t+1}\beta^{s+1}-\alpha^{k+1}\beta^{m+1}-\alpha^{m+1}\beta^{k+1}}{(\alpha-\beta)^{2}}\right)\\ &-\tilde{1}(\tilde{\mathcal{L}e}_{k}+\tilde{\mathcal{L}e}_{m}-\tilde{\mathcal{L}e}_{s}-\tilde{\mathcal{L}e}_{t})\\ = &\ \frac{4}{5}\tilde{\alpha}\tilde{\beta}(-(\alpha\beta)^{m+1}(\alpha^{k-m}+\beta^{k-m})+(\alpha\beta)^{t+1}(\alpha^{s-t}+\beta^{s-t})\\ &-\tilde{1}(\tilde{\mathcal{L}e}_{k}+\tilde{\mathcal{L}e}_{m}-\tilde{\mathcal{L}e}_{s}-\tilde{\mathcal{L}e}_{t}).\end{aligned}$$

Taking $\alpha\beta=-1$, $\alpha-\beta=\sqrt{5}$ and considering the Binet's formula for the Lucas sequence in Equation (4) gives

$$\tilde{\mathcal{L}e}_{k}\tilde{\mathcal{L}e}_{m}-\tilde{\mathcal{L}e}_{s}\tilde{\mathcal{L}e}_{t}=\frac{4}{5}\tilde{\alpha}\tilde{\beta}((-1)^{m}L_{k-m}-(-1)^{t}L_{s-t})-\tilde{1}(\tilde{\mathcal{L}e}_{k}+\tilde{\mathcal{L}e}_{m}-\tilde{\mathcal{L}e}_{s}-\tilde{\mathcal{L}e}_{t}),$$

which completes the proof. $\square$

Theorem 3.10. *For positive integers n and m, with $n \geq m$, the following identities hold:*

1. $\tilde{\mathcal{F}}_n \tilde{\mathcal{L}e}_m - \tilde{\mathcal{F}}_m \tilde{\mathcal{L}e}_n = 2\tilde{\alpha}\tilde{\beta}(-1)^m F_{n-m} - \tilde{1}(\tilde{\mathcal{F}}_n - \tilde{\mathcal{F}}_m)$,

2. $$\begin{aligned}\tilde{\mathcal{F}}_n \tilde{\mathcal{L}e}_m + \tilde{\mathcal{F}}_m \tilde{\mathcal{L}e}_n &= \frac{4}{5}(2\tilde{\mathcal{L}}_{n+m+1} - L_{n+m+1}) + \frac{4\mathfrak{p}}{5}(L_{n+m+3} + 2L_{n+m+5}\varepsilon) \\ &+ \frac{8}{5}L_{n+m+4}J\varepsilon - \frac{2}{5}\tilde{\alpha}\tilde{\beta}(-1)^m L_{n-m} - \tilde{1}(\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_m).\end{aligned}$$

Proof. 1. Writing the Binet's formulas for the $\mathcal{DGC}$ Fibonacci sequence in Equation (11) and the $\mathcal{DGC}$ Leonardo sequence in Equation (16) into the left-hand side, we have

$$\begin{aligned}\tilde{\mathcal{F}}_n \tilde{\mathcal{L}e}_m - \tilde{\mathcal{F}}_m \tilde{\mathcal{L}e}_n = &\ 2\tilde{\alpha}\tilde{\beta}\left(\frac{\alpha^n\beta^m(\alpha-\beta) - \alpha^m\beta^n(\alpha-\beta)}{(\alpha-\beta)^2}\right) - \tilde{1}(\tilde{\mathcal{F}}_n - \tilde{\mathcal{F}}_m) \\ = &\ 2\tilde{\alpha}\tilde{\beta}(\alpha\beta)^m\left(\frac{\alpha^{n-m}-\beta^{n-m}}{\alpha-\beta}\right) - \tilde{1}(\tilde{\mathcal{F}}_n - \tilde{\mathcal{F}}_m).\end{aligned}$$

The proof is completed by using $\alpha\beta = -1$ and the Binet's formula for the Fibonacci sequence (3).

2. Similarly, we first apply the Binet's formulas for the $\mathcal{DGC}$ Fibonacci sequence in (11) and the $\mathcal{DGC}$ Leonardo sequence in (16) to the left-hand side. We thus get

$$\begin{aligned}\tilde{\mathcal{F}}_n \tilde{\mathcal{L}e}_m + \tilde{\mathcal{F}}_m \tilde{\mathcal{L}e}_n = &\ \frac{4}{5}(\tilde{\alpha}^2\alpha^{n+m+1} + \tilde{\beta}^2\beta^{n+m+1}) \\ &-\frac{2\tilde{\alpha}\tilde{\beta}}{5}(\alpha^n\beta^m(\alpha+\beta) + \alpha^m\beta^n(\alpha+\beta)) - \tilde{1}(\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_m).\end{aligned} \tag{18}$$

Substituting $\alpha + \beta = 1, \alpha\beta = -1$, the Binet's formula for the Lucas sequence in Equation (4) and the followings

$$\begin{cases} \tilde{\alpha}^2 &= (1+\mathfrak{p}\alpha^2) + 2\alpha J + 2\alpha^2(1+\mathfrak{p}\alpha^2)\varepsilon + 4\alpha^3 J\varepsilon) \\ \beta^2 &= (1+\mathfrak{p}\beta^2) + 2\beta J + 2\beta^2(1+\mathfrak{p}\beta^2)\varepsilon + 4\alpha^3 J\varepsilon) \end{cases} \tag{19}$$

into Equation (18), we have

$$\begin{aligned}\tilde{\mathcal{F}}_n \tilde{\mathcal{L}e}_m + \tilde{\mathcal{F}}_m \tilde{\mathcal{L}e}_n = &\ \frac{4}{5}(L_{n+m+1} + L_{n+m+3}\mathfrak{p} + 2L_{n+m+2}J + (2L_{n+m+3} + 2L_{n+m+5}\mathfrak{p})\varepsilon \\ &+4L_{n+m+4}J\varepsilon) - \frac{2\tilde{\alpha}\tilde{\beta}}{5}(-1)^m L_{n-m} - \tilde{1}(\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_m).\end{aligned}$$

The proof is completed by considering $\mathcal{DGC}$ Lucas sequence statement in Equation (12). □

Theorem 3.11. *For positive integers k, m and s, with $m \geq k$ and $m \geq s$, the following identity holds:*

$$\begin{aligned}\tilde{\mathcal{L}e}_{m+k}\tilde{\mathcal{L}e}_{m-k} - \tilde{\mathcal{L}e}_{m+s}\tilde{\mathcal{L}e}_{m-s} &= 4\tilde{\alpha}\tilde{\beta}((-1)^{m-k}F_k^2 - (-1)^{m-s}F_s^2) \\ &+ \tilde{1}(\tilde{\mathcal{L}e}_{m+s} + \tilde{\mathcal{L}e}_{m-s} - \tilde{\mathcal{L}e}_{m+k} - \tilde{\mathcal{L}e}_{m-k}).\end{aligned}$$

Proof. Our proof starts again with the Binet's formula for the $\mathcal{DGC}$ Leonardo sequence. From $\alpha-\beta=\sqrt{5}$, it follows that

$$\begin{aligned}\tilde{\mathcal{L}}e_{m+k}\tilde{\mathcal{L}}e_{m-k}-\tilde{\mathcal{L}}e_{m+s}\tilde{\mathcal{L}}e_{m-s}=&\frac{4\tilde{\alpha}\tilde{\beta}}{5}(-(\alpha\beta)^{m-k+1}(\alpha^{2k}+\beta^{2k})+(\alpha\beta)^{m-s+1}(\alpha^{2s}+\beta^{2s}))\\&+\tilde{1}(\tilde{\mathcal{L}}e_{m+s}+\tilde{\mathcal{L}}e_{m-s}-\tilde{\mathcal{L}}e_{m+k}-\tilde{\mathcal{L}}e_{m-k}).\end{aligned}$$

By using $\alpha\beta=-1$ and the Binet's formula for the Lucas sequence in Equation (4), we have that

$$\begin{aligned}\tilde{\mathcal{L}}e_{m+k}\tilde{\mathcal{L}}e_{m-k}-\tilde{\mathcal{L}}e_{m+s}\tilde{\mathcal{L}}e_{m-s}=&\frac{4\tilde{\alpha}\tilde{\beta}}{5}\left((-1)^{m-k}L_{2k}-(-1)^{m-s}L_{2s}\right)\\&+\tilde{1}(\tilde{\mathcal{L}}e_{m+s}+\tilde{\mathcal{L}}e_{m-s}-\tilde{\mathcal{L}}e_{m+k}-\tilde{\mathcal{L}}e_{m-k}).\end{aligned}$$

Writing $L_{2k}=5F_k^2+2(-1)^k$ (see [24]) completes the proof. □

Theorem 3.12. *For n and m integers, with $n\geq 1$ and $m\geq 1$, the following identity holds:*

$$\begin{aligned}\tilde{\mathcal{L}}e_{m+1}\tilde{\mathcal{L}}e_{n+1}-\tilde{\mathcal{L}}e_{m-1}\tilde{\mathcal{L}}e_{n-1}=&\,4(2\tilde{\mathcal{F}}_{m+n+2}-F_{m+n+2})+4\mathfrak{p}(F_{m+n+4}+2F_{m+n+6}\varepsilon)\\&+8F_{m+n+5}J\varepsilon-\tilde{1}(\tilde{\mathcal{L}}e_m+\tilde{\mathcal{L}}e_n)-2(\tilde{1})^2.\end{aligned}$$

Proof. Let us first write the Binet's formula for the $\mathcal{DGC}$ Leonardo sequence. We thus get

$$\begin{aligned}\tilde{\mathcal{L}}e_{m+1}\tilde{\mathcal{L}}e_{n+1}-\tilde{\mathcal{L}}e_{m-1}\tilde{\mathcal{L}}e_{n-1}=&\,4\left(\frac{\tilde{\alpha}^2\alpha^{m+n+4}+\tilde{\beta}^2\beta^{m+n+4}-\tilde{\alpha}^2\alpha^{m+n}-\tilde{\beta}^2\beta^{m+n}}{(\alpha-\beta)^2}\right)\\&-4\tilde{\alpha}\tilde{\beta}\left(\frac{\alpha^{m+2}\beta^{n+2}+\alpha^{n+2}\beta^{m+2}-\alpha^m\beta^n-\alpha^n\beta^m}{(\alpha-\beta)^2}\right)\\&+\tilde{1}(\tilde{\mathcal{L}}e_{m-1}+\tilde{\mathcal{L}}e_{n-1}-\tilde{\mathcal{L}}e_{m+1}-\tilde{\mathcal{L}}e_{n+1}).\end{aligned}$$

We conclude from $\alpha\beta=-1$ that

$$\begin{aligned}\alpha^{m+2}\beta^{n+2}+\alpha^{n+2}\beta^{m+2}-\alpha^m\beta^n-\alpha^n\beta^m&=\alpha^m\beta^n((\alpha\beta)^2-1)+\alpha^n\beta^m((\alpha\beta)^2-1)\\&=0.\end{aligned}$$

Moreover, from Equation (14) we have

$$(\tilde{\mathcal{L}}e_{m-1}+\tilde{\mathcal{L}}e_{n-1}-\tilde{\mathcal{L}}e_{m+1}-\tilde{\mathcal{L}}e_{n+1})=-(\tilde{\mathcal{L}}e_m+\tilde{\mathcal{L}}e_n+\tilde{1}+\tilde{1}).$$

According to equations (19) and (4), we see that

$$\begin{aligned}\tilde{\alpha}^2\alpha^{m+n+4}+\tilde{\beta}^2\beta^{n+m+4}-\tilde{\alpha}^2\alpha^{m+n}-\tilde{\beta}^2\beta^{n+m}=&\,(L_{m+n+4}-L_{m+n})+\mathfrak{p}(L_{m+n+6}-L_{m+n+2})\\&+2(L_{m+n+5}-L_{m+n+1})J\\&+2\left((L_{m+n+6}-L_{m+n+2})\right.\\&\quad\left.+\mathfrak{p}(L_{m+n+8}-L_{m+n+4})\right)\varepsilon\\&+4(L_{m+n+7}-L_{m+n+3})J\varepsilon.\end{aligned}$$

The proof is completed by using $L_{n+r}-L_{n-r}=5F_nF_r$ for an even integer r (see [24]) and the definition of the $\mathcal{DGC}$ Fibonacci sequence. □

4 Conclusion

In this paper, the order-2 relations for the $\mathcal{DGC}$ Leonardo sequence are computed for $\mathfrak{p} \in \mathbb{R}$. The advantage of implementing this construction is that the dual-complex, the hyper-dual, and the dual-hyperbolic Leonardo sequences can be carried out for $\mathfrak{p} \in \{-1, 0, 1\}$, and $J \in \{\mathbf{i}, \boldsymbol{\epsilon}, \mathbf{j}\}$, respectively (see Table 3).

Table 3. Special cases of the $\mathcal{DGC}$ Leonardo sequence

The sequence type	**The Leonardo sequence**	J	$\mathfrak{p}$	**Condition**
Dual-complex	$Le_n + Le_{n+1}\mathbf{i} + Le_{n+2}\varepsilon + Le_{n+3}\mathbf{i}\varepsilon$	$\mathbf{i}$	-1	
Hyper-dual, [21]	$Le_n + Le_{n+1}\boldsymbol{\epsilon} + Le_{n+2}\varepsilon + Le_{n+3}\boldsymbol{\epsilon}\varepsilon$	$\boldsymbol{\epsilon}$	0	$\boldsymbol{\epsilon} \neq 0,\ \boldsymbol{\epsilon}\varepsilon \neq 0$
Dual-hyperbolic	$Le_n + Le_{n+1}\mathbf{j} + Le_{n+2}\varepsilon + Le_{n+3}\mathbf{j}\varepsilon$	$\mathbf{j}$	1	$\mathbf{j} \neq \pm 1$

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