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Some new results on the negative polynomial Pell's equation

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Abstract: We consider the negative polynomial Pell's equation $P^2(X) - D(X)Q^2(X) = -1$, where $D(X) \in \mathbb{Z}[X]$ be some fixed, monic, square-free, even degree polynomials. In this paper, we investigate the existence of polynomial solutions $P(X)$, $Q(X)$ with integer coefficients.
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1 Introduction

The classical *Pell's equation* is

$$x^2 - Dy^2 = 1, \tag{1}$$

where D is a square-free positive integer. Solving a Pell's equation for integers x and y is one of the classical problems in number theory. In 1768, Lagrange proved that the equation (1) has infinitely many solutions ([15, vol. XXIII, p. 272], [16, vol. XXIV, p. 236]). In fact, a classical result says that there exists a non-trivial solution (x_0, y_0) is called a fundamental solution such that any other solution takes the form $(x_0 + y_0\sqrt{D})^n$, $n \in \mathbb{Z}$.

On the other hand, the problem of solving a *negative Pell's equation* has not been understood satisfactorily. It is an equation of the form

$$x^2 - Dy^2 = -1, \tag{2}$$

where D is a square-free integer and x, y are integer solutions. There is no solution for equation (2) if D is a negative integer and the length of the period in the continued fraction expansion of $\sqrt{D}$ is even. However, if the length of the period in the continued fraction expansion of $\sqrt{D}$ is odd, then (2) has infinitely many integer solutions [26, Theorem 7.26]. Furthermore, the negative Pell's equation is not solvable for D with prime divisor congruent to $3 \bmod 4$ or D is divisible by 4. Moreover, Fouvry and Klüners [5] gave the upper and lower bounds for the long-lasting conjecture on the asymptotic formulae for the number of square-free integers D for which fundamental solution of the equation (2) has norm -1. Recently, the bound was further improved by Koymans and Pagano [14].

Similarly, we can consider the *polynomial Pell's equation*

$$P^2(X) - D(X)Q^2(X) = \pm 1, \tag{3}$$

where $D(X)$ is a given fixed, square-free polynomial with integer coefficients and $P(X), Q(X)$ are its integer polynomial solutions.

In 1976, Nathanson [20] proved that when $D(X) = X^2 + d \in \mathbb{Z}[X]$, the equation $P^2(X) - D(X)Q^2(X) = 1$ is solvable in $\mathbb{Z}[X]$ if and only if $d = \pm 1, \pm 2$. Moreover, such a polynomial solutions can be expressed in terms of Chebyshev polynomials [22].

In 2004, Dubickas and Steuding [4] extended Nathanson's result for polynomials of the form $D(X) = X^{2k} + d \in \mathbb{Z}[X]$, $k \in \mathbb{N}$. More precisely, they proved that the equation $P^2(X) - (X^{2k} + d)Q^2(X) = 1$ is solvable in $\mathbb{Z}[X]$ if and only if $d \in \{\pm 1, \pm 2\}$.

There are many results in positive polynomial Pell's equations, we slightly open its counterpart the *negative polynomial Pell's equation*,

$$P^2(X) - D(X)Q^2(X) = -1, \tag{4}$$

where $D(X)$ is a fixed, even degree, square-free polynomial with integer coefficients and $P(X)$, $Q(X)$ are its integer polynomial solutions. More precisely, we prove the following theorems:

Theorem 1.1. *Let d be an integer with $d \neq \pm 1,\ \pm 2$. Then the negative polynomial Pell's equation*

$$P^2(X) - (X^2 + d)Q^2(X) = -1 \tag{5}$$

has no non-trivial solutions over $\mathbb{Z}[i]$.

Theorem 1.2. *The equation* (5) *has non-trivial polynomial solutions over $\mathbb{Z}$ if and only if $d = 1$.*

The proof of Theorem 1.2 is very similar to the proof of the following theorem. Thus, the generalization of the above theorem is as follows:

Theorem 1.3. *The negative polynomial Pell's equation*

$$P^2(X) - (X^{2k} + d)Q^2(X) = -1, \tag{6}$$

where $d \in \mathbb{Z}$ and $k \in \mathbb{N}$, has non-trivial solutions in $\mathbb{Z}[X]$ if and only if $d = 1$.

1.1 The ABC conjecture for polynomials (Stothers and Mason)

Stothers [28] and Mason [19] independently proved the ABC conjecture for polynomials.

Let $n_0(P(X))$ denote the number of distinct complex zeros of a polynomial $P(X)$ (which does not vanish identically). If A, B, C are coprime polynomials over $\mathbb{C}$, not all constant polynomials satisfy $A + B = C$, then

$$\max\{\deg A, \deg B, \deg C\} < n_0(ABC). \tag{7}$$

In 1984, Silverman [24] gave a different proof with the help of Riemann–Hurwitz formula. Then Snyder [25] provided a slightly different proof of the Stothers–Mason theorem in 2000. The connection between the inequality (7) and the Fermat's last theorem for polynomials can be found in Lang's survey article [17]. The ABC conjecture for polynomials has notable applications to the polynomial Pell's equation.

2 Results

2.1 Proof of Theorem 1.1

We prove the theorem by contradiction. We first consider the equation (5) as a polynomial over $\mathbb{Z}[i]$. We suppose that the equation (5) has non-trivial solutions over $\mathbb{Z}[i]$. We choose a solution $P(X),\ Q(X)$ of (5) with $\deg P(X) > 0$ is minimal and we take a non-zero d with $\mid d \mid \geq 3$. We split the proof into two cases.

Case (i): If $d \neq -\alpha^2,\ \alpha \in \mathbb{Z}[i]$, then $X^2 + d$ is irreducible over $\mathbb{Z}[i]$. We now rewrite (5) as,

$$(P(X) + i)(P(X) - i) = (X^2 + d)Q^2(X). \tag{8}$$

Since $(X^2 + d)$ is irreducible over $\mathbb{Z}[i]$ and $\mathbb{Z}[i]$ is a unique factorization domain, it divides one of the $(P(X) + i)$ or $(P(X) - i)$. We assume that $(X^2 + d)$ divides $P(X) - i$. Therefore,

$$P(X) - i = (X^2 + d)P_1(X),$$

where $P_1(X)$ is a polynomial over $\mathbb{Z}[i]$.

Then

$$P(X) - i + 2i = P(X) + i = (X^2 + d)P_1(X) + 2i.$$

On substituting into the equation (8), we have

$$P_1(X)((X^2 + d)P_1(X) + 2i) = Q^2(X).$$

Since the greatest common divisor of $P_1(X)$ and $(X^2 + d)P_1(X) + 2i$ is 1 or 2, we must obtain at least one of the following conditions:

1. $(X^2 + d)P_1(X) + 2i = P_2^2(X), \quad P_1(X) = Q_2^2(X);$
2. $(X^2 + d)P_1(X) + 2i = -P_2^2(X), \quad P_1(X) = -Q_2^2(X);$
3. $(X^2 + d)P_1(X) + 2i = -iP_2^2(X), \quad P_1(X) = iQ_2^2(X);$
4. $(X^2 + d)P_1(X) + 2i = iP_2^2(X), \quad P_1(X) = -iQ_2^2(X);$
5. $(X^2 + d)P_1(X) + 2i = 2P_2^2(X), \quad P_1(X) = 2Q_2^2(X);$
6. $(X^2 + d)P_1(X) + 2i = -2P_2^2(X), \quad P_1(X) = -2Q_2^2(X);$
7. $(X^2 + d)P_1(X) + 2i = -2iP_2^2(X), \quad P_1(X) = 2iQ_2^2(X);$
8. $(X^2 + d)P_1(X) + 2i = 2iP_2^2(X), \quad P_1(X) = -2iQ_2^2(X).$

As $P_2(X)$ is a polynomial over $\mathbb{Z}[i]$. We substitute $X = \sqrt{-d}$ in conditions (1)–(8) and we see that the following possibilities are admissible: $(r + s\sqrt{-d})^2 = \pm 2i$ or $(r + s\sqrt{-d})^2 = \pm 2$ or $(r+s\sqrt{-d})^2 = \pm i$ or $(r+s\sqrt{-d})^2 = \pm 1$ for some $r,\ s \in \mathbb{Z}[i]$. We need the following arguments to sort out the impossible conditions.

We first consider that $(r + s\sqrt{-d})^2 = \pm 2i$ and $(r + s\sqrt{-d})^2 = \pm i$. Substituting $r = x + iy$, $s = u + iv$, where $x,\ y,\ u,\ v \in \mathbb{Z}$, we have

$$(x + iy)^2 - (u + iv)^2 d + 2i((x + iy)(u + iv))\sqrt{d} = \pm 2i,\ \pm i.$$

On equating real and imaginary parts, we get

$$x^2 - y^2 - (u^2 - v^2)d - 2\sqrt{d}(xv + yu) = 0, \tag{9}$$

$$xy - uvd + (xu - vy)\sqrt{d} = \pm 1,\ \pm 1/2. \tag{10}$$

By our choice of d, equation (9) can be separated as rational and irrational parts,

$$x^2 - y^2 - (u^2 - v^2)d = 0.$$

This could be possible only when d is a perfect square or $d = \pm 1$. This ends in a contradiction.

We now explore the equation $(r + s\sqrt{-d})^2 = \pm 2$. As we proceeded before, we equate real and imaginary parts and we obtain

$$x^2 - y^2 - (u^2 - v^2)d - 2\sqrt{d}(xv + yu) = \pm 2, \tag{11}$$

$$xy - uvd + (xu - vy)\sqrt{d} = 0. \tag{12}$$

Again we repeat the same procedure as separating rational and irrational parts,

$$x^2 - y^2 - (u^2 - v^2)d = \pm 2, \tag{13}$$

$$xv + yu = 0, \tag{14}$$

$$xy - uvd = 0, \tag{15}$$

$$xu - vy = 0. \tag{16}$$

By solving the simultaneous equations (14) and (16), we get either $y = 0$ or $u^2 + v^2 = 0$. We first assume that $y = 0$ and $x \neq 0$, then $u = v = 0$. Therefore $x = \pm\sqrt{2}$ or $\pm i\sqrt{2}$. Since x is an integer, both can not be possible. On the other hand, if we assume both x and y are zero, then $uv = 0$ (by using (15)). Again a contradiction. Hence we conclude that y should be a non-zero and $u^2 + v^2 = 0$. Here the only possibility is $u = v = 0$. Thus we end with $x = 0$ (by using (12)) and the values of y are $\pm\sqrt{2}$ or $\pm i\sqrt{2}$. This is again a contradiction.

Now we take $(r + s\sqrt{-d})^2 = \pm 1$. As we did in the previous arguments, we first deal with the equation

$$x^2 - y^2 - (u^2 - v^2)d = 1. \tag{17}$$

There are two cases either $y = 0$ or $u^2 + v^2 = 0$ (by using (14) and (16)). At first, we suppose to consider both x and y are zero. Then we obtain $uv = 0$ (by using (15)). So we omit it. If we assume $y = 0$ and $x \neq 0$, then $u = v$. Thus $x = \pm 1$ and the value of r is ± 1. On the other side, if $u^2 + v^2 = 0$, then $u = v = 0$. Therefore value of $s = 0$.

Finally, we consider the equation

$$x^2 - y^2 - (u^2 - v^2)d = -1.$$

Again by the same procedure as we deal with the equation (17), we end with $y = \pm 1$ and $u = v = x = 0$. Thus $r = \pm i,\ s = 0$. Among eight conditions, only (7) and (8) are possible. We now rewrite the condition (7) as

$$P_2^2(X) - (X^2 + d)(iQ_2(X))^2 = -1, \tag{18}$$

and condition (8) as

$$(iP_2(X))^2 - (X^2 + d)Q_2^2(X) = -1. \tag{19}$$

But in both equations (18) and (19), $2\deg(P_2(X)) = 2 + \deg(P_1(X)) = \deg(P(X))$. It leads to a contradiction on the minimality of $\deg(P(X))$. Therefore, equation (5) has no non-trivial solutions if $d\,(\neq \pm 1,\ \pm 2) \neq -\alpha^2,\ \alpha \in \mathbb{Z}[i]$.

Case (ii): Let $d = -\alpha^2$, α be a non-unit in $\mathbb{Z}[i]$ and $N(\alpha) > 2$. The constant term of the solution polynomials $P(X)$ and $Q(X)$ are $\pm i$, 0, respectively. Suppose that $P(0) = i$. Then $P(X) = i + XP_1(X)$ and $Q(X) = XQ_1(X)$. We substitute $P(X)$, $Q(X)$ into equation (5) and we obtain

$$P_1(X)(XP_1(X) + 2i) = X(X^2 - \alpha^2)Q_1^2(X). \tag{20}$$

Since $P_1(X)$ is a polynomial without a constant term, we write $P_1(X) = XP_2(X)$. We now rewrite (20) as

$$P_2(X)(X^2P_2(X) + 2i) = (X^2 - \alpha^2)Q_1^2(X). \tag{21}$$

We suppose that $X \pm \alpha$ divides $X^2P_2(X) + 2i$. Then we put $X = \mp\alpha$ and we get $\alpha^2P_2(\mp\alpha) = -2i$. Thus α^2 divides $2i$. Since $N(\alpha) > 2$, this is not possible. Therefore, both $X + \alpha$
and $X - \alpha$ should divide $P_2(X)$. We can say $P_2(X) = (X^2 - \alpha^2)P_3(X)$. On substituting in (21), we obtain

$$P_3(X)(X^2(X^2 - \alpha^2)P_3(X) + 2i) = Q_1^2(X).$$

The greatest common divisor of $P_3(X)$ and $X^2(X^2 - \alpha^2)P_3(X) + 2i$ is 1 or 2. Again we repeat the same procedure as in Case (i).

This completes the proof of Theorem 1.1. □

2.2 Continued fraction expansion of $\sqrt{D(X)}$

We here adopt the same method used for irrationals $\sqrt{D}$ in [21].
The continued fraction expansion of $\sqrt{D(X)}$ is of the form

$$[a_0(X), \overline{a_1(X), a_2(X), \ldots, a_{r-1}(X), 2a_0(X)}]$$

with convergents $H_n(X)/K_n(X)$ and $a_i(X)$ being a non-constant polynomial in $\mathbb{Z}[X]$. Let r be the length of the shortest period in the continued fraction expansion of $\sqrt{D(X)}$.

We define

$$\zeta_0(X) = \frac{M_0(X) + \sqrt{D(X)}}{N_0(X)}$$

with $N_0(X) = 1$ and $M_0(X) = 0$.

In general, we define

$$\begin{aligned} a_i(X) &= [\zeta_i(X)], \\ \zeta_i(X) &= \frac{M_i(X) + \sqrt{D(X)}}{N_i(X)}, \\ M_{i+1}(X) &= a_i(X)N_i(X) - M_i(X), \\ N_{i+1}(X) &= \frac{D(X) - M_{i+1}^2(X)}{N_i(X)}, \end{aligned}$$

where $[.]$ denotes the rational part of the polynomial in terms of X. Since r is the length of the period, we write $\zeta_0 = \zeta_r = \zeta_{2r} = \cdots$. Thus for all $j \geq 0$ we write

$$\frac{M_{jr}(X) + \sqrt{D(X)}}{N_{jr}(X)} = \zeta_{jr}(X) = \zeta_0(X) = \frac{M_0(X) + \sqrt{D(X)}}{N_0(X)}.$$

Theorem 2.1. *If $D(X)$ is a square-free polynomial in $\mathbb{Z}[X]$ with a period length of r, then $H_n^2(X) - D(X)K_n^2(X) = (-1)^{n-1}N_{n+1}(X)$.*

Proof. The well-known classical result [21, Theorem 7.3] says that

$$\begin{aligned}\zeta_0(X) &= [a_0(X), a_1(X), a_2(X), \ldots, a_n(X), \zeta_{n+1}(X)]\\ &= \frac{\zeta_{n+1}(X)H_n(X) + H_{n-1}(X)}{\zeta_{n+1}(X)K_n(X) + K_{n-1}(X)}\\ &= \frac{\left(\frac{M_{n+1}(X)+\sqrt{D(X)}}{N_{n+1}(X)}\right)H_n(X) + H_{n-1}(X)}{\left(\frac{M_{n+1}(X)+\sqrt{D(X)}}{N_{n+1}(X)}\right)K_n(X) + K_{n-1}(X)}\\ \sqrt{D(X)} &= \frac{\left(M_{n+1}(X) + \sqrt{D(X)}\right)H_n(X) + H_{n-1}(X)N_{n+1}(X)}{\left(M_{n+1}(X) + \sqrt{D(X)}\right)K_n(X) + K_{n-1}(X)N_{n+1}(X)}.\end{aligned}$$

We separate it as a rational and an irrational part, and equate each part to zero.

$$-M_{n+1}(X)H_n(X) + K_n(X)D(X) - H_{n-1}(X)N_{n+1}(X) = 0, \tag{22}$$

$$M_{n+1}(X)K_n(X) + N_{n+1}(X)K_{n-1}(X) - H_n(X) = 0. \tag{23}$$

We eliminate $M_{n+1}(X)$ from the above equations (22) and (23). Then we write

$$H_n^2(X) - D(X)K_n^2(X) = (H_n(X)K_{n-1}(X) - K_n(X)H_{n-1}(X))N_{n+1}(X).$$

Then by using the result $H_n(X)K_{n-1}(X) - K_n(X)H_{n-1}(X) = (-1)^{n-1}$ [21, Theorem 7.5], we now obtain

$$H_n^2(X) - D(X)K_n^2(X) = (-1)^{n-1}N_{n+1}(X). \tag{24}$$

This completes the proof. □

Corollary 2.1. *Let r be the length of the period in the continued fraction expansion of $\sqrt{D(X)}$. Then for $n \geq 0$, the equation* (24) *becomes*

$$H_{nr-1}^2(X) - D(X)K_{nr-1}^2(X) = (-1)^{nr}N_{nr}(X) = (-1)^{nr}.$$

Proof. We replace n by $nr - 1$ in equation (24).

$$\begin{aligned}H_{nr-1}^2(X) - D(X)K_{nr-1}^2(X) &= (-1)^{nr}N_{nr}(X)\\ &= (-1)^{nr}N_0(X)\\ &= (-1)^{nr}.\end{aligned}$$

□

The following lemma is an analogous result of [4, Theorem 1] for the negative polynomial Pell's equation.

Lemma 2.1. *If $n_0(D(X))$, where $D(X) \in \mathbb{C}[X]$ is less than or equal to $1/2 \deg D(X)$, then the negative polynomial Pell's equation* (4) *has no non-trivial solutions in $\mathbb{C}[X]$.*

Proof. We consider $A = P^2(X)$, $B = -D(X)Q^2(X)$, $C = -1$.
We note that $\max\{\deg A, \deg B, \deg C\} = \deg B$ and $n_0(P(X)) \leq \deg P(X)$, $n_0(Q(X)) \leq \deg Q(X)$.
By using the ABC conjecture for polynomials, we write

$$\begin{aligned}
\deg D(X)Q^2(X) &< n_0(P^2(X)D(X)Q^2(X)) \\
&= n_0(P(X)D(X)Q(X)), \\
\deg D(X) &< n_0(P(X)) + n_0(D(X)) + n_0(Q(X)) - 2\deg Q(X), \\
\deg D(X) &< \deg P(X) - \deg Q(X) + n_0(D(X)), \\
1/2 \deg D(X) &< n_0(D(X)).
\end{aligned}$$

This completes the proof. □

We need the following lemma to prove Theorem 1.3.

Lemma 2.2. *Let $D(X)$ be a polynomial in $\mathbb{C}[X]$ with a degree of $2k$. Then the fundamental solutions $(U(X), V(X))$ in $\mathbb{C}[X]$ of equation* (4) *satisfying $\deg U(X) = 1/2 \deg D(X)$ and $\deg V(X) = 0$ is minimal.*

Proof. Firstly, let us consider $D(X)$ be a quadratic polynomial in $\mathbb{C}[X]$. We observe that the non-trivial solutions of (4) exists only if $D(X)$ has distinct roots. Let γ, δ be the roots of $D(X)$. Then we write $D(X) = c(X-\gamma)(X-\delta)$, $c \in \mathbb{C}, \gamma \neq \delta$.
We set

$$U(X) = \frac{2X - (\gamma+\delta)}{\sqrt{-1}(\gamma-\delta)}; \;\; V(X) = \frac{2}{\sqrt{-c}(\gamma-\delta)}.$$

For the general case, we assume the contrary. Suppose that $\deg U(X) < 1/2 \deg D(X)$ and $\deg V(X) > 0$. Since $\deg D(X) = 2\deg P(X) - 2\deg Q(X)$ and $\deg P(X)$ must be at least 1 greater than the $\deg Q(X)$.

Thus

$$\deg D(X) = 2\deg U(X) - 2\deg V(X) < \deg D(X) - 2t,$$

for some positive integer t. This completes the proof. □

2.3 Proof of Theorem 1.3

We use the method of continued fraction expansion of $\sqrt{X^{2k}+d}$, $d \in \mathbb{Z}$, i.e.,

$$\sqrt{X^{2k}+d} = [X^k, \overline{2X^k/d, 2X^k}].$$

By using Lemma 2.2, the fundamental solution over $\mathbb{C}$ is $\left(\frac{X^k}{\sqrt{d}}, \frac{1}{\sqrt{d}}\right)$, $d \in \mathbb{Z}$. The integer polynomial solution is possible only for odd periodic lengths.

Thus

$$\left(\frac{X^k+\sqrt{X^{2k}+d}}{\sqrt{d}}\right)^{2n-1} = \frac{1}{d^{(2n-1)/2}}\left(X^k+\sqrt{X^{2k}+d}\right)^{2n-1}$$
$$= P_{2n-1}(X)+\sqrt{X^{2k}+d}Q_{2n-1}(X), \;\; n\in\mathbb{N}.$$

We now expand the powers. Thus, to show the existence of non-trivial solutions in $\mathbb{Z}[X]$ for the negative polynomial Pell's equation (6), it is enough to show that the leading coefficient of $P_{2n-1}(X)$ is an integer.

Hence, the coefficient of $X^{k(2n-1)}$ in $P_{2n-1}(X)$ is

$$\frac{1}{d^{(2n-1)/2}}\left(1+\binom{2n-1}{2}+\binom{2n-1}{4}+\cdots\right)=\frac{2^{(2n-2)}}{d^{(2n-1)/2}}.$$

The integer solutions exist if and only if $d=1$. This completes the proof of the theorem. □

The following theorems are some of other negative polynomial Pell's equations.

Theorem 2.2. *The negative polynomial Pell's equation*

$$P^2(X)-(X^{2k}+aX+b)Q^2(X)=-1, \tag{25}$$

where $a,\ b\in\mathbb{Z}$ *has no non-trivial solutions in* $\mathbb{Z}[X]$.

Theorem 2.3. *The negative polynomial Pell's equation*

$$P^2(X)-(X^{2k}+aX^k+b)Q^2(X)=-1, \tag{26}$$

where $a,\ b\in\mathbb{Z}$ *and* $k\in\mathbb{N}$ *has no non-trivial solutions in* $\mathbb{Z}[X]$ *except for* $b=a^2/4+1$.

Since the length of the period in the continued fraction expansions of both $\sqrt{X^{2k}+aX+b}$ and $\sqrt{X^{2k}+aX^k+b}$ (except for $b=a^2/4+1$) are 2, then by Corollary 2.1 the negative polynomial Pell's equations (25) and (26) have no non-trivial solutions in $\mathbb{Z}[X]$.

3 Continued fraction expansions of some other polynomials

Mathematicians have recently focused on degenerate special numbers and polynomials, including Bernoulli, Euler, Stirling numbers, Bell polynomials, harmonic numbers, and hyperharmonic numbers [7–12]. We specifically focused on harmonic numbers.

The harmonic numbers are defined by

$$H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}, \quad (n\in\mathbb{N})$$

with $H_0=0$ (see [3]). The generating function of the harmonic numbers is given by

$$-\frac{\log(1-t)}{1-t}=\sum_{n=0}^{\infty}H_nt^n.$$

Recently, the degenerate harmonic numbers were defined by

$$-\frac{\log_{\lambda}(1-t)}{1-t}=\sum_{n=0}^{\infty}H_{n,\lambda}t^{n},$$

where $\log_{\lambda}$ is the degenerate logarithm, which is the compositional inverse of e_{λ} (see [12, 13]).

Now we write

$$H_{n,\lambda}=\sum_{k=1}^{n}\frac{(1)_{k,1/\lambda}\lambda^{k-1}(-1)^{k-1}}{k!},\quad H_{0,\lambda}=0,$$

where $(x)_{0,\lambda}=1$; $(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$, $n\geq 1$. We note that $\lim_{\lambda\to 0}H_{n,\lambda}=H_{n}$, $n\geq 1$. The continued fraction expansion of any number is as follows [6]:

Definition 3.1. *An expression of the form*

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ddots}}}$$

is called a continued fraction expansion. The values a_i $(i=0,1,\ldots)$ are called partial quotients which are integers, real or complex numbers or functions of variables.

Let $\alpha=\alpha_0$ be any real number and we define

$$\begin{cases} a_k & =\lfloor\alpha_k\rfloor \qquad \textit{for } k=0,1,2,\ldots \\ \alpha_{k+1} & =\dfrac{1}{\alpha_k-a_k} \qquad \textit{if } \alpha_k \textit{ is not an integer.}\end{cases}$$

Moreover, the k-th convergent of α_0 is a rational number. i.e., let $\frac{p_k}{q_k}$ is the k-th convergent with $\gcd(p_k,q_k)=1$. We write

$$\frac{p_k}{q_k}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{\ddots+\cfrac{1}{a_k}}}}}$$

The convergents $\frac{p_k}{q_k}$ of α are defined as follows:

$$\begin{aligned} p_{-1}&=1, & p_0&=a_0, & p_k&=a_kp_{k-1}+p_{k-2},\\ q_{-1}&=0, & q_0&=1, & q_k&=a_kq_{k-1}+q_{k-2}.\end{aligned}\tag{27}$$

for $k\geq 1$ [6, p. 250].

The following theorem is due to Seidel and Stern [23, 27].

Theorem 3.1. [1, 18] *If $a_n>0$, then $[a_0,a_1,a_2,\ldots]$ converges if and only if $\sum a_n$ diverges.*

We note that the harmonic series $\sum 1/n$ diverges. Then by the Seidel–Stern Theorem 3.1, the infinite continued fraction $[\frac{t}{1}, \frac{t}{2}, \frac{t}{3}, \ldots]$ converges for any positive real number t.

Definition 3.2. *The harmonic continued fractions are denoted by*

$$HCF(t) = \frac{t}{1} + \cfrac{1}{\frac{t}{2} + \cfrac{1}{\frac{t}{3} + \cfrac{1}{\frac{t}{4} + \cfrac{1}{\ddots}}}}$$

When $t = 1,\ HCF(1) = \frac{2}{\pi - 2}$*, when* $t = 2,\ HCF(2) = \frac{1}{2\ln 2 - 1}$ *(see* [2]*).*

We now rewrite the degenerate harmonic numbers as

$$H_{n,\lambda} = 1 + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\prod_{i=2}^{n}\frac{\lambda-(i-1)}{i}\right).$$

Thus, we define the degenerate harmonic continued fractions are as follows:

$$\cfrac{1}{1 - \cfrac{-\frac{\lambda-1}{2}}{1 + (-\frac{\lambda-1}{2}) - \cfrac{-\frac{\lambda-2}{3}}{1 + (-\frac{\lambda-2}{3}) - \cfrac{-\frac{\lambda-3}{4}}{1 + (-\frac{\lambda-3}{4}) - \cfrac{-\frac{\lambda-4}{5}}{\ddots}}}}}$$

Hence the degenerate harmonic continued fractions can be written as $[1, -\frac{\lambda-1}{2}, -\frac{\lambda-2}{3}, -\frac{\lambda-3}{4}, \ldots]$. Similarly, we shall attempt to define continuous fraction expansions of more degenerate polynomials in the future.

4 Conclusion

In this paper, we considered the negative polynomial Pell's equation and proved a necessary and sufficient condition for it to have a solution. Moreover, we have discussed the existence of integer polynomial solutions with the help of continued fraction expansions and the ABC conjecture for polynomials. Finally, as an application we defined the degenerate harmonic continued fractions.

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References

[1] Beardon, A. F., & Short, I. (2010). The Seidel, Stern, Stolz and Van Vleck Theorem on continued fractions. *Bulletin of the the London Mathematical Society*, 42(3), 457–466.

[2] Bunder, M., & Tonien, J. (2017). Closed form expressions for two harmonic continued fractions. *The Mathematical Gazette*, 101(552), 439–448.

[3] Comte, L. (1974). *Advanced Combinatorics: The Art of Finite and Infinite Expansions* (Revised and enlarged edition). Dordrecht: D. Reidel Publishing Co, Boston.

[4] Dubickas, A., & Steuding, J. (2004). The polynomial Pell equation. *Elemente der Mathematik*, 59(4), 133–143.

[5] Fouvry, É., & Klüners, J. (2010). On the negative Pell equation. *Annals of Mathematics*, 172(3), 2035–2104.

[6] Hua, L. K. (1982). *Introduction to Number Theory*. Springer, Berlin.

[7] Kim, D. S., & Kim, T. (2020). A note on a new type of generate Bernoulli numbers. *Russian Journal of Mathematical Physics*, 27(2), 227–235.

[8] Kim, T., & Kim, D. S. (2020). Degenerate polyexponential functions and degenerate Bell polynomials. *Journal of Mathematical Analysis and Applications*, 487(2), Article 124017.

[9] Kim, D. S., Kim, T., Kwon, J., & Lee, H. (2020). A note on λ-Bernoulli numbers of the second kind. *Advanced Studies in Contemporary Mathematics*, 30(2), 187–195.

[10] Kim, T., & Kim, D. S. (2020). Some relations of two type 2 polynomials and discrete harmonic numbers and polynomials. *Symmetry*, 12(6), 905.

[11] Kim, T., & Kim, D. S. (2023). Combinatorial identities involving degenerate harmonic and hyperharmonic numbers. *Advances in Applied Mathematics*, 148, Article 102535.

[12] Kim, T., & Kim, D. S. (2023). Some identities involving degenerate Stirling numbers associated with several degenerate polynomials and numbers. *Russian Journal of Mathematical Physics*, 30(1), 62–75.

[13] Kim, T., & Kim, D. S. (2023). Some identities on degenerate hyperharmonic numbers. *Georgian Mathematical Journal*, 30(2), 255–262.

[14] Koymans, P., & Pagano, C. (2020). *Higher Rédei reciprocity and integral points on conics*. arXiv preprint: 2005.14157.

[15] Lagrange, J. L. (1767). *Sur la solution des problèmes indeterminés du second degré*. Vol. XXIII, Berlin.

[16] Lagrange, J. L. (1768). *Histoire de ĺ Académie de Berlin*. Vol. XXIV, Berlin.

[17] Lang, S. (1993). Die abc-Vermutung. *Elemente der Mathematik*, 48, 89–99.

[18] Lorentzen, L., & Waadeland, H. (2008). *Continued Fractions. Volume 1: Convergence Theory*. Atlantis Press.

[19] Mason, R. C. (1984). *Diophantine equations over function fields*. LMS Lecture Notes, 96, Cambridge University Press.

[20] Nathanson, M. B. (1976). Polynomial Pell's equation. *Proceedings of the American Mathematical Society*, 56, 89–92.

[21] Niven, I., Zuckerman, H. S., & Montgomery, H. L. (1991). *An Introduction to the Theory of Numbers*. (5th ed.). University of Michigan, John Wiley & Sons, New York.

[22] Pastor, A. V. (2001). Generalized Chebyshev polynomials and the Pell–Abel equation. *Fundamentalnaya i Prikladnaya Matematika*, 7(4), 1123–1145.

[23] Seidel, L. (1846). Untersuchungen über die Konvergenz und Divergenz der Kettenbrüche, Habilschrift, München.

[24] Silverman, J. H. (1984). The S-unit equation over function fields. *Mathematical Proceedings of the Cambridge Philosophical Society*, 95(1), 3–4.

[25] Snyder, N. (2000). An alternate proof of Mason's theorem. *Elemente der Mathematik*, 55(3), 93–94.

[26] Stark, H. M. (1978). *An Introduction to Number Theory*. MIT Press, Cambridge (MA).

[27] Stern, M. A. (1848) Über die Kennzeichen der Konvergenz eins Kettenbruchs. *Journal für die Reine und Angewandte Mathematik*, 1848(37), 255–272.

[28] Stothers, W. W. (1981). Polynomial identities and Hauptmoduln. *The Quarterly Journal of Mathematics*, 32(3), 349–370.