

MIN-turns and MAX-turns in k -Dyck paths: A pure generating function approach

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Abstract: k -Dyck paths differ from ordinary Dyck paths by using an up-step of length k . We analyze at which level the path is after the s -th up-step and before the $(s+1)$ -st up-step. In honour of Rainer Kemp who studied a related concept 40 years ago the terms MAX-terms and MIN-terms are used. Results are obtained by an appropriate use of trivariate generating functions; practically no combinatorial arguments are used.

Keywords: Dyck paths, Generating functions, Kernel method, Lagrange inversion.

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1 Introduction

The objects of our interest in this paper are k -Dyck paths, having up-steps $(1, k)$ and down-steps $(1, -1)$, and never go below the x -axis. At the end, they reach the x -axis, but we also need versions that end at a prescribed level different from 0. Much material about such paths can be found in [9], [7] and in many standard textbooks on combinatorics.

We consider MAX-turns, where each up-step ends and MIN-turns, where each up-step starts. The Figure 1 explains the concept readily:



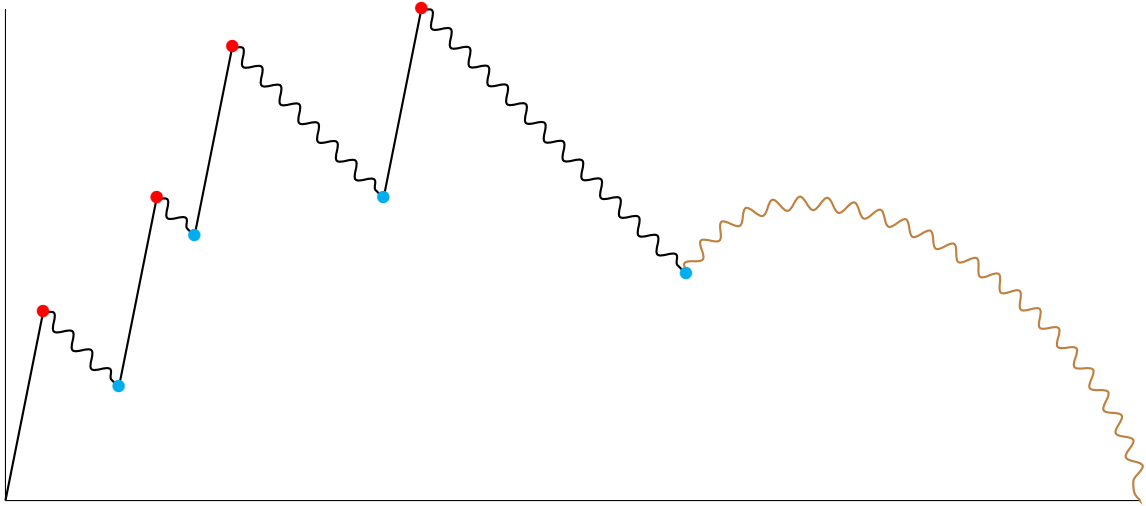


Figure 1. The first 4 MAX-turns and the first 4 MIN-turns are shown.

The k -Dyck paths can only exist for a length of the form $(k + 1)N$, which is clear for combinatorial reasons or otherwise. We want to know the average level of the s -th MAX-turn resp. MIN-turn, among all k -Dyck of the same length. In order to do this, we sum the level of the s -th MAX-turn resp. MIN-turn, over all k -Dyck paths of length $(k + 1)N$. The generating functions we obtain, enumerate the (partial) k -Dyck paths that end at a prescribed level j as well; this is of interest in [2] and later in [7], in the section on Hoppy's walks.

To get the average, one only needs to divide by the total number of such k -Dyck paths. Our main achievement is to get the *fully explicit* functions

$$\text{MAX}(z, w) = \sum_{N \geq 0, s \geq 1} z^{(k+1)N} w^s \left[\begin{array}{l} \text{cumulative level of the } s\text{-th MAX-turn} \\ \text{in all } k\text{-Dyck paths of length } (k + 1)N \end{array} \right]$$

and

$$\text{MIN}(z, w) = \sum_{N \geq 0, s \geq 1} z^{(k+1)N} w^s \left[\begin{array}{l} \text{cumulative level of the } s\text{-th MIN-turn} \\ \text{in all } k\text{-Dyck paths of length } (k + 1)N \end{array} \right].$$

As a bonus we get $\text{OSC}(z, w) := \text{MAX}(z, w) - \text{MIN}(z, w)$; this cumulates the lengths of the wavy line between the s -th MAX-turn and s -th MIN-turn, see Figure 1. The function $\text{OSC}(z, w)$ is somewhat simpler than $\text{MIN}(z, w)$ and $\text{MAX}(z, w)$, because of cancellations. In this way, we recover some of the results from [1] without resorting to any bijective combinatorics. Note that such a wavy line might have length zero as well, if two up-steps follow each other immediately. Rainer Kemp in [5] has considered MAX- and MIN-turns for Dyck paths, although his definitions were slightly different (peaks and valleys).

The key to the success of our method is the simple but perhaps unusual identity

$$\sum_{i \geq 0} ([z^i] f(z)) \cdot y^i = f(y).$$

2 Some basic observations

As we will see soon, the equation $u = z + zuw^{k+1}$ plays a major role when enumerating k -Dyck paths. The equation is of the form $u = z\Phi(u)$, with $\Phi(u) = 1 + wu^{k+1}$, so it is amenable to the Lagrange inversion [4], and the coefficients of the inverse function $u = u(z)$ can be computed:

$$\begin{aligned} [z^{(k+1)N+1}]u &= \frac{1}{(k+1)N+1} [u^{(k+1)N}] (1 + wu^{k+1})^{(k+1)N+1} \\ &= \frac{1}{(k+1)N+1} w^N \binom{(k+1)N+1}{N}. \end{aligned}$$

One solution of the equation, which is analytic at the origin and has combinatorial significance, is called \bar{u} . We will use the formula (again by the Lagrange inversion formula)

$$\bar{u} = z \sum_{N \geq 0} w^N z^{(k+1)N} \frac{1}{kN+1} \binom{(k+1)N}{N}. \quad (1)$$

We also need the specialization where $w = 1$, which we call \hat{u} , which satisfies the equation $u = z + zu^{k+1}$:

$$\hat{u} = z \sum_{\lambda \geq 0} \frac{1}{(k+1)\lambda+1} \binom{1+(k+1)\lambda}{\lambda} z^{(k+1)\lambda} \quad (2)$$

and (again by the Lagrange inversion formula)

$$\hat{u}^{-k} = z^{-k} - z \sum_{\lambda \geq 0} \frac{k}{\lambda+1} \binom{(k+1)\lambda}{\lambda} z^{(k+1)\lambda}. \quad (3)$$

3 MIN-turns

After these classical observations about the implicit function, we move to the treatment of MIN-turns. Our method of choice is *adding a new slice*. The following substitution is essential for adding a new slice (which is an up-step, followed by a maximal sequence of down-steps):

$$u^j \longrightarrow zw \sum_{0 \leq i \leq j+k} z^i u^{j+k-i} = \frac{z w u^{k+1}}{u-z} u^j - \frac{w z^{k+2}}{u-z} z^j.$$

The substitution tells us what can become of a partial k -ary path landing at level j when adding (attaching) another slice landing on level $j+k-i$. The technique of adding-a-new slice is described in [3].

Now let $F_m(u) = F_m(u; z)$ be the generating function according to m slices; z refers to the lengths and u to the level of the m -th MIN-turn. The substitution leads to

$$F_{m+1}(u) = \frac{z w u^{k+1}}{u-z} F_m(u) - \frac{w z^{k+2}}{u-z} F_m(z), \quad F_0(u) = 1.$$

Let $F(u) = \sum_{m \geq 0} F_m(u)$, so that we do not care about the number m anymore, since the variable w takes care of it; then

$$F(u) = F(u; z, w) = 1 + \frac{z w u^{k+1}}{u - z} F(u) - \frac{w z^{k+2}}{u - z} F(z),$$

or

$$F(u) = \frac{u - z - w z^{k+2} F(z)}{u - z - z w u^{k+1}}.$$

$u = \bar{u}$ is a factor of the denominator, but for z and u small, we have $u \sim z$, so this factor must cancel in the numerator as well. This is what one learns from the kernel method [6]. We find

$$F(z) = \frac{\bar{u} - z}{w z^{k+2}}$$

and further

$$F(u) = F(u; z, w) = \frac{u - \bar{u}}{u - z - z w u^{k+1}}.$$

That ends the computation of the “left” part of the k -Dyck path. For the right one, we start at level h with an up-step and end eventually at the zero level. We do not use the variable w here. The kernel method could also be used, but there is a simpler way. Reading the path from right to left, there is a decomposition when the path leaves a level and never comes back to it. Recall that \bar{u}/z is the generating function of k -Dyck paths. With this, we find for the enumeration of the “right” part

$$\left(\frac{\bar{u}}{z}\right)^h z^h \left(\frac{\bar{u}}{z} - 1\right) = \frac{(\bar{u} - z)\bar{u}^h}{z};$$

the minus 1 term happens since the reversed path must end with a down-step. The formula works only for $h \geq 1$, but the instance $h = 0$ is not needed (although easy). The fact that this generating function is essentially a *power* is part of our successful approach. We now have

$$\begin{aligned} \text{MIN}(z, w) &= \sum_{h \geq 1} h[u^h] F(u) \cdot \frac{(\bar{u} - z)\bar{u}^h}{z} \\ &= \sum_{h \geq 1} [u^{h-1}] \frac{d}{du} F(u) \cdot \frac{(\bar{u} - z)\bar{u}^h}{z} \\ &= \frac{(\bar{u} - z)\bar{u}}{z} \cdot \frac{-zu + uz w u^{k+1} k + \bar{u}u - \bar{u}z w u^{k+1} k - \bar{u}z w u^{k+1}}{u(u - z - z w u^{k+1})^2} \Big|_{u=\bar{u}} \\ &= \frac{(\bar{u} - z)\bar{u}}{z} \cdot \frac{-z\hat{u} + uz w \hat{u}^{k+1} k + \bar{u}\hat{u} - \bar{u}z w \hat{u}^{k+1} k - \bar{u}z w \hat{u}^{k+1}}{\hat{u}(\hat{u} - z - z w \hat{u}^{k+1})^2}. \end{aligned}$$

A simplification that only uses $\hat{u}^{k+1} = \frac{\hat{u}-z}{z}$ eventually leads to

$$\text{MIN}(z, w) = \frac{k w \hat{u}}{z(1-w)^2} + \frac{(\bar{u} - z)}{z^2(1-w)^2 \hat{u}^k} - \frac{(k+1)\bar{u}w}{z(1-w)^2}$$

which can be checked easily, even by hand.

Theorem 3.1. *The generating function $\text{MIN}(z, w)$ where the coefficient of $z^{(k+1)N} w^s$ refers to the cumulative levels of the s -th MIN-turn, is given by*

$$\text{MIN}(z, w) = \frac{k w \hat{u}}{z(1-w)^2} + \frac{(\bar{u} - z)}{z^2(1-w)^2 \hat{u}^k} - \frac{(k+1)\bar{u}w}{z(1-w)^2}.$$

The next step is to expand this function:

$$\begin{aligned}
[w^s]\text{MIN}(z, w) &= [w^s] \frac{k w \widehat{u}}{z(1-w)^2} - [w^s] \frac{(k+1)\bar{u}w}{z(1-w)^2} + [w^s] \frac{(\bar{u}-z)}{z^2(1-w)^2 \widehat{u}^k} \\
&= \frac{sk\widehat{u}}{z} - (k+1) \sum_{i=0}^{s-1} (s-i) [w^i] \frac{\bar{u}}{z} \\
&\quad + \sum_{i=0}^s (s+1-i) [w^i] \frac{z \sum_{N \geq 0} w^N z^{(k+1)N} \frac{1}{kN+1} \binom{(k+1)N}{N}}{z^2 \widehat{u}^k} - z \\
&= \frac{sk\widehat{u}}{z} - (k+1) \sum_{i=0}^{s-1} (s-i) [w^i] \frac{\bar{u}}{z} \\
&\quad + \sum_{i=0}^s (s+1-i) [w^i] \frac{\sum_{N \geq 1} w^N z^{(k+1)N} \frac{1}{kN+1} \binom{(k+1)N}{N}}{z \widehat{u}^k} \\
&= \frac{sk\widehat{u}}{z} - (k+1) \sum_{i=0}^{s-1} (s-i) z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \\
&\quad + \sum_{i=1}^s (s+1-i) \frac{1}{z \widehat{u}^k} z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \\
&= \frac{sk\widehat{u}}{z} - (k+1) \sum_{i=0}^{s-1} (s-i) z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \\
&\quad + \sum_{i=1}^s (s+1-i) z^{(k+1)(i-1)} \frac{1}{ki+1} \binom{(k+1)i}{i} \\
&\quad - \sum_{i=1}^s (s+1-i) z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \sum_{\lambda \geq 0} \frac{k}{\lambda+1} \binom{(k+1)\lambda}{\lambda} z^{(k+1)\lambda}.
\end{aligned}$$

And now we read off the coefficient of $z^{(k+1)N}$; we assume that $N \geq s$, otherwise a path would not have an s -th MIN-turn:

$$\begin{aligned}
[w^s z^{(k+1)N}]\text{MIN}(z, w) &= sk \frac{1}{kN+1} \binom{(k+1)N}{N} \\
&\quad - \sum_{i=1}^s (s+1-i) \frac{1}{ki+1} \binom{(k+1)i}{i} \frac{k}{(N-i)+1} \binom{(k+1)(N-i)}{(N-i)}.
\end{aligned}$$

Theorem 3.2. *The sum of levels of the s -th MIN-turns in all the k -Dyck paths of length $(k+1)N$ is given by*

$$\frac{sk}{kN+1} \binom{(k+1)N}{N} - \sum_{i=1}^s (s+1-i) \frac{1}{ki+1} \binom{(k+1)i}{i} \frac{k}{(N-i)+1} \binom{(k+1)(N-i)}{(N-i)}.$$

This formula is fully explicit and covers all instances of $N \geq s$.

4 MAX-turns

It is easy to go from a MIN-turn to the next MAX-turn, just by doing *one* up-step. On the level of generating functions, this means

$$G(u; z, w) = F(u; z, w)wzu^k.$$

The right part of a part (everything from a given MAX-turn until the end) is even easier than before, since level h must be reached without further restriction. The result for the “right” side of the path is then

$$\left(\frac{\widehat{u}}{z}\right)^{h+1} z^h = \widehat{u}^{h-1} \frac{\widehat{u}^2}{z}.$$

After these preliminaries, we can engage into the big computation:

$$\begin{aligned} \text{MAX}(z, w) &= \sum_{h \geq 1} h[u^h]G(u) \cdot \widehat{u}^{h-1} \frac{\widehat{u}^2}{z} = \frac{\widehat{u}^2}{z} \sum_{h \geq 1} [u^{h-1}] \frac{d}{du} G(u) \cdot \widehat{u}^{h-1} \\ &= \frac{wk\widehat{u}}{z(1-w)^2} - \frac{wk\bar{u}}{z(1-w)^2} + \frac{w(\bar{u}-z)}{z^2\widehat{u}^k(1-w)^2} - \frac{w^2\bar{u}}{z(1-w)^2}. \end{aligned}$$

The same type of simplifications as before has been applied.

And now we go to the coefficients of this, in a similar style as it was done for the MIN-turns:

$$\begin{aligned} [w^s]\text{MAX}(z, w) &= s \frac{k\widehat{u}}{z} - \sum_{i=0}^{s-1} (s-i)[w^i] \frac{k\bar{u}}{z} \\ &\quad + \frac{1}{z\widehat{u}^k} \sum_{i=0}^{s-1} (s-i)[w^i] \frac{(\bar{u}-z)}{z} - \sum_{i=0}^{s-2} (s-1-i)[w^i] \frac{\bar{u}}{z} \\ &= sk \sum_{N \geq 0} z^{(k+1)N} \frac{1}{kN+1} \binom{(k+1)N}{N} \\ &\quad - \sum_{i=0}^{s-1} (s-i) z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \\ &\quad + \sum_{i=1}^{s-1} (s-i) z^{(k+1)(i-1)} \frac{1}{ki+1} \binom{(k+1)i}{i} \\ &\quad - \sum_{\lambda \geq 0} \frac{k}{\lambda+1} \binom{(k+1)\lambda}{\lambda} z^{(k+1)\lambda} \sum_{i=1}^{s-1} (s-i) z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \\ &\quad - \sum_{i=0}^{s-2} (s-1-i) z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i}. \end{aligned}$$

And, again for $N \geq s$, we read off the coefficient of $z^{(k+1)N}$:

$$\begin{aligned} [w^s z^{(k+1)N}]\text{MAX}(z, w) &= sk \frac{1}{kN+1} \binom{(k+1)N}{N} \\ &\quad - \sum_{i=1}^{s-1} (s-i) \frac{1}{ki+1} \binom{(k+1)i}{i} \frac{k}{N-i+1} \binom{(k+1)(N-i)}{N-i}. \end{aligned}$$

Theorem 4.1. *The generating function $\text{MIN}(z, w)$ where the coefficient of $z^{(k+1)N}w^s$ refers to the cumulative levels of the s -th MAX-turn, is given by*

$$\text{MAX}(z, w) = \frac{wk\widehat{u}}{z(1-w)^2} - \frac{wk\bar{u}}{z(1-w)^2} + \frac{w(\bar{u}-z)}{z^2\widehat{u}^k(1-w)^2} - \frac{w^2\bar{u}}{z(1-w)^2}.$$

The coefficient $[w^s z^{(k+1)N}] \text{MAX}(z, w)$ is for $N \geq s$ given by

$$\frac{sk}{kN+1} \binom{(k+1)N}{N} - \sum_{i=1}^{s-1} (s-i) \frac{1}{ki+1} \binom{(k+1)i}{i} \frac{k}{N-i+1} \binom{(k+1)(N-i)}{N-i}.$$

5 The oscillation

The cumulative function of the s -th oscillation (= total length of the s -th wavy line, cf. Figure 1 in all paths of length $(k+1)N$) is

$$\text{OSC}(z, w) := \text{MAX}(z, w) - \text{MIN}(z, w).$$

The simplication is easy:

$$\begin{aligned} \text{OSC}(z, w) &= \frac{wk\widehat{u}}{z(1-w)^2} - \frac{wk\bar{u}}{z(1-w)^2} + \frac{w(\bar{u}-z)}{z^2\widehat{u}^k(1-w)^2} - \frac{w^2\bar{u}}{z(1-w)^2} \\ &\quad - \left[\frac{kw\widehat{u}}{z(1-w)^2} + \frac{(\bar{u}-z)}{z^2(1-w)^2\widehat{u}^k} - \frac{(k+1)\bar{u}w}{z(1-w)^2} \right] \\ &= \frac{w(\bar{u}-z)}{z^2\widehat{u}^k(1-w)^2} - \frac{(\bar{u}-z)}{z^2\widehat{u}^k(1-w)^2} - \frac{w^2\bar{u}}{z(1-w)^2} + \frac{w\bar{u}}{z(1-w)^2} \\ &= \frac{w\bar{u}}{z(1-w)} - \frac{(\bar{u}-z)}{z^2\widehat{u}^k(1-w)}. \end{aligned}$$

Further, by reading off coefficients ($s \geq 1$),

$$\begin{aligned} [w^s] \text{OSC}(z, w) &= \sum_{i=1}^s [w^i] \frac{\bar{u}}{z} - \sum_{i=0}^s [w^i] \frac{(\bar{u}-z)}{z^2\widehat{u}^k} \\ &= \sum_{i=1}^s z^{(k+1)i} \frac{1}{kN+1} \binom{(k+1)i}{i} - \frac{1}{z\widehat{u}^k} \sum_{i=0}^s [w^i] \frac{(\bar{u}-z)}{z} \\ &= \sum_{i=1}^s z^{(k+1)i} \frac{1}{kN+1} \binom{(k+1)i}{i} - \frac{1}{z\widehat{u}^k} \sum_{i=1}^s [w^i] \sum_{N \geq 1} w^N z^{(k+1)N} \frac{1}{kN+1} \binom{(k+1)N}{N} \\ &= \sum_{i=1}^s z^{(k+1)i} \frac{1}{kN+1} \binom{(k+1)i}{i} - \frac{1}{z\widehat{u}^k} \sum_{i=1}^s z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i}. \end{aligned}$$

The next step is to read off $[w^s z^{(k+1)N}] \text{OSC}(z, w)$ (for $N \geq s$):

$$\begin{aligned}
[w^s z^{(k+1)N}] \text{OSC}(z, w) &= [z^{(k+1)N}] \sum_{i=1}^s z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \\
&\quad - [z^{(k+1)N}] \frac{1}{z \widehat{u}^k} \sum_{i=1}^s [w^i] z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \\
&= -[z^{(k+1)N}] \left[z^{-k-1} - \sum_{\lambda \geq 0} \frac{k}{\lambda+1} \binom{(k+1)\lambda}{\lambda} z^{(k+1)\lambda} \right] \sum_{i=1}^s z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \\
&= [z^{(k+1)N}] \sum_{i=1}^s z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \sum_{\lambda \geq 0} \frac{k}{\lambda+1} \binom{(k+1)\lambda}{\lambda} z^{(k+1)\lambda} \\
&= \sum_{i=1}^s z^{(k+1)i} \frac{1}{ki+1} \binom{(k+1)i}{i} \frac{k}{N-i+1} \binom{(k+1)(N-i)}{N-i}.
\end{aligned}$$

We summarize the results.

Theorem 5.1. *The cumulative (summed over all k -Dyck paths of length $(k+1)N$) difference of the s -th MAX turn and the $(s+1)$ -st MIN turn (nicknamed the s -th wavy line) is given by the explicit formula (for $N \geq s \geq 1$)*

$$\sum_{i=1}^s \frac{1}{ki+1} \binom{(k+1)i}{i} \frac{k}{N-i+1} \binom{(k+1)(N-i)}{N-i}. \quad (4)$$

In this paper the results have been derived by manipulating power series only; some equivalent ones were obtained in [1] by other (mostly bijective) methods.

6 Application: Hoppy walks

Deng and Mansour [2], compare also [7], introduce a rabbit named Hoppy and let him move according to certain rules. While the story about Hobby is charming and entertaining, we do not need this here and move straight ahead to the enumeration issues. Eventually, the enumeration problem is one about k -Dyck paths ($k \geq 1$). The up-steps are $(1, k)$ and the down-steps are $(1, -1)$.

The question is about the length of the sequence of down-steps printed in red, see Figure 2. Or, phrased differently, how many k -Dyck paths end on level j , after m up-steps, the last step being an up-step. The recent paper [8] contains similar computations, although without the restriction that the last step must be an up-step.

This question clearly belongs to the realm of the questions discussed here.

In fact, we are looking at the last oscillation ('wavy line'). If a k -Dyck path has length $(k+1)N$, the last oscillation occurs for $s = N$. The formula (5) leads for $s = N$ to

$$\sum_{i=1}^N \frac{1}{ki+1} \binom{(k+1)i}{i} \frac{k}{N-i+1} \binom{(k+1)(N-i)}{N-i}. \quad (5)$$

Further considerations are in [7].

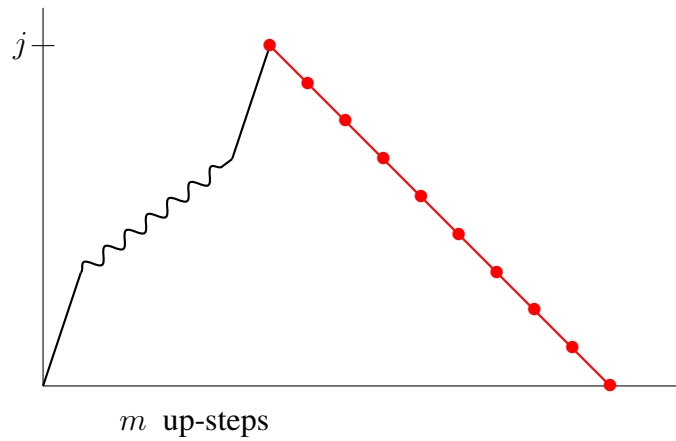


Figure 2. The number of final down-steps

7 Application: Sequences from The On-Line Encyclopedia of Integer Sequences

Now we move to instances of formula (5) that appear in [10]. The principal reference is [1].

$k = 2$	$s = N$	A334680
$k = 3$	$s = N$	A334682
$k = 4$	$s = N$	A334719
$k = 2$	$s = 1$	A007226
$k = 3$	$s = 1$	A007228
$k = 4$	$s = 1$	A124724
$k = 2$	$s = 2$	A334640
$k = 3$	$s = 2$	A334645
$k = 4$	$s = 2$	A334646

Tables of other similar sequences can be constructed easily with a computer algebra program, using explicit formulæ in Theorems 1, 2, 3.

References

- [1] Asinowski, A., Hackl, B., & Selkirk, S. (2022). Down-step statistics in generalized Dyck paths. *Discrete Mathematics & Theoretical Computer Science*, 24(1), Article 17.
- [2] Deng, E. Y. P., & Mansour, T. (2008). Three Hoppy path problems and ternary paths. *Discrete Applied Mathematics*, 156(5), 770–779.
- [3] Flajolet, P., & Prodinger, H. (1987). Level number sequences for trees. *Discrete Mathematics*, 65(2), 149–156.

- [4] Flajolet, P., & Sedgewick, R. (2009). *Analytic Combinatorics*. Cambridge University Press, Cambridge.
- [5] Kemp, R. (1982). On the average oscillation of a stack. *Combinatorica*, 2(2), 157–176.
- [6] Prodinger, H. (2003 / 2004). The kernel method: A collection of examples. *Séminaire Lotharingien de Combinatoire*, 50, Article B50f.
- [7] Prodinger, H. (2023). A walk through my lattice path garden. *Séminaire Lotharingien de Combinatoire*, 87b, Article #1.
- [8] Prodinger, H. (2024). On k -Dyck paths with a negative boundary. *Journal of Combinatorial Methods and Combinatorial Computing*, 119, 3–12.
- [9] Selkirk, S. J. (2019). *On a generalisation of k -Dyck paths*. Master's thesis, Stellenbosch University.
- [10] Sloane, N. J. A., & The OEIS Foundation Inc. (2023). The On-Line Encyclopedia of Integer Sequences. Available online at: <https://oeis.org/>.