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Metallic means and Pythagorean triples

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Abstract: In this article, we study the connection between Pythagorean triples and metallic means. We derive several interconnecting identities between different metallic means. We study the Pythagorean triples in the three-term recurrent sequences corresponding to different metallic means. Further, we relate different families of primitive Pythagorean triples to the corresponding metallic means.
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1 Introduction

Historically, golden ratio and silver ratios have been studied for composing music, paintings, construction of temples, and ancient places of importance. Popular books [4] and [5] explore historical and mathematical aspects of the golden ratio. In [1] the silver ratio arising in the construction of Castel del Monte (a castle in Italy) is discussed. In [11], Vera W. de Spinadel introduced n^{th} metallic means as the limiting ratio of consecutive terms of a sequence $\{G_k\}_{k\geq 0}$ generated by the three-term recurrence $G_{k+1} = nG_k + G_{k-1}$. In the article [12], she studied the design aspects of the metallic means. A notion of an associated polygon with the metallic means is analyzed in [10, 13]. In a recent article [14], the author relates how different taffy (a candy made by pulling sticky sugar base) pullers are related to golden and silver ratios. The taffy puller pulls the taffy base using the moving rods. In certain pullers length of the taffy at the $(k+1)^{\text{st}}$ step is twice the length at k^{th} step added to the length at $(k-1)^{\text{st}}$ step. Thus relating to the silver ratio. Recently, [7] studied the connection between right-angle triangles and metallic means. In [7] connection between Metallic means with Pythagorean triples is explored. [8] introduced the triads of metallic means. In [9], author gave the relation between different metallic means and the corresponding sequences. Here, we generalize the identities in [8], prove and extend identities in [7], prove the results in [9], and further study the Pythagorean triples in the recursively defined sequences whose limiting ratio is a metallic mean.

In this article, we use metallic means, metallic ratios or metallic numbers as synonyms. A metallic mean of order n is the real positive solution of the quadratic equation,

$$x^2 - nx - 1 = 0. \tag{1}$$

It is given by $\delta_n = \dfrac{n+\sqrt{n^2+4}}{2}$. We say δ_n is a metallic mean of integer order when n is an integer, and a metallic mean of fractional order when n is a rational number. Unless mentioned we are referring to metallic means of integer order. From the expression of δ_n, we get

$$(2\delta_n - n)^2 = n^2 + 4. \tag{2}$$

From the equation (2) we can look at a right triangle of sides 2,n,H with hypotenuse $H = 2\delta_n - n$. Then $\delta_n = \dfrac{H+n}{2}$.

1.1 Notations

Let $\mathtt{i} = \sqrt{-1}$. The set $[m] = \{1, 2, \ldots, m\}$. Define the symmetric polynomials for a set of numbers $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ as,

$$\begin{aligned}
\sigma_{-1}(\mathbf{x}) &= 0,\\
\sigma_0(\mathbf{x}) &= 1,\\
\sigma_1(\mathbf{x}) &= \sum_{i=1}^{n} x_i,\\
\sigma_j(\mathbf{x}) &= \sum_{s\subset[n],|s|=j} \prod_{q\in s} x_q,\\
\sigma_n(\mathbf{x}) &= \prod_{i=1}^{n} x_i.
\end{aligned}$$

Let $\gcd(a, b)$ denote the greatest common divisor of the integrs a and b.

2 Results

2.1 Metallic means and cotangent relations:

In this section, we give identities relating to different integer metallic means.

Consider set of integers $\boldsymbol{k}_n = \{k_1, k_2, \ldots, k_n\}$, let the set of corresponding metallic means $\boldsymbol{\delta} = \{\delta_{k_1}, \delta_{k_2}, \ldots, \delta_{k_n}\}$. When n is odd, let $t_1 = t_2 = \lfloor \frac{n}{2} \rfloor$, and when n is even, let $t_1 = \frac{n}{2}$ and $t_2 = \frac{n}{2} - 1$.

Theorem 2.1. *For* $k = \frac{\sum_{j=0}^{t_1} \sigma_{n-2j}(\boldsymbol{k}_n) 4^j (-1)^j}{\sum_{j=0}^{t_2} \sigma_{n-(2j+1)}(\boldsymbol{k}_n) 4^j (-1)^j}$, *we have*

$$\delta_k = \frac{\sum_{j=0}^{t_1} \sigma_{n-2j}(\boldsymbol{\delta})(-1)^j}{\sum_{j=0}^{t_2} \sigma_{n-(2j+1)}(\boldsymbol{\delta})(-1)^j}.$$

Proof. Consider $\theta = \sum_{j=1}^{n} \theta_j$, for angles θ_j such that $k_j = 2\cot 2\theta_j$ for $k_j \in \boldsymbol{k}_n$, then we have $\delta_{k_j} = \cot\theta_j$. This can be seen by the elementary trigonometric relation (in comparison to Equation (1), with $x = \cot\theta$),

$$\cot^2\theta - 2\cot 2\theta \cot\theta - 1 = 0. \tag{3}$$

To find $\cot\theta$ in terms of $\cot\theta_j$, consider the complex numbers $z_j = (\cot\theta_j + \mathtt{i})$. Note that z_j has an argument θ_j. Now consider the complex number $z = \prod_{j=1}^{n} z_j$,

$$z = \prod_{j=1}^{n} (\cot\theta_j + \mathtt{i}). \tag{4}$$

Note that the argument of complex number z is $\theta = \sum_{j=1}^{n} \theta_j$. Thus $\cot\theta$ is obtained by taking the ratio of the real part and imaginary parts of the product (4).

Let $\mathcal{C}(\Theta) = \{\cot\theta_1, \cot\theta_2, \ldots, \cot\theta_n\}$. Then we have from (4),

$$z = \sum_{j=0}^{n} \sigma_{n-j}(\mathcal{C}(\Theta))\mathtt{i}^j.$$

Separating real and imaginary parts we have

$$\mathrm{Re}(z) = \sum_{j=0}^{t_1} \sigma_{n-2j}(\mathcal{C}(\Theta))(-1)^j,$$
$$\mathrm{Im}(z) = \sum_{j=0}^{t_2} \sigma_{n-(2j+1)}(\mathcal{C}(\Theta))(-1)^j.$$

Here when n is odd, $t_1 = t_2$ and equal to $\lfloor \frac{n}{2} \rfloor$. When n is even, $t_1 = \frac{n}{2}$ and $t_2 = \frac{n}{2} - 1$. Thus by taking the ratio of real and imaginary parts of z,

$$\cot(\theta) = \frac{\sum\limits_{j=0}^{t_1} \sigma_{n-2j}(\mathcal{C}(\Theta))(-1)^j}{\sum\limits_{j=0}^{t_2} \sigma_{n-(2j+1)}(\mathcal{C}(\Theta))(-1)^j}.$$

With $\mathcal{C}(2\Theta) = \{\cot 2\theta_1, \cot 2\theta_2, \ldots, \cot 2\theta_n\}$, we have

$$\begin{aligned}
2\cot(2\theta) &= \left(\frac{2^n}{2^{n-1}}\right) \frac{\sum\limits_{j=0}^{t_1} \sigma_{n-2j}(\mathcal{C}(2\Theta))(-1)^j}{\sum\limits_{j=0}^{t_2} \sigma_{n-(2j+1)}(\mathcal{C}(2\Theta))(-1)^j}, \\
&= \frac{\sum\limits_{j=0}^{t_1} \sigma_{n-2j}(\boldsymbol{k}_n)4^j(-1)^j}{\sum\limits_{j=0}^{t_2} \sigma_{n-(2j+1)}(\boldsymbol{k}_n)4^j(-1)^j}.
\end{aligned}$$

Thus we have

$$\delta_k = \frac{\sum\limits_{j=0}^{t_1} \sigma_{n-2j}(\delta_{k_i})(-1)^j}{\sum\limits_{j=0}^{t_2} \sigma_{n-(2j+1)}(\delta_{k_i})(-1)^j}, \tag{5}$$

for

$$k = \frac{\sum\limits_{j=0}^{t_1} \sigma_{n-2j}(\boldsymbol{k}_n)4^j(-1)^j}{\sum\limits_{j=0}^{t_2} \sigma_{n-(2j+1)}(\boldsymbol{k}_n)4^j(-1)^j}. \tag{6}$$

This completes the proof. □

Corollary 2.1. *For positive integers m, n, k, if $k = \dfrac{mn-4}{m+n}$, then we have $\delta_k = \dfrac{\delta_n\delta_m - 1}{\delta_m + \delta_n}$.*

Proof. Let $\theta_1 + \theta_2 = \theta$, with $m = 2\cot\theta_1$ and $n = 2\cot\theta_2$, and from (5) and (6) the proof follows. □

Note that by rearranging

$$k = \frac{mn-4}{m+n} \implies n = \frac{4+mk}{m-k}, \tag{7}$$

thus by choosing $k = m-1$, we get integer $n = 4 + m(m-1)$. Thus we have infinitely many solutions of positive integers m, n, k such that $k = \dfrac{mn-4}{m+n}$.

Example 2.1. *Take $m = 5$, then we have $(k, m, n) = (4, 5, 24)$ such that $4 = \dfrac{5 \times 24 - 4}{29}$. We have $\delta_4 = \dfrac{4+\sqrt{20}}{2}$, $\delta_5 = \dfrac{5+\sqrt{29}}{2}$ and $\delta_{24} = \dfrac{24+\sqrt{580}}{2}$. They satisfy $\delta_4 = \dfrac{\delta_5\delta_{24} - 1}{\delta_5 + \delta_{24}}$.*

Corollary 2.2. *For positive integers m, n, k, if $k = \frac{mn+4}{m-n}$, then we have $\delta_k = \frac{\delta_n\delta_m+1}{\delta_n-\delta_m}$.*

Proof. Similar to the arguments in the previous proof, let $\theta_1 - \theta_2 = \theta$, with $m = 2\cot\theta_1$ and $n = 2\cot\theta_2$, and from (5) and (6) the proof follows. □

By choosing $n = m - 1$, we have integer $k = 4 + m(m-1)$.

In general (6) is a fraction for any $\boldsymbol{k}_n$, we look for the choice of $\boldsymbol{k}_n$ such that k is an integer. The following theorem says there are infinitely many $\boldsymbol{k}_n$ for which k is an integer.

Theorem 2.2. *There are infinite sets of integers* $\{k, \boldsymbol{k}_n\}$ *which satisfy* (6)*.*

Proof. Note that we are not imposing the condition that all of the integers have to be positive. We prove this existence by induction on $n = |\boldsymbol{k}_n|$.

Our induction proposition P_n: There are integers $\boldsymbol{k}_n$ that make the denominator of (6) equal to one.

Therefore, $\boldsymbol{k}_n$ is such that

$$\left(\sum_{j=0}^{t_2} \sigma_{n-(2j+1)}(\boldsymbol{k}_n) 4^j (-1)^j\right) = 1.$$

Let $\boldsymbol{k}_n = \boldsymbol{k}_{n-1} \cup k_n$, we assume P_{n-1} is true. Then we have

$$\left(\sum_{j=0}^{t_2} (k_n \sigma_{n-(2j+2)}(\boldsymbol{k}_{n-1}) + \sigma_{n-(2j+1)}(\boldsymbol{k}_{n-1})) 4^j (-1)^j\right) = 1.$$

$$\left(\sum_{j=0}^{t_2} (k_n \sigma_{(n-1)-(2j+1)}(\boldsymbol{k}_{n-1}) + \sigma_{n-(2j+1)}(\boldsymbol{k}_{n-1})) 4^j (-1)^j\right) = 1,$$

$$k_n = 1 - \sum_{j=0}^{t_2} \sigma_{n-(2j+1)}(\boldsymbol{k}_{n-1}) 4^j (-1)^j.$$

Thus we have a $\boldsymbol{k}_n$ which makes P_n true. For the base case, when $n = 2$, we have the denominator, $k_1 + k_2$, choosing $k_2 = -(k_1 - 1)$ will make the denominator value to be one.

Now for every value of k_1 we have uniquely determined $\boldsymbol{k}_n$, therefore we have infinitely many $\boldsymbol{k}_n$. □

Example 2.2. *According to Theorem 2.2 we take initial condition* 7 *and* $\boldsymbol{k}_4 = \{7, -6, 47, 2167\}$ *gives integer* $k_5 = 4693727$*. Similarly, if we take* $\boldsymbol{k}_6 = \{2, -1, 7, 47, 2167, 4693727\}$ *this gives integer* $k_7 = 22031068456807$.

The above Example 2.2 suggests that similar to the rearrangement (7), we get a positive integer solution for (6) by rearranging it.

2.2 Metallic means and the three-term recurrent sequences:

The Fibonacci numbers are given by the recurrence

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}. \tag{8}$$

with the initial condition $\begin{bmatrix} F_1 & F_0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

The characteristic polynomial of the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is given by $\lambda^2 - \lambda - 1 = 0$. Thus Fibonacci numbers are the first metallic numbers.

In general, for the p^{th} metallic number, we have the sequence $\{G_n\}$ given by

$$\begin{bmatrix} G_n \\ G_{n-1} \end{bmatrix} = \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{n-1} \\ G_{n-2} \end{bmatrix}. \tag{9}$$

with the initial condition $\begin{bmatrix} G_1 & G_0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

In this section, we derive the relation for odd n,

$$\arctan \delta_p^{n+2} + \arctan \delta_p^n = 2 \arctan \delta_{2G_{n+1}}.$$

The characteristic polynomial for the matrix in (9) is given by $\lambda^2 - p\lambda - 1 = 0$. And the eigenvalues $\delta_p = \frac{p+\sqrt{P^2+4}}{2}$ and $\delta_p' = \frac{p-\sqrt{P^2+4}}{2}$.

An eigenvector of the matrix is given by

$$\begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \delta_p \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then we have

$$\begin{aligned} px + y &= \delta_p x \\ x &= \delta_p y. \end{aligned}$$

Taking $y = 1$, we get $x = \delta_p$. We also have $(\delta_p') = \frac{-1}{\delta_p}$. The eigenvalue decomposition is given by

$$\begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \delta_p & (\delta_p') \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \delta_p & 0 \\ 0 & (\delta_p') \end{bmatrix} \frac{1}{(\delta_p - \delta_p')} \begin{bmatrix} 1 & -(\delta_p') \\ -1 & \delta_p \end{bmatrix}.$$

Thus we have

$$\begin{aligned} \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix}^n &= \begin{bmatrix} \delta_p & (\delta_p') \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \delta_p^n & 0 \\ 0 & (\delta_p')^n \end{bmatrix} \frac{1}{(\delta_p - \delta_p')} \begin{bmatrix} 1 & -(\delta_p') \\ -1 & \delta_p \end{bmatrix}, \\ &= \frac{1}{(\delta_p - \delta_p')} \begin{bmatrix} \delta_p^{n+1} & (\delta_p')^{n+1} \\ \delta_p^n & (\delta_p')^n \end{bmatrix} \begin{bmatrix} 1 & -(\delta_p') \\ -1 & \delta_p \end{bmatrix}, \\ &= \frac{\delta_p}{(\delta_p^2 + 1)} \begin{bmatrix} \frac{\delta_p^{2(n+1)} + (-1)^n}{\delta_p^{n+1}} & \frac{\delta_p^{2n} - (-1)^n}{\delta_p^n} \\ \frac{\delta_p^{2n} - (-1)^n}{\delta_p^n} & \frac{\delta_p^{2(n-1)} - (-1)^{n-1}}{\delta_p^{n-1}} \end{bmatrix}. \end{aligned}$$

Thus we have for odd n, $G_{n+1} = \frac{\delta_p}{(\delta_p^2 + 1)} \frac{\delta_p^{2n+2} - 1}{\delta_p^{n+1}}$.

Let $\tan \alpha = \delta_p^n$ and $\tan \beta = \delta_p^{n+2}$. Then,

$$\begin{aligned} G_{n+1} &= \frac{\tan \alpha \tan \beta - 1}{\tan \alpha + \tan \beta}, \\ \frac{1}{G_{n+1}} &= -\tan(\alpha + \beta). \end{aligned}$$

Thus

$$\tan((\beta+\alpha)/2) = \delta_{2G_{n+1}}.$$

Therefore when angles are in the third quadrant, we have the relation,

$$\arctan \delta_p^{n+2} + \arctan \delta_p^n = 2\arctan \delta_{2G_{n+1}}. \tag{10}$$

Example 2.3. *Let F_n be the n^{th} Fibonacci number, as a result of the* (10)*, for even n, we can represent the $2F_n$ th metallic number in terms of golden ratio δ_1 as,*

$$\delta_{2F_n} = \tan\left(\frac{\arctan \delta_1^{n+1} + \arctan \delta_1^{n-1}}{2}\right).$$

Take $F_4 = 3$, we have $\delta_6 = 6.162277660168380$, by evaluating $\arctan \delta_1^5 = 1.480869576898658$ and $\arctan \delta_1^3 = 1.338972522294493$ which further give $\tan(1.409921049596575) = \delta_6$.

2.3 Pythagorean triples in the sequences corresponding to metallic means

For the Fibonacci sequence with initial condition $F_0 = 0, F_1 = 1$ we have the relation:

$$(F_{n-1}F_{n+2})^2 + (2F_nF_{n+1})^2 = (F_{2n+1})^2. \tag{11}$$

This relation is illustrated in [6]. Further [3] proved for any Pythagorean triple there are initial conditions of (8) such that the triple occurs as (11). In [2] authors investigated Pell number triples.

Example 2.4. *From the Fibonacci sequence, we have $F_3 = 2$, $F_4 = 3$ and $F_5 = 5$, $F_6 = 8$ and $F_9 = 34$, then we have the triple (16,30,34).*

In general, in the Euclids generating parametric expression of Pythagorean triples, consider $(G_{n+1}^2 - G_n^2, 2G_nG_{n+1}, G_{n+1}^2 + G_n^2)$.

From the recurrence relation, we have

$$\begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix} = \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus we have

$$\begin{aligned} G_{n+1}^2 + G_n^2 &= \left\| \begin{bmatrix} G_{n+1} & G_n \end{bmatrix}^T \right\|^2, \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix}^T \right)^n \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix}^{2n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = G_{2n+1}. \end{aligned}$$

Further, we use the generating function of G_n,

$$G_n = \frac{1}{\sqrt{p^2+4}} \left(\delta_p^n - (\delta_p')^n\right).$$

with $\delta_p = \frac{p+\sqrt{P^2+4}}{2}$ and $(\delta_p') = \frac{p-\sqrt{P^2+4}}{2}$.

Then evaluating

$$G_{n+1}^2 - G_n^2 = \frac{1}{p^2+4}\left(\delta_p^{2n+2} + (\delta_p')^{2n+2} - \delta_p^{2n} - (\delta_p')^{2n} + 4(-1)^n\right).$$

Note that $\delta_p + \delta_p' = p$, $p^2 + 2 = \delta_p^2 + (\delta_p')^2$ and $(p^2+1)^2 + 2p^2 + 1 = \delta_p^4 + (\delta_p')^4$. Thus we get

$$\begin{aligned}G_{n+1}^2 - G_n^2 &= pG_{n+1}G_{n-1} + \frac{1}{p^2+4}((-1)^{n-1}((\delta_p)^4 + (\delta_p')^4) + (-1)^n((\delta_p)^2 + (\delta_p')^2) + 4(-1)^n),\\ &= pG_{n+1}G_{n-1} + \frac{(-1)^{n-1}}{p^2+4}((p^2+1)^2 + 2p^2 + 1 - (p^2+2) - 4),\end{aligned}$$

With further algebraic simplifications, we find that

$$G_{n+1}^2 - G_n^2 = pG_{n+2}G_{n-1} + (-1)^{n-1}(p^2-1).$$

Thus in general

$$(pG_{n+2}G_{n-1} + (-1)^{n-1}(p^2-1))^2 + (2G_nG_{n+1})^2 = G_{2n+1}^2.$$

Example 2.5. *From the Pell-number sequence (which corresponds to $p = 2$) starting with $P_0 = 0$, $P_1 = 1$, we have*

$$P = \{0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741\}.$$

$P_2 = 2$ and $P_3 = 5$ correspond to the triple $(2 \times 12 - (2^2 - 1) = 21, 20, 29)$. *Similarly $P_3 = 5$ and $P_4 = 12$ correspond to* $(2 \times 2 \times 29 + (2^2 - 1) = 119, 120, 169)$.

2.4 Relating metallic means to the primitive Pythagorean triples

Metallic means are related to the right angle triangles as discussed in the introduction. In this section, we further explore the relation between metallic means and Pythagorean triples. We have the following lemma,

Lemma 2.1. *Primitive Pythagorean triples $a < b < c$ form integer $s = 2\sqrt{\frac{c+b}{c-b}}$ if and only if $c - b \in \{1, 2, 8\}$.*

Proof. The bidirectional statement implies for primitive Pythagorean triples $a < b < c$, if s is an integer, then $c - b \in \{1, 2, 8\}$. On the other hand, if $c - b \in \{1, 2, 8\}$, then s is an integer.

"If" part: From Euclid's generating function(with $(m^2 - n^2, 2mn, m^2 + n^2)$), to generate primitive Pythagorean triples assume $\gcd(m, n) = 1$ and both m and n cannot be odd. Also assume $m^2 - n^2 < 2mn$, in this case $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$.

$$\begin{aligned}c + b &= m^2 + n^2 + 2mn\\ &= (m+n)^2,\\ c - b &= m^2 + n^2 - 2mn,\\ &= (m-n)^2.\end{aligned}$$

Thus we get $s = 2\frac{m+n}{m-n}$, In this case, have integer s when $m-n$ is either 1 or 2. But if $m-n=2$, we have $a = 2(m+n)$ and $b = 2n^2+4$. Thus $\gcd(a,b) = 2$, so a contradiction. Therefore we get $c-b=1$.

Now if we assume $m^2-n^2 > 2mn$, in this case $(a,b,c) = (2mn, m^2-n^2, m^2+n^2)$. From the generating triples

$$\begin{aligned} c+b &= m^2+n^2+m^2-n^2, \\ &= 2m^2, \\ c-b &= m^2+n^2-m^2+n^2, \\ &= 2n^2. \end{aligned}$$

Thus we get, $s = 2\,\dfrac{m}{n}$. In this case, we have integer s, when n either 1 or 2. This gives possible $c-b$ values to be $2, 8$.

"Only if" part: For primitive Pythagorean triples assume $c-b \in \{1,2,8\}$. when $c-b=1$ we have only possibility $m^2-n^2 < 2mn$ and $s = 2\dfrac{m+n}{m-n} = 2(m+n)$ is an integer. When $c-b=2$ or 8 we have only possibility $m^2-n^2 > 2mn$ and $s = 2\dfrac{m}{n}$ is either $2m$ or m, is an integer. $\square$

Define the families of triples $F_j = \{(a,b,c) \ : \ c-b=j\}$ for $j \in \{1,2,8\}$. These families of triples are known as F_1 , F_2, and F_8 families. From Lemma 2.1, we can derive the generating functions of these families. Let $l = c-b$. When we have $l=1$, we have $m^2-n^2 < 2mn$ from the proof of Lemma 2.1. $s = 2(m+n)$. In this case, we also have $l = (m-n)^2 = 1$ implying $m-n=1$ (because $m>n$). Thus we get $m = \dfrac{s+2}{4}$, $n = \dfrac{s-2}{4}$, and the triples

$$(a,b,c) = \frac{1}{16}(8s, 2(s^2-4), 2s^2+8). \tag{12}$$

Take $s = 4t+2$ in (12), and we get the generating function of F_1 :

$$(a,b,c) = (2t+1, 2t^2+2t, 2t^2+2t+1), \quad \text{for} \quad t = 1,2,\ldots \tag{13}$$

In case of $l=2$ or 8, we have $m^2-n^2 > 2mn$. Further, $s = \dfrac{2m}{n}$. For $n=1$, we have $l=2$. and generating function of F_2 is

$$(a,b,c) = (2m, m^2-1, m^2+1) \quad \text{for} \quad m = 2,3,\ldots \tag{14}$$

In the case of $l=8$, we have $n=2$. Thus we get generating function of F_8 as

$$(a,b,c) = (4m, m^2-4, m^2+4) \quad \text{for} \quad m = 3,4,\ldots \tag{15}$$

Example 2.6. *Interestingly, triple* $(12,5,13)$ *corresponds to* $m=3$ *in* (15) *and* $t=2$ *in* (13). *Similarly, triple* $(3,4,5)$ *corresponds to* $t=1$ *in* (13) *and* $m=2$ *in* (14). *The triple* $(20,21,29)$ *belongs to* F_8 *and Fermat family (where* $|b-a|=1$ *in the triple* (a,b,c)*). The families* F_1 *are known as Pythagoras family, and* F_2 *are known as Plato family,* [15].

For a right-angle triangle with sides having primitive Pythagorean triples $a<b<c$, let θ be the angle between the sides having length b and c.

Theorem 2.3. *If $c-b \in \{1,2,8\}$, we have $\delta_k = \cot\frac{\theta}{4}$ for positive integer $k = 2\sqrt{\frac{b+c}{c-b}}$.*

Proof. By considering $\frac{\theta}{4}$ in the (3),

$$\cot^2\frac{\theta}{4} - 2\cot\frac{\theta}{2}\cot\frac{\theta}{4} - 1 = 0.$$

We have $\delta_{2\cot\frac{\theta}{2}} = \cot\frac{\theta}{4}$. With $\cot\theta = \frac{b}{a}$, we have $2\cot\frac{\theta}{2} = 2\sqrt{\frac{b+c}{c-b}}$. Also from Lemma 2.1, we have $k = 2\sqrt{\frac{b+c}{c-b}}$ as a positve integer. Thus $\delta_k = \cot\frac{\theta}{4}$. $\square$

Example 2.7. *Consider the triple $(20, 21, 29)$, we get $k = 2\sqrt{\frac{50}{8}} = 5$. Thus $\delta_5 = \frac{5+\sqrt{29}}{2} =$ 5.192582403567252. It is also given by $\cot\frac{\operatorname{arccot}\left(\frac{21}{20}\right)}{4} = 5.192582403567252$.*

2.4.1 Metallic means of fractional order taking fractional and integer values and primitive Pythagorean triples

By re-arranging the metallic mean expression (2) for δ_n, we get

$$\left(\frac{2\delta_n - n}{2}\right)^2 = \left(\frac{n}{2}\right)^2 + 1.$$

If we take a rational number $f = \dfrac{2b}{a}$ (where a, b are from a Pyhtagorean triple, of the form $a^2 + b^2 = c^2$), then

$$\begin{aligned}\left(\frac{2\delta_f - f}{2}\right)^2 &= \left(\frac{f}{2}\right)^2 + 1,\\ &= \left(\frac{c}{a}\right)^2.\end{aligned}$$

So $\delta_f = \left(\dfrac{c}{a} + \dfrac{f}{2}\right) = \dfrac{c+b}{a}$ can be regarded as a rational metallic number taking rational value.

Example 2.8. *Consider $a = 4, b = 3, c = 5$. Then $f = \dfrac{3}{2}$. $\delta_{\frac{3}{2}} = 2$ is an integer.*

Example 2.9. *Consider triple $a < b < c$ in F_1, then $f = \dfrac{2b}{a}$ corresponds to $\delta_f = \dfrac{(2t+1)^2}{2t+1} = 2t+1$ (here t as in generating function (13)) is an integer for all t.*

Example 2.10. *Consider triple $a < b < c$ in F_2, then $f = \dfrac{2b}{a}$ corresponds to $\delta_f = \dfrac{2m^2}{2m} = m$ (here m as in generating function (14)) is an integer for all m.*

Example 2.11. *Consider triple $a < b < c$ in F_8, then $f = \dfrac{2b}{a}$ corresponds to $\delta_f = \dfrac{2m^2}{4m} = \dfrac{m}{2}$ (here m as in generating function (15)) is an integer for an even m.*

3 Conclusion

This article relates metallic means and Pythagorean triples. Several interconnecting relations between metallic means are given using trigonometric identities. Pythagorean triples in the sequences corresponding to metallic means are investigated.

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