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Two arithmetic functions related to Euler's and Dedekind's functions

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To Prof. Taekyun Kim for his 60th birthday!

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Abstract: Two new arithmetic functions are introduced. In some sense, they are modifications of Euler's and Dedekind's functions. Some properties of the new functions are studied.
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1 Introduction

Euler's and Dedekind's functions are two from the most important arithmetic functions (see, e.g. [4,5,7]). They have different modifications (see, e.g., [6]). Here, two new of their modifications are introduced.

Let the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i} \tag{1}$$

be given, where $k, \alpha_1, \ldots, \alpha_k, k \geq 1$ are natural numbers and $p_1, \ldots, p_k$ are different primes.

For the above n, Euler's and Dedekind's functions are defined as follows (see, e.g., [4,5,7]:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1),\ \varphi(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1),\ \psi(1) = 1.$$

Below, we will use also functions (see [1,2,6]):

$$\underline{\text{set}}(n) = \{p_1, \ldots, p_k\},$$

$$\underline{\text{mult}}(n) = \prod_{i=1}^{k} p_i.$$

2 Main results

Let us define for the above n:

$$\overline{\varphi}(n) = \prod_{i=1}^{k}(p_i^{\alpha_i} - 1),\ \overline{\varphi}(1) = 1,$$

$$\overline{\psi}(n) = \prod_{i=1}^{k}(p_i^{\alpha_i} + 1),\ \overline{\psi}(1) = 1.$$

Obviously, for each natural number n:

$$\varphi(n) \leq \overline{\varphi}(n) \leq n - 1 < n < n + 1 \leq \overline{\psi}(n) \leq \psi(n). \tag{2}$$

Theorem 2.1. *For each natural number n:*

$$\psi(n) - \overline{\psi}(n) + \varphi(n) - \overline{\varphi}(n) \geq 0. \tag{3}$$

Proof. When $n = 1$, (3) is obvious. When $n \geq 2$ is a prime number, then we obtain directly that

$$\psi(n) - \overline{\psi}(n) + \varphi(n) - \overline{\varphi}(n) = 0.$$

Let us assume that (3) is valid for some natural number n and let $2 \leq p \notin \underline{\text{set}}(n)$ is a prime number. Then by the induction assumption

$$\begin{aligned} \psi(np) - \overline{\psi}(np) + \varphi(np) - \overline{\varphi}(np) &= (p+1)(\psi(n) - \overline{\psi}(n)) + (p-1)(\varphi(n) - \overline{\varphi}(n)) \\ &\geq (p-1)(\psi(n) - \overline{\psi}(n) + \varphi(n) - \overline{\varphi}(n)) \geq 0. \end{aligned}$$

Let $p \in \underline{\text{set}}(n)$. Therefore, $n = mp^a$ for some natural numbers $a, m \geq 1$ and prime p, and

$$\begin{aligned} &\psi(np) - \overline{\psi}(np) + \varphi(np) - \overline{\varphi}(np) \\ &= \psi(m)(p^{a+1} + p^a) - \overline{\psi}(m)(p^{a+1} + 1) + \varphi(m)(p^{a+1} - p^a) - \overline{\varphi}(n)(p^{a+1} - 1) \\ &= p^{a+1}(\psi(m) - \overline{\psi}(m) + \varphi(m) - \overline{\varphi}(m)) + p^a(\psi(m) - \varphi(m)) - \overline{\psi}(m) + \overline{\varphi}(m) \\ &\geq (p^a - 1)(\psi(m) - \varphi(m)) \geq 0. \end{aligned}$$

that proves the Theorem. □

Theorem 2.2. *For each natural number n:*

$$\overline{\varphi}(n) + \overline{\psi}(n) \geq 2n. \tag{4}$$

Proof. When $n = 1$, (4) is obvious. When $n \geq 2$ is a prime number, then we obtain directly that

$$\overline{\varphi}(n) + \overline{\psi}(n) = 2n.$$

Let us assume that (4) is valid for some natural number n and let $2 \leq p \notin \underline{\text{set}}(n)$ is a prime number. Then, by the induction assumption we obtain

$$\begin{aligned}\overline{\varphi}(np) + \overline{\psi}(np) - 2np &= \overline{\varphi}(n)(p-1) + \overline{\psi}(n)(p+1) - 2np \\ &\geq p(\overline{\varphi}(n) + \overline{\psi}(n) - 2n) + \overline{\psi}(n) - \overline{\varphi}(n) \geq 0.\end{aligned}$$

Let $p \in \underline{\text{set}}(n)$. Therefore, as above, $n = mp^a$, where $a, m \geq 1$ are natural numbers, and

$$\begin{aligned}\overline{\varphi}(np) + \overline{\psi}(np) - 2np &= \overline{\varphi}(m)(p^{a+1} - 1) + \overline{\psi}(m)(p^{a+1} + 1) - 2mp^{a+1} \\ &= (\overline{\varphi}(m) + \overline{\psi}(m) - 2m)p^{a+1} + \overline{\psi}(m) - \overline{\varphi}(m) \geq 0.\end{aligned}$$

□

In [3], the author introduced the arithmetic function (for n given by (1)):

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.$$

For it is valid:

Theorem 2.3. *For each natural number $n \geq 2$:*

$$\overline{\varphi}(n)\overline{\psi}(n) \leq n^2 - RF(n). \tag{5}$$

Proof. When $n \geq 2$ is a prime number, then we obtain directly that

$$\overline{\varphi}(n)\overline{\psi}(n) = n^2 - 1 = n^2 - RF(n).$$

Let us assume that (5) is valid for some natural number n and let $2 \leq p \notin \underline{\text{set}}(n)$ is a prime number. Then, from $p \geq 2$,

$$RF(np) = RF(n)$$

and by the induction assumption we obtain

$$\begin{aligned}(np)^2 - RF(np) - \overline{\varphi}(np)\overline{\psi}(np) &= n^2p^2 - RF(n) - \overline{\varphi}(n)\overline{\psi}(n)(p^2 - 1) \\ &\geq n^2p^2 - RF(n) - (n^2 - RF(n))(p^2 - 1) \\ &= (p^2 - 2)RF(n) + n^2 > 0.\end{aligned}$$

Let $p \in \underline{\text{set}}(n)$. Therefore, as above $n = mp^a$, where $a, m \geq 1$ are natural numbers,

$$RF(np) = RF(m)p^a$$

and

$$
\begin{aligned}
&(np)^2 - RF(np) - \overline{\varphi}(np)\overline{\psi}(np)\\
&= m^2p^{2a+2} - RF(m)p^a - \overline{\varphi}(m)\overline{\psi}(m)(p^{a+1}-1)(p^{a+1}+1)\\
&= m^2p^{2a+2} - RF(m)p^a - \overline{\varphi}(m)\overline{\psi}(m)(p^{2a+2}-1)\\
&= (m^2 - RF(m) - \overline{\varphi}(m)\overline{\psi}(m))p^{2a+2} + RF(m)(p^{2a+2}-p^a) + \overline{\varphi}(m)\overline{\psi}(m)\\
&> 0. \qquad \square
\end{aligned}
$$

that proves the Theorem.

From (2) it follows that for each natural number n:

$$\overline{\varphi}(n)\psi(n) \geq \varphi(n)\overline{\psi}(n)$$

and

$$\frac{\overline{\varphi}(n)}{\overline{\psi}(n)} \geq \frac{\varphi(n)}{\psi(n)}.$$

The following assertion is more interesting:

Theorem 2.4. *For each natural number n:*

$$\overline{\varphi}(n)\overline{\psi}(n) \geq \varphi(n)\psi(n). \tag{6}$$

Proof. When $n = 1$, (6) is obvious. When $n \geq 2$ is a prime number, then we obtain directly that

$$\overline{\varphi}(n)\overline{\psi}(n) = n^2 - 1 = \varphi(n)\psi(n).$$

When n has the form of (1), then

$$\overline{\varphi}(n)\overline{\psi}(n) = \prod_{i=1}^{k}(p_i^{2\alpha_i} - 1) \geq \prod_{i=1}^{k} p_i^{2\alpha_i-2}(p_i^2 - 1) = \varphi(n)\psi(n).$$

$\square$

It is well-known that for each natural number $n \geq 2$:

$$\varphi(n)\psi(n) \leq n^2 - 1,$$

but (2) and (6) yield

$$\varphi(n)\psi(n) \leq n^2 - RF(n), \tag{7}$$

i.e., a reinforcement of this inequality.

3 Conclusion

Two new arithmetic functions were introduced and some of their properties were discussed. In the future, other properties of theirs will be studied and relations between the new and existing arithmetic functions will be explored.

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