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Factorial polynomials and associated number families

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Abstract: Two doubly indexed families of polynomials in several indeterminates are considered. They are related to the falling and rising factorials in a similar way as the potential polynomials (introduced by L. Comtet) are related to the ordinary power function. We study the inversion relations valid for these factorial polynomials as well as the number families associated with them.
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1 Basic notions

In the following, we denote by $\mathcal{F}$ the algebra $\mathcal{K}[[x]]$ of formal power series in x with coefficients in a fixed commutative field $\mathcal{K}$ of characteristic zero. The elements of $\mathcal{F}$ will be called *functions*, those of $\mathcal{K}$ *constants* (or *numbers*). For any functions f, g and constant c, the sum $f + g$, the scalar product cf, and the product $f \cdot g$ are, as customary, assumed to be defined coordinate-wise and by Cauchy convolution, respectively. Furthermore, we consider the composition $\circ$ with $(f \circ g)(x) := f(g(x))$ which is a partial operation on $\mathcal{F}$, but well-defined, for example, if the leading coefficient of g is zero or, otherwise, f is a (Laurent) polynomial in x^{-1}, x with coefficients in $\mathcal{K}$. The identity element is, of course, $\iota = \iota(x) := x$, satifying $f \circ \iota = \iota \circ f = f$.

The ordinary algebraic derivation D on $\mathcal{K}[x]$ (with $D(\iota) = 1$) can be extended, in a unique way, to a derivation on $\mathcal{F}$ (here also denoted by D) for which the known rules for addition, multiplication and composition apply [13, p. 15, 61]. Iterating D leads, in the usual way, to derivatives of higher order $D^n(f)$. By setting $f_n := D^n(f)(0)$, we obtain, as is well known, the representation of $f(x)$ in the form of a Taylor series expansion:

$$f(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}.$$

The constants $f_0, f_1, f_2, \ldots \in \mathcal{K}$ are called *Taylor coefficients of* f.

2 Potential polynomials

Of special interest is the task of finding the general Taylor coefficient of a composite function $f \circ g$. Its well known solution is given by a famous formula of Faà di Bruno (cf. [3, p. 137] and [11, eqs. (1.3) and (4.1)]), which in modern notation is

$$D^n(f \circ g)(0) = \sum_{k=0}^{n} D^k(f)(g_0) B_{n,k}(g_1, \ldots, g_{n-k+1}). \tag{2.1}$$

Here $B_{n,k}$ are the *partial Bell polynomials* (or: *exponential polynomials*) that can be represented in the form of a 'diophantine' sum

$$B_{n,k} = \sum \frac{n!}{r_1! r_2! \cdots (1!)^{r_1} (2!)^{r_2} \cdots} X_1^{r_1} X_2^{r_2} \cdots X_{n-k+1}^{r_{n-k+1}}$$

to be taken over all sequences of integers $r_1, r_2, r_3, \ldots \geq 0$ such that $r_1 + r_2 + r_3 + \cdots = k$ and $r_1 + 2r_2 + 3r_3 + \cdots = n$. Since these polynomials had been introduced by E. T. Bell [1] in 1934, an enormous number of studies have been carried out on their properties, applications and variants (see, e. g., [3, 5, 9, 13, 15]).

By replacing the Taylor coefficients g_j with indeterminates X_j on the right-hand side of (2.1), one obtains a polynomial dependent on f (in [12, 13] called the *Faà di Bruno polynomial of* f). An important special case is $f = \iota^k$, $k \in \mathbb{Z}$, which leads to the *potential polynomials*

$$P_{n,k} = \sum_{j=0}^{n} k^{\underline{j}} X_0^{k-j} B_{n,j}, \tag{2.2}$$

introduced by Comtet [3, p. 141] and extensively studied in [3, 12, 13]. Here $k^{\underline{j}}$ is D. Knuth's symbol for the falling factorial power $k(k-1)\cdots(k-j+1)$, and $k^{\underline{j}} = 1$ for $j = 0$.

If $g \in \mathcal{F}$ is (compositionally) invertible, then $g_0 = g(0) = 0$, and we obtain from (2.1) $D^n(g^k)(0) = k! B_{n,k}(g_1, \ldots, g_{n-k+1})$, that is, $B_{n,k}(g_1, \ldots, g_{n-k+1})$ is the n-th Taylor coefficient of $g(x)^k/k!$. For completeness, we note that there also exists a polynomial expression $A_{n,k}(g_1, \ldots, g_{n-k+1})$ for the n-th Taylor coefficient of $\overline{g}(x)^k/k!$, where $\overline{g}$ denotes the unique inverse of g (with $g \circ \overline{g} = \overline{g} \circ g = \iota$). The fundamental properties (recurrences, inverse relations, reciprocity laws) of these two families of polynomials are treated in detail in [11–13]. Here we make use of the fact that the (lower triangular) matrices $(A_{n,k})$ and $(B_{n,k})$ are inverses of each other with respect to matrix multiplication, more precisely:

$$\sum_{j=k}^{n} A_{n,j} B_{j,k} = \delta_{nk} \quad (1 \le k \le n), \tag{2.3}$$

where $\delta_{nn} = 1$, $\delta_{nk} = 0$ if $n \neq k$ (Kronecker symbol); see [13, p. 29 and p. 82].

Let now Q_n be any sequence of polynomials from $\mathcal{K}[X_1, X_2, \ldots]$. We then call the sequence of numbers $q(n) := Q_n(1, 1, \ldots)$, obtained by replacing each indeterminate occurring in Q_n by $1 \in \mathcal{K}$, *associated with* Q_n. Thus $q(n)$ is equal to the sum of the coefficients of Q_n. For example, it is well known that the family of numbers $s_2(n,k) := B_{n,k}(1, \ldots, 1)$ associated with the partial Bell polynomials consists just of the Stirling numbers of the second kind [3, Thm. B, p. 135]. Together with the signed Stirling numbers of the first kind $s_1(n,k)$, they satisfy the inverse relation $\sum_{j=k}^{n} s_1(n,j) s_2(j,k) = \delta_{nk}$ (see, for example, [14, Prop. 1.4.1(a)]). Therefore, specializing all indeterminates to 1 in (2.3), $A_{n,k}(1, \ldots, 1)$ turns out to be equal to $s_1(n,k)$.

In his famous treatise *Methodus differentialis* (London, 1730) J. Stirling introduced the number family $s_1(n,k)$ by expanding $x^{\underline{n}}$ into a polynomial in standard form

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s_1(n,k) x^k. \tag{2.4}$$

Due to the inverse relationship of the Stirling numbers of the first and second kind [14, Prop. 1.4.1(b)] we get from (2.4)

$$x^n = \sum_{k=0}^{n} s_2(n,k) x^{\underline{k}}. \tag{2.5}$$

Comparing now (2.5) with (2.2), we find that $P_{n,k}(1, \ldots, 1) = k^n$ holds, i.e., k^n is the number family associated with the potential polynomials $P_{n,k}$.

Remark 2.1. The power terms $k^n, k^{\underline{n}}, k^{\overline{n}}$ have an obvious combinatorial meaning that entails a natural combinatorial interpretation of equations (2.4) and (2.5) (see, for example, [14, p. 33–35]). Also the Stirling numbers have a combinatorial meaning: $s_2(n,k)$ counts the ways n objects can be divided into k non-empty subsets ('subset numbers'), whereas the signless expression $c(n,k) := |s_1(n,k)| = (-1)^{n-k} s_1(n,k)$ represents the number of permutations of n objects having k cycles ('cycle numbers').

3 Factorial polynomials

Analogous to the way in which the potential polynomials were defined as Faà di Bruno polynomials of ι^k, let us now introduce *lower and upper factorial polynomials* as Faà di Bruno polynomials of the falling and rising factorial power functions $\iota^{\underline{k}}$ and $\iota^{\overline{k}}$, respectively:

$$P_{n,\underline{k}} := \sum_{j=0}^{n} D^j(\iota^{\underline{k}})(X_0)B_{n,j}(X_1,\ldots,X_{n-j+1}) \tag{3.1}$$

$$P_{n,\overline{k}} := \sum_{j=0}^{n} D^j(\iota^{\overline{k}})(X_0)B_{n,j}(X_1,\ldots,X_{n-j+1}) \tag{3.2}$$

We first consider (3.1). Since D^j is a linear operator, applying (2.5) to $\iota^{\underline{k}}$ and observing (2.2) yields

$$\begin{aligned} P_{n,\underline{k}} &= \sum_{j=0}^{n}\sum_{r=0}^{k} s_1(k,r)D^j(\iota^r)(X_0)B_{n,j} \\ &= \sum_{r=0}^{k} s_1(k,r)\sum_{j=0}^{n} r^{\underline{j}}X_0^{r-j}B_{n,j} \\ &= \sum_{r=0}^{k} s_1(k,r)P_{n,r} \end{aligned} \tag{3.3}$$

which can be reversed to

$$P_{n,k} = \sum_{r=0}^{k} s_2(k,r)P_{n,\underline{r}}. \tag{3.4}$$

In a similar way (3.2) can be evaluated. The only thing to keep in mind is that $\iota^{\overline{k}} = (-1)^k(-\iota)^{\underline{k}}$ and therefore by (2.4)

$$\iota^{\overline{k}} = (-1)^k\sum_{r=0}^{k} s_1(k,r)(-\iota)^r = \sum_{r=0}^{k} c(k,r)\iota^r,$$

whence we readily obtain

$$P_{n,\overline{k}} = \sum_{r=0}^{k} c(k,r)P_{n,r}. \tag{3.5}$$

To be able to establish also the connection between $P_{n,\overline{k}}$ and $P_{n,\underline{k}}$, recall the numbers $l(n,k)$ introduced by I. Lah [8], [10, p. 43–44]:

$$l(n,k) := (-1)^n\frac{n!}{k!}\binom{n-1}{k-1} \quad \text{with} \quad x^{\overline{n}} = \sum_{k=0}^{n} l(n,k)x^{\underline{k}}. \tag{3.6}$$

Remark 3.1. The unsigned *Lah numbers* $|l(n,k)|$ count the ways a set of n objects can be partitioned into k non-empty linearly ordered subsets.

Two well-known remarkable properties of the signed Lah numbers come into play in what follows: their representability in terms of the Stirling numbers, and the fact that they are inverses of themselves (see [10, p. 44] and [13, p. 31, 91]):

$$\sum_{j=k}^{n}(-1)^j s_1(n,j)s_2(j,k) = l(n,k), \tag{3.7}$$

$$\sum_{j=k}^{n} l(n,j)l(j,k) = \delta_{nk}. \tag{3.8}$$

Theorem 3.1. *For all nonnegative integers n,k we have*

(i) $$P_{n,\overline{k}} = \sum_{j=0}^{k}(-1)^k l(k,j)P_{n,\underline{j}},$$

(ii) $$P_{n,\underline{k}} = \sum_{j=0}^{k}(-1)^j l(k,j)P_{n,\overline{j}}.$$

Proof. We have

$$\begin{aligned}
P_{n,\overline{k}} &= \sum_{r=0}^{k} c(k,r)P_{n,r} && \textit{(by (3.5))}\\
&= \sum_{r=0}^{k}\sum_{j=0}^{r} c(k,r)s_2(r,j)P_{n,\underline{j}} && \textit{(by (3.4))}\\
&= \sum_{r=0}^{k}\sum_{j=0}^{r}(-1)^{k-r}s_1(k,r)s_2(r,j)P_{n,\underline{j}} && \textit{(by Remark 2.1)}\\
&= \sum_{j=0}^{k}(-1)^k\left(\sum_{r=j}^{k}(-1)^r s_1(k,r)s_2(r,j)\right)P_{n,\underline{j}}. && \textit{(rearranging the double sum)}
\end{aligned}$$

Now, applying (3.7) to the inner sum, we obtain assertion (i).
Assertion (ii) follows from (i) with respect to (3.8) as follows:

$$\begin{aligned}
\sum_{j=0}^{k}(-1)^j l(k,j)P_{n,\overline{j}} &= \sum_{j=0}^{k}(-1)^j l(k,j)\sum_{r=0}^{j}(-1)^j l(j,r)P_{n,\underline{r}}\\
&= \sum_{r=0}^{k}\left(\sum_{j=r}^{k}(-1)^{2j} l(k,j)l(j,r)\right)P_{n,\underline{r}}\\
&= \sum_{r=0}^{k}\delta_{kr}P_{n,\underline{r}} = P_{n,\underline{k}}.
\end{aligned}$$

□

4 Associated number families

This section deals with the number families associated with the upper and lower factorial polynomials:

$$[n]_{\overline{k}} := P_{n,\overline{k}}(1,\ldots,1) = \sum_{r=0}^{k} c(k,r)r^n, \tag{4.1}$$

$$[n]_{\underline{k}} := P_{n,\underline{k}}(1,\ldots,1) = \sum_{r=0}^{k} s_1(k,r)r^n. \tag{4.2}$$

To be more consistent with the combinatorial interpretation of the Stirling numbers (as explained in Remark 2.1), we will use in some cases the notation of J. Karamata recommended by Knuth [6]:

$$s_2(n,k) = \left\{ {n \atop k} \right\} \qquad \text{and} \qquad c(n,k) = \left[{n \atop k} \right].$$

We start with some preparatory remarks. From (2.5) we readily obtain $r^n = \sum_{j=0}^n j!\binom{r}{j} s_2(n,j)$ which implies an identity that reduces a power sum with arbitrary weights $\gamma_{k,r}$, $1 \le r \le k$, and integer exponent $n \ge 1$ as follows:

$$\sum_{r=1}^{k} \gamma_{k,r} r^n = \sum_{j=1}^{\min(k,n)} j! s_2(n,j) \sum_{r=j}^{k} \binom{r}{j} \gamma_{k,r}. \tag{4.3}$$

There are quite a few cases allowing the inner sum on the right-hand side of (4.3) to be simplified significantly, that is, we would obtain an upper summation rule such as

$$\sum_{r=j}^{k} \binom{r}{j} \gamma_{k,r} = f(k,j) \tag{4.4}$$

with a more or less closed expression $f(k,j)$.

Remark 4.1. The most simple example is $\gamma_{k,r} \equiv 1$, yielding in (4.4) the term $f(k,j) = \binom{k+1}{j+1}$. With this, (4.3) becomes the familar identity

$$1^n + 2^n + \cdots + k^n = \sum_{j=1}^{\min(k,n)} j! \left\{ {n \atop j} \right\} \binom{k+1}{j+1}. \tag{4.5}$$

The reader may find in Hsu [4] a great variety of choices for the $\gamma_{k,r}$. However, the case of Stirling numbers as weights has not been considered in those discussions.

Boyadzhiev [2] has evaluated (4.3) for the cycle numbers $\gamma_{k,r} = c(k,r)$ thereby taking $f(k,j) = c(k+1,j+1)$, which turns (4.4) into a well-known identity (see, for example, [7, p. 68, eq. (51)]). This gives immediately the following nice result [2, Prop. 2.7] that has a remarkable analogy to (4.5):

Proposition 4.1. *For any positive integers* n,k *we have*

$$[n]_{\overline{k}} = \sum_{r=1}^{k} \left[{k \atop r} \right] r^n = \sum_{j=1}^{\min(k,n)} j! \left\{ {n \atop j} \right\} \left[{k+1 \atop j+1} \right].$$

Remark 4.2. Instead of $s_2(n,j)$ Boyadzhiev makes use of a Stirling *function* $S(n,j)$ whose first argument is allowed to be a complex number $n \neq 0$. In this case, the upper summation limit $\min(k,n)$, which appears in (4.3) and in Proposition 4.1, has to be replaced by k.

From now on, the exponent n is assumed to be any positive integer.

Let us turn finally to the question of what result we get from (4.3) when the *signed* Stirling numbers of the first kind $s_1(k,r)$ are chosen as weights. The answer is given in the following theorem.

Theorem 4.2. *For any positive integers n, k we have*

$$[n]_{\underline{k}} = \sum_{r=1}^{k} (-1)^{k-r} \begin{bmatrix} k \\ r \end{bmatrix} r^n = \sum_{j=1}^{\min(k,n)} (-1)^{k-j} j! \begin{Bmatrix} n \\ j \end{Bmatrix} \left(\begin{bmatrix} k-1 \\ j-1 \end{bmatrix} - \begin{bmatrix} k-1 \\ j \end{bmatrix} \right).$$

Proof. Evaluating (4.3) for $\gamma_{k,r} = s_1(k, r)$ requires a new upper summation formula:

$$\sum_{r=j+1}^{k} \binom{r}{j} s_1(k, r) = k s_1(k-1, j). \tag{4.6}$$

First, we show by induction on k that (4.6) holds for every $k \geq 1$ and $0 \leq j \leq k-1$. In the case of $k = 1$ we have $j = 0$, and both sides of (4.6) are equal to 1.

The induction step ($k \to k+1$) makes repeated use of the familiar recurrence formula for the Stirling numbers of the first kind (see, e. g., [7, p. 67]):

$$s_1(k+1, r) = s_1(k, r-1) - k s_1(k, r).$$

Replacing $s_1(k+1, r)$ by $s_1(k, r-1) - k s_1(k, r)$ we obtain

$$\sum_{r=j+1}^{k+1} s_1(k+1, r) \binom{r}{j} = \sum_{r=j+1}^{k+1} s_1(k, r-1) \binom{r}{j} - \sum_{r=j+1}^{k+1} k s_1(k, r) \binom{r}{j}. \tag{4.7}$$

By induction hypothesis, the second sum on the right-hand side of (4.7) is equal to $k^2 s_1(k-1, j)$. The first sum can be splitted into two parts by applying the familiar basic addition formula for the binomial coefficients. We then have, again by induction hypothesis,

$$\sum_{r=j+1}^{k+1} s_1(k, r-1) \binom{r-1}{j-1} = \sum_{r=j}^{k} s_1(k, r) \binom{r}{j-1} = k s_1(k-1, j-1),$$
$$\sum_{r=j+1}^{k+1} s_1(k, r-1) \binom{r-1}{j} = \sum_{r=j}^{k} s_1(k, r) \binom{r}{j} = s_1(k, j) + k s_1(k-1, j).$$

Combining these results we obtain for the sum on the left-hand side of (4.7)

$$\begin{aligned} &k s_1(k-1, j-1) + s_1(k, j) + k s_1(k-1, j) - k^2 s_1(k-1, j) \\ &\quad = k \cdot [s_1(k-1, j-1) - (k-1) s_1(k-1, j)] + s_1(k, j) \\ &\quad = k s_1(k, j) + s_1(k, j) \\ &\quad = (k+1) s_1(k, j), \end{aligned}$$

which completes the induction.

Finally, it follows from (4.6):

$$\begin{aligned} \sum_{r=j}^{k} \binom{r}{j} s_1(k, r) &= s_1(k, j) + k s_1(k-1, j) \\ &= s_1(k-1, j-1) + s_1(k-1, j) \\ &= (-1)^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} - (-1)^{k-j} \begin{bmatrix} k-1 \\ j \end{bmatrix}. \end{aligned}$$

This yields the asserted equation. □

Table 1. Numbers $[n]_{\underline{k}}$ for $1 \leq n, k \leq 7$

$k =$	1	2	3	4	5	6	7
$n = 1$	1	1	-1	2	-6	24	-120
$n = 2$	1	3	-1	0	4	-28	188
$n = 3$	1	7	5	-16	54	-222	1098
$n = 4$	1	15	35	-60	124	-280	440
$n = 5$	1	31	149	-88	-186	1914	-13350
$n = 6$	1	63	539	420	-2996	13832	-66592
$n = 7$	1	127	1805	4664	-15546	43578	-98862

We conclude this section with some examples for small exponents $n = 1, 2, 3$ which demonstrate that the given power sum indeed undergoes a substantial simplification through the right-hand side expression in the statement of Theorem 4.2.

Let $k \geq 2$, then, cancelling $(-1)^k$ in each equation we obtain the identities:

$$\begin{aligned}
\sum_{r=1}^{k}(-1)^r \begin{bmatrix} k \\ r \end{bmatrix} r \ &= \begin{bmatrix} k-1 \\ 1 \end{bmatrix}, \\
\sum_{r=1}^{k}(-1)^r \begin{bmatrix} k \\ r \end{bmatrix} r^2 &= 3\begin{bmatrix} k-1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} k-1 \\ 2 \end{bmatrix}, \\
\sum_{r=1}^{k}(-1)^r \begin{bmatrix} k \\ r \end{bmatrix} r^3 &= 7\begin{bmatrix} k-1 \\ 1 \end{bmatrix} - 12\begin{bmatrix} k-1 \\ 2 \end{bmatrix} + 6\begin{bmatrix} k-1 \\ 3 \end{bmatrix}.
\end{aligned}$$

5 Concluding remarks

The aim of the present paper was to supplement the potential polynomials introduced by Comtet [3] with the analogously defined Faà di Bruno polynomials of the falling and rising power functions. As the results in Section 3 show, both types of factorial polynomials introduced in this way can be expressed in terms of the potential polynomials, and vice versa. Of course, the relevant relationships yield also directly their specialized version for the associated number families discussed in Section 4. In this way, for example, the equations (3.3) and (3.4) immediately become the following inverse relation pair (with respect to the Stirling numbers of the first and second kind):

$$[n]_{\underline{k}} = \sum_{r=0}^{k} s_1(k,r) r^n, \qquad k^n = \sum_{r=0}^{k} s_2(k,r)[n]_{\underline{r}}.$$

The reader may compare this with the 'classical' identities (see (2.4) and (2.5)):

$$n^{\underline{k}} = \sum_{r=0}^{k} s_1(k,r) n^r, \qquad n^k = \sum_{r=0}^{k} s_2(k,r) n^{\underline{r}}.$$

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References

[1] Bell, E. T. (1934). Exponential polynomials. *Annals of Mathematics*, 35, 258–277.

[2] Boyadzhiev, K. N. (2008/2009). Power sum identities with generalized Stirling numbers. *The Fibonacci Quarterly*, 46/47(4), 326–330.

[3] Comtet, L. (1974). *Advanced Combinatorics* (Rev. and enlarged ed.). Reidel, Dordrecht, Holland.

[4] Hsu, L. C. (1993). A summation rule using Stirling numbers of the second kind. *The Fibonacci Quarterly*, 31(3), 256–262.

[5] Kim, D. S., Kim., T, Lee, S. H., & Park, J.-W. (2021). Some new formulas of complete and incomplete degenerate Bell polynomials. *Advances in Difference Equations*, 2021, Article 326.

[6] Knuth, D. E. (1992). Two notes on notation. *The American Mathematical Monthly*, 99, 403–422.

[7] Knuth, D. E. (1997). *The Art of Computer Programming. Vol. 1: Fundamental Algorithms* (3rd ed.). Addision-Wesley, Reading, MA.

[8] Lah, I. (1955). Eine neue Art von Zahlen, ihre Eigenschaften und Anwendung in der mathematischen Statistik. *Mitteilungsblatt für mathematische Statistik*. 7, 203–212.

[9] Mihoubi, M. (2010). Partial Bell polynomials and inverse relations. *Journal of Integer Sequences*, 13, Article 10.4.5.

[10] Riordan, J. (2002). *Introduction to Combinatorial Analysis*. Reprint, Dover Publ., Inc., Mineola, New York.

[11] Schreiber, A. (2015). Multivariate Stirling polynomials of the first and second kind. *Discrete Mathematics*, 338, 2462–2484.

[12] Schreiber, A. (2020). Inverse relations and reciprocity laws involving partial Bell polynomials and related extensions. *Enumerative Combinatorics and Applications*, 1(1), Article S2R3.

[13] Schreiber, A. (2021). *Stirling Polynomials in Several Indeterminates*. Logos Verlag Berlin GmbH, (155 pp.).

[14] Stanley, R. P. (1986). *Enumerative Combinatorics, Vol. 1*. Wadsworth & Brooks/Cole, Monterey.

[15] Todorov, P. G. (1985). New differential recurrence relations for Bell polynomials and Lie coefficients. *Comptes rendus de l'Académie bulgare des Sciences*, 38(1), 43–45.