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Characterization of prime and composite numbers using the notion of successive sum of integers and the consequence in primality testing

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Abstract: In this paper, we give a characterization of primes and composite natural numbers using the notion of the sum of successive natural numbers. We prove essentially that an odd natural number $N \geq 3$ is prime if and only if the unique decomposition of N as a sum of successive natural numbers is the trivial decomposition $N = a + (a + 1)$ with $a = (N - 1)/2$.

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1 Introduction

The first idea of this work is due to the first author of this paper in giving some numerical observations about primes and composite numbers: these observations noted by the author can be resumed as follows: every odd prime number N admits a single representation as sum of two successive natural numbers $N = a + (a + 1)$ with $a = (N - 1) / 2$, called the trivial representation. For example,

$$5 = 2 + 3,$$
$$7 = 3 + 4,$$
$$101 = 50 + 51,$$
$$331 = 165 + 166, \ldots\text{etc.}$$

While, a composite number admits other representations than the trivial representation, at least one representation of the form $N = a + (a + 1) + \cdots + (a + k)$ with $k \geq 2$. For example,

$$9 = 4 + 5 = 2 + 3 + 4,$$
$$15 = 7 + 8 = 1 + 2 + 3 + 4 + 5 = 4 + 5 + 6,$$
$$77 = 38 + 39 = 8 + 9 + 10 + 11 + 12 + 13 + 14 = 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12.$$

The theoretical confirmations of these results were made by the other authors of this paper.

It is well known that an integer N is said to be prime number, if it admits only two divisors: the integer 1 and the number N itself. In this paper, we use an alternative characterization of a prime number, as established in such works as [3, 5, 7], using the previous notion of the sum of successive natural numbers and we prove that: *An odd natural number $N \geq 3$ is prime if and only if the unique decomposition of N as a sum of successive natural numbers is the trivial decomposition $N = a + (a + 1)$ with $a = (N - 1) / 2$.*

2 Preliminaries

The set of natural numbers $\mathbb{N} := \{1, 2, \ldots, n, \ldots\}$ is well ordered using the usual ordering relation denoted by $\leq$. For the natural numbers a, b, we say that a divides b, denoted $a \mid b$, if there is a natural number q such that $b = aq$. The proper divisors ($d \mid N$, with $d \neq 1$ and $d \neq N$) of the natural number N cannot exceed $(N / 2)$, i.e., $d \leq (N / 2)$, if $d \mid N$. A natural number $p > 1$ is called prime if it is not divisible by any natural number other than 1 and itself, in other words, if $p = nm$, then $n = 1$ or $m = 1$. A natural number that is not prime is called composite.

Euclid's Theorem ensures that there are infinitely many primes. The Fundamental Theorem of Arithmetic confirms that any natural number $n > 1$ can be written, in a unique way up to reordering, as the product of primes:

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_h^{\alpha_h},$$

where p_i are unique and $\alpha_i \geq 1$ are positive integers.

The natural numbers a and $a + 1$ are called successive natural numbers.

All necessary notions of elementary number theory are from [2, 4] and those of computational number theory are from [1, 6].

3 The main result of the paper

In this paper, we give an alternative characterization of prime and composite numbers using the concept of the sum of successive natural numbers, as mentioned in the following theorem.

Theorem 1. *An odd natural number $N \geq 3$ is prime number if and only if the unique decomposition of N as a sum of successive natural numbers is the trivial decomposition*

$$N = a + (a + 1)$$

with $a = (N - 1) / 2$.

Proof. It is clear that every odd natural number $N = 2a + 1$ can be written in the form $N = a + (a + 1)$ as the sum of two successive natural numbers with $a = (N - 1) / 2$.

Suppose that N is prime and can be written as

$$N = a + (a+1) + (a+2) + \cdots + (a+k), \qquad \text{with } k \geq 1 \text{ and } a \geq 1 \tag{$\star$}$$

We will prove in this case, that k must be equal to 1, i.e., $N = a + (a + 1)$.
Suppose that, on the contrary, we have $k \geq 2$. We have from ($\star$)

$$N = (k+1)a + k(k+1)/2 = (k+1)(a + k/2).$$

Two cases can be considered depending on whether the integer k is even or odd.

<u>The first case</u>: If k is odd, then $N = ((k + 1) / 2)(2a + k)$. Thus $(k + 1) / 2$ is a divisor of N. Since N is a prime, then $(k + 1) / 2$ must equal to 1 or $2a + k = 1$.

If $(k + 1) / 2 = 1$, then $k = 1$ and we have $N = 2a + 1 = a + (a + 1)$, which confirms the assertion. If $2a + k = 1$, then $k = 1 - 2a$, and this case is excluded because the integers a and k must be positive integers with respect to the representation ($\star$).

<u>For the second case</u>, if k is even, then $N = (k + 1)(2a + k) / 2$. Hence $k + 1 = 1$ or $(2a + k) / 2 = 1$ (since N is prime). The case $(k + 1) = 1$ gives $k = 0$, which is in contradiction with the fact that $k \geq 1$. The case $(2a + k) / 2 = 1$ implies $2a + k = 2$, then $k = 2 - 2a$, which is negative or null because $a \geq 1$, and this case is excluded. Then k must be equal to 1, the only decomposition of the prime N is the trivial representation

$$N = a + (a+1).$$

<u>Conversely</u>, we will show that, if the natural number N is not prime, then N admits several representations as the sum of consecutive natural numbers other than the trivial representation. Suppose that the natural number N is not prime. Let $N = n \times m$ be a decomposition of the number N in product of natural numbers with $n \neq 1$ and $m \neq 1$. Since N is odd, then the numbers n and m must be odd natural numbers. The number N can be written as

$$N = \underbrace{m + \cdots + m}_{n \text{ times}} + \underbrace{n + \cdots + n}_{m \text{ times}}.$$

As n is an odd natural number, $(n - 1) / 2$ is a natural number and we have

$$n = (n-1)/2 + 1 + (n-1)/2.$$

The number N can be decomposed as

$$m+\cdots+m+(m)+m+\cdots+m.$$

From this remark, we can write N as follows:

$$N=m-(n-1)/2+\left(m-(n-1)/2+1\right)+\cdots+(m-1)+(m)+(m+1)+\cdots+\left(m+(n-1)/2\right)$$

and these numbers are consecutive numbers from the number $a = m - (n - 1) / 2$ to the number $m + (n - 1) / 2$ with $k = n$ and the difference is 1 between any two consecutive numbers. The second representation can be obtained in considering $N = n + \cdots + n$ to obtain:

$$N=n-\left((m-1)/2\right)+\left(n-\left((m-1)/2\right)+1\right)+\cdots+(n-1)+(n)+(n+1)+\cdots+\left(n+(m-1)/2\right).$$

In this case, $a = n - (m - 1) / 2$ and $k = m$.

Example 2. Take $N = 35 = 5 \times 7 = 7 + 7 + (7) + 7 + 7$. Then N can be written as a sum of consecutive natural numbers as follows:

$$(7-2)+(7-2+1)+(7)+(7+1)+(7+2)=5+6+7+8+9=35$$

and we also have

$$N=35=5\times 7=5+5+5+(5)+5+5+5=2+3+4+(5)+6+7+8.$$

The other representation is the trivial decomposition as $35 = a + (a + 1) = 17 + 18$, with $a = (35 - 1) / 2 = 17$.

Since the number 5 is prime, the only decomposition of 5 as sum of consecutive numbers is $5 = 2 + (2 + 1)$.

The same case is true for the number $7 = 3 + (3 + 1)$, and for any prime number p we have the only decomposition as the sum of the two consecutive natural numbers

$$p=(p-1)/2+\left((p-1)/2+1\right).$$

Remark 3. *The number of the decompositions of the natural N as a sum of consecutive numbers depends on the number of the representations of the number N as a product of two factors.*

Corollary 4. *If $N = n_1m_1 = n_2m_2 = \cdots = n_tm_t$ are the different decompositions of N as a product of two factors, then the number of representations of N as sum of consecutive numbers is equal to* $2t + 1$.

Proof. From the above theorem, for each decomposition of $N = nm$ there are two different representations of N as a sum of consecutive numbers. Consequently, for t different decom-positions in the product there are $2t$ representations as sums of consecutive numbers plus the trivial representation $N = (N - 1) / 2 + ((N - 1) / 2 + 1)$. □

Remark 5. *The trivial decompositions* $N=N\times 1=\underbrace{1+\cdots+1}_{N\text{ times}}=1\times N=\underbrace{N}_{1\text{ times}}$ *are excluded in Theorem* 1 *because they cannot be written as sums of natural consecutive numbers.*

If the number N is prime, then it admits the trivial representation

$$N = (N - 1) / 2 + ((N - 1) / 2 + 1).$$

If the number N is a perfect square, i.e., $N = n^2 = n \times n$, then the number N admits two representations in a sum of natural consecutive numbers:

- *the trivial representation*

$$N = (n-1)/2 + \left((n-1)/2+1\right),$$

- *and the representation*

$$N = \left(n-(n-1)/2\right) + \left(n-\left((n-1)/2\right)+1\right) + \cdots + \left(n+(n-1)/2\right).$$

When $N = p^2 = p \times p$, with p prime, this case is a particular one of the case $N = n^2 = n \times n$.

Example 6. The number 63 can be decomposed in product of two factors as 63 = 7 × 9 = 21 × 3, then the number 63 admits 2 + 2 + 1 = 5 representations as a sum of consecutive numbers which are:

$$63 = 6+7+8+9+10+11+12,$$

$$63 = 3+4+5+6+7+8+9+10+11,$$

$$63 = 20+21+22,$$

$$\begin{aligned} 63 &= -7-6-5-4-3-2-1+0+1+2+3+4+5+6+7+8+9+10+11+12+13 \\ &= 8+9+10+11+12+13, \end{aligned}$$

$$63 = (63-1)/2 + \left((63-1)/2+1\right) = 31+32.$$

The number 25 = 5 × 5 admits only the trivial representation 25 = 12 + 13 and the representation 25 = 3 + 4 + 5 + 6 + 7.

Since the statement of Corollary 4 is based on the number *t* of different decompositions of $N = n \times m$ as a product of two factors, the following theorem shows how to compute *t* as a function of the prime factors of the natural number $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_h^{\alpha_h}$.

Theorem 7. *If $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_h^{\alpha_h}$ is the decomposition of N on prime factors with $N \geq 3$, then the number t of non-trivial different decompositions of N as a product of two factors is*

$$t = \frac{\tau(N)}{2} - 1$$

with $\tau(N) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_h+1)$ being the number of positive divisors of N and x being the least positive integer greater than or equal to the real number $x \in \mathbb{R}$.

Proof. Let $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_h^{\alpha_h}$ be the decomposition of *N* on prime factors with $N \geq 3$. To each decomposition of *N* into a product of two factors $N = A \times B$ there corresponds a couple of divisors (*A*, *B*) with $A = p_1^{a_1} p_2^{a_2} \cdots p_h^{a_h}$ and $B = p_1^{b_1} p_2^{b_2} \cdots p_h^{b_h}$ such that $0 \leq a_i, b_i \leq \alpha_i$ with $a_i + b_i = \alpha_i$, for $1 \leq i \leq h$.

Let X be the set of pairs $(A, B) \in \mathbb{N}^2$ verifying $N = A \times B$, i.e.,

$$X = \{(A, B) \in \mathbb{N}^2 \mid N = A \times B\},$$

and let $D(N)$ be the set of all divisors of N.

Consider the following mapping

$$\varphi : X \to D(N) \text{ with } \varphi(A, B) = A,$$

which is one to one and onto. In fact, if $\varphi(A, B) = \varphi(C, D)$, then $A = C$, which yields $B = D$. Thus, $(A, B) = (C, D)$ and φ is injective. And for each $d \in D(N)$, the pair $\left(d, \frac{N}{d}\right)$ is in X with

$$\varphi\left(d, \frac{N}{d}\right) = d,$$

which asserts that φ is surjective.

Hence $|X| = |D(N)| = \tau(N)$.

Since $A \times B = B \times A$, the decompositions $N = A \times B$ and $N = B \times A$ of N with $A \neq B$ are the same, the number of different decompositions of N as a product of two factors (including the trivial case when $N = N \times 1 = 1 \times N$) is equal to $\frac{|X|}{2} = \frac{\tau(N)}{2}$. Consequently, the number t of the *non-trivial* different decompositions of N as a product of two factors is

$$t = \frac{\tau(N)}{2} - 1.$$

Example 8.

1. For $N = 36 = 2^2 \times 3^2$ we have:

$$X = \{(1,36),(36,1),(2,18),(18,2)(3,12),(12,3)(4,9),(9,4),(6,6)\}$$

and then $N = 1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6$, which gives us

$$|X| = \tau(N) = (2+1)(2+1) = 9 \text{ and } t = \frac{9}{2} - 1 = 4.$$

2. For $N = 4 = 2^2$ we have : $X = \{(1,4),(4,1),(2,2)\}$ and then $N = 1 \times 4 = 2 \times 2$, which gives us $|X| = \tau(N) = (2+1) = 3$ and then $t = \frac{3}{2} - 1 = 1$.

4 Conclusion

In this paper, an alternative characterization of primality for an odd natural number greater than or equal to 3 is obtained using the notion of representation in sum of consecutive natural numbers. This notion allowed us to derive an algorithm of complexity in $O(\sqrt{n})$ slower than the classical sieve of Eratosthenes which is of complexity in $O(n \log\log n)$.

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