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Generalized perfect numerical semigroups

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Abstract: In this work, we study the isolated gaps for generalized numerical semigroups, introduce generalized perfect numerical semigroups, and exemplify these semigroups. In particular, we reveal the effects of the perfectness condition on a generalized Weierstrass semigroup.
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1 Introduction

In this work, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It can be easily seen that $\mathbb{N}_0$ is a monoid by the operation $+$.

Let S be a submonoid of $\mathbb{N}_0$ such that $H(S) = \mathbb{N}_0 \backslash S$ be a finite set, then S is called a *numerical semigroup.* The number of elements of $H(S)$ is called genus of S, and it is denoted by $g(S)$. The largest element of $H(S)$ is called the *Frobenius number* of S and denoted by $F(S)$. The smallest integer $x \in S$ such that $x + n \in S$ for every $n \in \mathbb{N}_0$ is called the *conductor* of S. The conductor of S is denoted by $C(S)$. The elements of S which are smaller than $C(S)$ are called the *small elements* of S, and the number of the small elements of S is denoted by $n(S)$.

For details on numerical semigroups, see [13]. If S has $n = n(S)$ small elements, it is customary to list them as $s_0 = 0 < s_1 < \cdots < s_{n-1}$ and write $S = \{s_0 = 0, s_1, \ldots, s_{n-1}, s_n = C(S), \rightarrow\}$, the arrow at the end meaning that all subsequent integers belong to S.

Moreno-Frias and Rosales introduce the concept of perfect numerical semigroup in [11]. It is a class of numerical semigroups that comes from topology, precisely the concept of a perfect set.

Let $S \subseteq \mathbb{N}_0$ be a numerical semigroup and let $h \in H(S)$. If $\{h-1, h+1\} \subset S$, then h is called an *isolated gap* of S. If S has no isolated gaps, then S is called a *perfect numerical semigroup.*

For example, we take $S = \{0, 5, 7, 10, \rightarrow\}$. Since $6 \in H(S)$ is an isolated gap, S is not a perfect numerical semigroup. On the other hand, $N = \{0, 4, 5, \rightarrow\}$ has not an isolated gap and N is a perfect numerical semigroup.

In [12], the authors characterize all perfect semigroups with embedding dimension three. In [14], the author investigate some properties of isolated gaps of a numerical semigroup, and they exhibit relations between the set of isolated gaps and the Apéry set. Using these properties, the authors give methods for finding all isolated gaps of the semigroup of embedding dimensions two and three.

Let $S \subseteq \mathbb{N}_0^d$ be a monoid. If $H(S) = \mathbb{N}_0^d \backslash S$ is a finite set, then S is called a *generalized numerical semigroup,* and the cardinality of $\mathbb{N}_0^d \backslash S$ is called as the genus of S. This is a natural generalization of the concept of the numerical semigroup. Let $N_{g,d}$ denote the number of generalized numerical semigroups $S \subseteq \mathbb{N}_0^d$ of genus g. In [6], the authors compute $N_{g,d}$ for small values of g, d and provide asymptotic bounds on $N_{g,d}$ for large values of g, d.

Several distinct generalizations of numerical semigroups have been made and studied extensively. For example, the concept of Weierstrass semigroup of d distinct rational point on an algebraic curve is one of them, and it has attracted authors because of applications, see [1, 9, 10]

Many important results have been obtained by applying the Weierstrass semigroup on curves to linear codes. In addition to the Hermitian curve, the Weierstrass semigroup of distinct rational points was studied on the Suzuki and GK curves. The authors construct algebraic geometry codes and improve the bounds on the minimum distance of the linear codes, $d \geq 2$, see [2, 3, 7–10].

In this work, we search isolated gaps of generalized numerical semigroups S in $\mathbb{N}_0^d$, $d \geq 2$, and study the generalized perfect numerical semigroups and we exhibit several examples. We examine how the perfectness condition affects the generalizations of some numerical group families, a special case is the Weierstrass semigroup of d distinct rational points.

The paper is organized as follows. In Section 2, we fix the notation that will be used throughout the paper and we give the necessary background on the generalized numerical semigroups. In Section 3, we deal with the isolated gaps of a generalized numerical semigroup, and we give conditions for determining isolated gaps in $\mathbb{N}_0^d$. The generalized perfect numerical semigroups are explained in the same section and these semigroups are illustrated with several examples. In Section 4, we investigate the effects of the perfectness condition on a Weierstrass semigroup of d distinct rational points on a curve.

2 Background

The purpose of this section is to recall the basic properties of a generalized numerical semigroup. These properties are generalizations of their counterparts in the classical numerical semigroup theory.

Let e_i denote the element of $\mathbb{N}_0^d$ such that its i-th component is 1 and other components are zero, for $i = 1, \dots, d$. If $z \in \mathbb{N}_0^d$, then the i-th component is usually denoted by $z^{(i)}$. We express the natural partial ordering on $\mathbb{N}_0^d$ by $\leq$. i.e., $x, y \in \mathbb{N}_0^d$, $x \leq y$ if and only if $x^{(i)} \leq y^{(i)}$ for every $i = 1, \dots, d$.

For $x \in \mathbb{N}_0^d$ and $A \subseteq \mathbb{N}_0^d$, we define $x + A = \{x + a : a \in A\}$.

If $A \subseteq \mathbb{N}_0^d$, then the submonoid of $\mathbb{N}_0^d$ generated by A is

$$\langle A \rangle = \{\lambda_1 a_1 + ... + \lambda_n a_n : \lambda_1, \dots, \lambda_n \in \mathbb{N}_0, a_1, \dots, a_n \in A, n \in \mathbb{N}_0\}.$$

If $S \subseteq \mathbb{N}_0^d$ is a monoid and if $H(S) = \mathbb{N}_0^d \backslash S$ is a finite set, then S is called a *generalized numerical semigroup.* The elements in $H(S)$ are called *gaps* of S and the number $g(S) = \#H(S)$ is called the *genus* of S.

If S is a generalized numerical semigroup and $S = \langle A \rangle$, then A is called a *generator system* of S, and we say S is generated by A. If A is a generator system of S and if any proper subset of A is not a generator system, then A is called *the minimal generator system* of S, and it is denoted by $G(S)$. Every generalized numerical semigroup has a finite minimal generator system and such set is also unique, for details we refer [4].

For $t \in \mathbb{N}_0^d$, we define the $\pi(t) = \{n \in \mathbb{N}_0^d : n \leq t\}$, where $\leq$ denotes the natural partial order in $\mathbb{N}_0^d$. Recall that for every $t \in \mathbb{N}_0^d$, the set $\pi(t)$ is finite.

Lemma 2.1. ([4]) *Let $S \subseteq \mathbb{N}_0^d$ be a monoid. Then S is a generalized numerical semigroup if and only if there exists $t \in \mathbb{N}_0^d$ such that for all elements $s \notin \pi(t)$, then $s \in S$.*

Lemma 2.1 is useful for the characterization of a generator system of a generalized numerical semigroup in $\mathbb{N}_0^d$.

Theorem 2.1. ([4]) *Let $d \geq 2$ and let $S = \langle A \rangle$ be the monoid generated by a set $A \subseteq \mathbb{N}_0^d$. Then S is a generalized numerical semigroup if and only if the set A fulfills each one of the following conditions:*

1. For all $j = 1, 2, \dots, d$ there exist $a_1^{(j)} e_j$, $a_2^{(j)} e_j, \dots, a_{r_j}^{(j)} e_j \in A$, $r_j \in \mathbb{N}$ such that $gcd(a_1^{(j)}, a_2^{(j)}, \dots, a_{r_j}^{(j)}) = 1$ (that is, the elements $a_i^{(j)}$, $1 \leq i \leq r_j$ generate a numerical semigroup).

2. For every i, k; $1 \leq i < k \leq d$ there exist $x_{ik}, x_{ki} \in A$ such that $x_{ik} = e_i + n_i^{(k)} e_k$ and $x_{ki} = e_k + n_k^{(i)} e_i$ with $n_i^{(k)}, n_k^{(i)} \in \mathbb{N}_0$.

Even if all generalized numerical semigroups are finitely generated submonoids of $\mathbb{N}_0^d$, for $d > 1$, not all finitely generated submonoids of $\mathbb{N}_0^d$ are generalized numerical semigroups, for details see [4].

The most important difference between numerical semigroups and generalized numerical semigroups becomes apparent when we consider the Frobenius element and generators. In $\mathbb{N}_0^d$

there is not a natural total order so it is not clear how to define the Frobenius element for a generalized numerical semigroups.

A total order $<_m$ in $\mathbb{N}_0^d$ is called a monomial order if it satisfies:

1) If $v, w \in \mathbb{N}_0^d$ with $v <_m w$, then $v + u <_m w + u$ for every $u \in \mathbb{N}_0^d$.

2) If $v \in \mathbb{N}_0^d$ and $v \neq 0$, then $0 <_m v$.

If "$<_m$ " is replaced by "$\prec$ " and the condition (1) is replaced with the condition " If $v, w \in \mathbb{N}_0^d$ with $v \prec w$, then $v \prec w + u$ for every $u \in \mathbb{N}_0^d$ ", then $\prec$ is called a relaxed monomial order.

For $v, w \in \mathbb{N}_0^d$, $v \prec w$ with respect to the lexicographic order if and only if the first nonzero coordinate of $w - v$ is positive. The lexicographic order is a relaxed monomial order, it is also a monomial order.

Let $S \subseteq \mathbb{N}_0^d$ be a generalized numerical semigroup, and $H(S)$ be the set of gaps of S. Given a relaxed monomial order $\prec$ in $\mathbb{N}_0^d$, we define the greatest element in $H(S)$ with respect to $\prec$ as the Frobenius element of S and it is denoted by $F(S)_\prec$. The smallest nonzero element of S with respect to $\prec$ is called multiplicity of S, and denoted by $m(S)_\prec$.

The Frobenius element of a generalized numerical semigroup is uniquely determined with respect to the defined relaxed monomial order, see [5,6].

Let $S \subseteq \mathbb{N}_0^d$ be a monoid and $\mathbf{n} \in S$. The Apéry set of S with respect to $\mathbf{n}$ is defined by

$$Ap(S, \mathbf{n}) = \{s \in S : s - \mathbf{n} \notin S\}$$

with $s - \mathbf{n}$ being the difference in $\mathbb{Z}_0^d$.

3 Main results

In this section, our aim is to search for isolated gaps of a generalized numerical semigroup S in $\mathbb{N}_0^d$, $d \geq 2$, and examine some properties of the generalized perfect numerical semigroups. Furthermore, we illustrate our results with some interesting examples.

Definition 3.1. *Let $S \subseteq \mathbb{N}_0^d$ be a generalized numerical semigroup, $d \geq 1$, and $h \in H(S)$. If for each $j \in \{1, \ldots, d\}$,*

$$\{h - e_j,\ h + e_j\} \cap \mathbb{N}_0^d \subset S,$$

then h is called an isolated gap of S and the set of isolated gaps of S is denoted by $\operatorname{iso}(S)$. *If* $\operatorname{iso}(S)$ *is the empty set, then S is called a generalized perfect numerical semigroup in $\mathbb{N}_0^d$.*

First, we consider the case of $d = 2$. Let $S \subseteq \mathbb{N}_0^2$ be a generalized numerical semigroup generated by A. There exist $a_1^{(1)}e_1, a_2^{(1)}e_1, \ldots, a_r^{(1)}e_1, a_1^{(2)}e_2, a_2^{(2)}e_2, \ldots, a_r^{(2)}e_2 \in A$, such that $\gcd(a_1^{(1)}, a_2^{(1)}, \ldots, a_r^{(1)}) = 1$ and $\gcd(a_1^{(2)}, a_2^{(2)}, \ldots, a_r^{(2)}) = 1$, by Theorem 2.1.

Let $S_1 = \langle a_1^{(1)}, a_2^{(1)}, \ldots, a_r^{(1)} \rangle$, $S_2 = \langle a_1^{(2)}, a_2^{(2)}, \ldots, a_r^{(2)} \rangle$. Hence, we obtain that

$$\mathbb{N}_0^2 = \{S_1 \times S_2\} \cup \{H(S_1) \times S_2\} \cup \{S_1 \times H(S_2)\} \cup \{H(S_1) \times H(S_2)\}.$$

If $x \in S_1$ and $y \in S_2$, then $(x, 0), (0, y) \in S$ and we get $(x, y) \in S_1 \times S_2 \subset S$.

Here, we note that $H(S_1) \times H(S_2)$ has important role (contains generators) for the generalization of the Weierstrass semigroup, see [8–10].

Let $S \subseteq \mathbb{N}_0^2$ be submonoid with the gap set $H(S) = \mathbb{N}_0^2 \backslash S$ and $h = (i, j) \in H(S)$. If $\{(i, j-1), (i, j+1), (i-1, j), (i+1, j)\} \cap \mathbb{N}_0^2 \subset S$, then h is an isolated gap of S.

Example 3.1. *Let $S \subseteq \mathbb{N}_0^2$ be a generalized numerical semigroup with the gap set $H(S) = \{(1,0), (1,1), (0,1)\}$. In this case, $G(S) = \{(0,2), (0,3), (1,2), (1,3), (2,0), (2,1), (3,0), (3,1)\}$ is the minimal generator system of S. Each $h \in H(S)$ is not an isolated gap, so S is a generalized perfect numerical semigroup.*

Remark 3.2. *The semigroup S given in Example 3.1 has not isolated gaps, but the numerical semigroups S_1 and S_2 have isolated gaps. Thus, a generalized numerical semigroup can be perfect even though the component semigroups (S_1 and S_2) are not perfect.*

Example 3.3. *Let $S \subseteq \mathbb{N}_0^2$ be a generalized numerical semigroup with the gap set $H(S) = \{(0,1), (1,0), (1,1), (1,2), (1,3), (1,5), (2,1), (3,1), (5,1)\}$. In this case, $G(S) = \{(2,0), (0,2), (3,0), (0,3), (1,4), (4,1)\}$ is the minimal generator system of S. Thus, $(1,5), (5,1) \in H(S)$ are the isolated gaps of S. Therefore, S is not a generalized perfect numerical semigroup. In Figure 1, the blue colored dots are elements of S, the red colored dots are gaps of S, the black colored dots are the isolated gaps of S.*

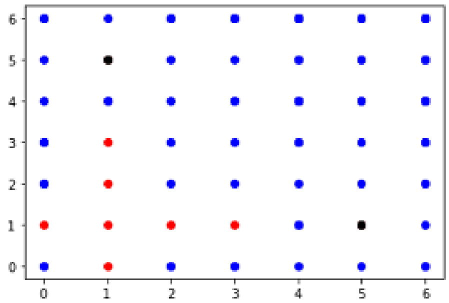

Figure 1. The generalized numerical semigroup in Example 3.3

Example 3.4. *Let $S \subseteq \mathbb{N}_0^2$ be a generalized numerical semigroup with the minimal generator system $G(S) = \{(0,2), (0,5), (3,0), (4,0), (1,4), (4,1)\}$. In this case,*

$$H(S) = \left\{ \begin{array}{l} (0,1),(0,3),(1,0),(1,1),(1,2),(1,3),(1,5),(1,7),(2,0),(2,1),(2,2), \\ (2,3),(2,4),(2,5),(2,6),(2,7),(2,9),(2,11),(3,1),(3,3), \\ (5,0),(5,1),(5,2),(5,3),(6,1),(6,3),(9,1),(9,3) \end{array} \right\}$$

is the gap set of S. Here, $\mathrm{iso}(S) = \{(2,9), (2,11), (9,1), (9,3)\}$, and considering $\prec$ the lexicographic order the multiplicity of S is $m(S)_\prec = (0,2)$, and

$$Ap(S,(0,2)) = \left\{ \begin{array}{l} (0,0),(3,0),(4,0),(0,5),(1,4),(1,9), \\ (2,8),(2,11),(3,5),(4,1)(5,4),(5,5), \\ (6,5),(9,5),(7,1),(8,1) \end{array} \right\} \cup \{(n,0),(m,1) : n \geq 6, m \geq 10\}.$$

The Frobenius element of S is $F(S)_\prec = (9,3)$. Since $\{(9,2),(8,3)\} \cap \mathbb{N}_0^2 \subset S$, we get that $F(S)_\prec$ is an isolated gap of S. In Figure 2, the green colored dots are the minimal generators of S, the red colored dots are gaps of S, the black colored dots are the isolated gaps of S, and the purple colored dots are the elements of $Ap(S,(0,2))$, not belonging to the set of minimal generators.

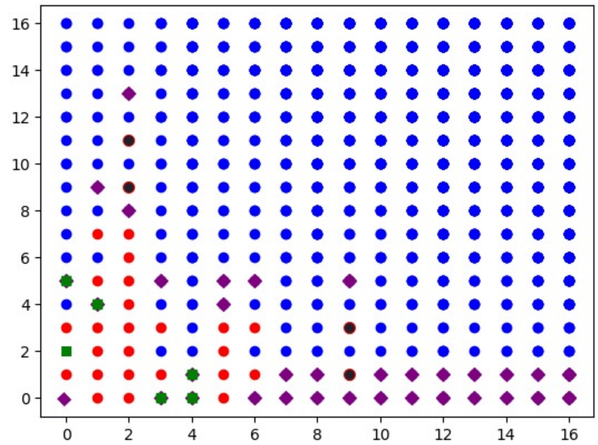

Figure 2. The generalized numerical semigroup with generator set $G(S)$ in Example 3.4.

Theorem 3.5. *Let $S = \langle (a,0),(1,b),(0,1),(2,0)\rangle$ be a generalized numerical semigroup, where $a,b \in \mathbb{N}$, a is odd and $b \geq 2$. Then, $F(S)_{\prec} = (a-2, b-1)$ with respect to lexicographic order and S is a generalized perfect numerical semigroup.*

Proof. First, we will show that $(a-2,b-1)$ is the largest element of $H(S)$. If $(a-2,b-1) \in S$, then there are $\alpha, \beta, \gamma, \lambda \in \mathbb{N}_0$ such that $(a-2,b-1) = \alpha(a,0) + \beta(1,b) + \gamma(0,1) + \lambda(2,0)$. In the case of $\alpha > 0$ or $\beta > 0$, we get $a = -\dfrac{\beta + 2\lambda + 2}{\alpha - 1} < 0$, $b = -\dfrac{\gamma+1}{\beta-1} < 0$, but these contradict $a, b > 0$. If $\alpha = \beta = 0$, then $2\lambda + 2 = a$, this is a contradiction since a is odd.

Suppose that $h = (h^{(1)}, h^{(2)}) \in H(S)$ and $(a-2,b-1) \prec h$ with respect to lexicographic order. We obtain $(h^{(1)}, h^{(2)}) - (a-2,b-1) = (h^{(1)} - a + 2, h^{(2)} - b + 1)$. Hence,

i) If $h^{(1)} - a + 2 = 0$, then $h^{(1)} = a-2$ and $h^{(2)} > b-1$. Then, $h = (a-2,0) + (0,h^{(2)}) \in S$.

ii) If $h^{(1)} > a-2$, then $(h^{(1)},0) \in S$. In particular $h = (h^{(1)},0) + (0,h^{(2)}) \in S$.

There are contradictions in both cases. Now, we will show that none of the elements of $H(S)$ are isolated points. Let define the set

$$\mathcal{K} = \{(x,t) : 1 \leq x \leq a-2, 0 \leq t \leq b-1 \text{ and } x \text{ is odd}\}.$$

Let $h = (h^{(1)}, h^{(2)}) \in H(S)$ and $h \notin \mathcal{K}$. Note that $h^{(1)}$ can not be even. Then, $h = (h^{(1)},0) + (0,h^{(2)})$, where $h^{(1)} > a-2$ and $h^{(2)} > b-1$. Since $F(S)_{\prec} \prec h$, it is a contradiction to $h \in H(S)$. Therefore, $H(S) \subseteq \mathcal{K}$. It is clear that $\mathcal{K} \subseteq H(S)$. Hence, we obtain $H(S) = \mathcal{K}$.

Assume that $h \in H(S)$ is an isolated points of S. Since $h^{(1)} \pm 1$ are even numbers, $(h^{(1)} \pm 1, h^{(2)}) \in S$. It can be seen that at least one of $(h^{(1)}, h^{(2)} \pm 1)$ is an element of $H(S)$. This is a contradiction. □

Corollary 3.1. *Let $S = \langle (a,0),(1,b),(0,1),(2,0)\rangle$ be a generalized numerical semigroup, where $a,b \in \mathbb{N}$, a is odd and $b \geq 2$ and considering $\prec$ the lexicographic order. Then, the following statements hold:*

1. $m(S)_{\prec} = (0,1)$, $g(S)_{\prec} = b(\frac{a-1}{2})$.
2. $Ap(S, m(S)_{\prec}) = \left\{(0,0),(2k,0),(2k-1,b),(a+n,0) : 1 \leq k \leq \frac{a-1}{2}, n \in \mathbb{N}_0\right\}$.

Proof. The proof follows from Theorem 3.5. □

Example 3.6. *Let $S \subseteq \mathbb{N}_0^2$ be a generalized numerical semigroup with the minimal generator system $G(S) = \{(2,0),(13,0),(0,1),(1,8)\}$. In this case, $m(S)_\prec = (0,1)$, $g(S) = 42$ and*

1. $H(S) = \{(x,t) : 1 \le x \le 11,\ 0 \le t \le 7$ *and x is odd*$\}$.
2. $Ap(S, m(S)_\prec) = \{(0,0),(2k,0),(2k-1,8),(13+n,0) : 1 \le k \le 6, n \in \mathbb{N}_0\}$.
3. *S is generalized perfect numerical semigroup.*

Example 3.7. *It is easy to see that $(2,1)$ is an isolated gap of $S = \langle (a,0),(0,a),(2,0),(0,2),(1,1)\rangle$, where $3 \le a \in \mathbb{N}$ and a is odd . Then, S is not generalized perfect numerical semigroup.*

Example 3.8. *Let $S \subseteq \mathbb{N}_0^4$, be a generalized numerical semigroup with the minimal generator system $A = \{(1,0,0,0),\ (1,0,0,1),\ (0,1,0,0),\ (0,1,0,1),\ (0,0,1,0),\ (0,0,2,1),\ (0,0,0,2),\ (0,0,1,3),\ (0,0,0,5)\}$. Then, $H(S) = \{(0,0,0,1),\ (0,0,0,3),\ (0,0,1,1))\}$. Thus $\mathrm{iso}(S) = \{(0,0,0,1),(0,0,0,3)\}$ and S is not perfect.*

Proposition 3.9. *Let $S \subset \mathbb{N}_0^d$ be a generalized numerical semigroup with the Frobenius element $F(S)_\prec = (x^{(1)}, \ldots, x^{(d)})$ according to a fixed relaxed mononomial order. $F(S)_\prec$ is an isolated gap of S if and only if*

$$\{F(S)_\prec - e_j : 1 \le j \le d\} \cap \mathbb{N}_0^d \subset S. \tag{1}$$

Proof. (:⇒) Suppose $F(S)_\prec$ be an isolated gap of S. By the definition of the concept of isolated gap, we have $\{F(S)_\prec \pm e_j : 1 \le j \le d\} \cap \mathbb{N}_0^d \subset S$. Thus, the assertions hold.

(⇐:) Let $F(S)_\prec$ be a Frobenius element and (1) hold. If $\{F(S)_\prec + e_j : 1 \le j \le d\} \cap \mathbb{N}_0^d \subset S$ is not true, then $F(S)_\prec + e_j$ becomes a gap, for some $j \le d$, and this is a contradiction with the definition of the Frobenius element. Therefore, $F(S)_\prec$ is an isolated gap. □

Example 3.10. *Let $S \subseteq \mathbb{N}_0^3$ be a generalized numerical semigroup with minimal generator system*

$$A = \left\{ \begin{array}{l} (2,0,0),(3,0,0),(0,2,0),(0,3,0),(0,0,2),(0,0,3), \\ (1,0,3),(3,0,1),(1,2,0),(2,1,0),(0,1,3),(0,3,1) \end{array} \right\}.$$

In this case, the gap set of S is

$$H(S) = \left\{ \begin{array}{l} (1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1),(1,0,1), \\ (2,0,1),(1,0,2),(3,1,0),(1,3,0),(0,2,1),(0,1,2), \\ (1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,1,5),(1,1,7), \\ (1,2,1),(1,3,1),(1,3,2),(1,4,1),(1,6,1),(2,1,1), \\ (2,2,1),(3,1,1),(3,1,2),(4,1,1),(4,0,1),(1,0,4), \\ (0,4,1),(0,1,4),(6,1,1) \end{array} \right\}.$$

Then, $\mathrm{iso}(S) = \{(1,1,7),(1,6,1),(6,1,1)\}$. With respect to lexicographic order, the Frobenius element of S is $F(S)_\prec = (6,1,1)$. Hence, $(6,1,1)$ is an isolated gap of S since $\{(5,1,1),(6,0,1),(6,1,0)\} \cap \mathbb{N}_0^3 \subset S$.

Proposition 3.11. *Let $S \subseteq \mathbb{N}_0^d$ be a generalized numerical semigroup, $h = (x^{(1)}, x^{(2)}, \ldots, x^{(d)})$ be a gap of S with $x^{(j_1)} \cdot x^{(j_2)} \cdots x^{(j_r)} \neq 0$, $1 \leq r \leq d$. If h is an isolated gap of S, then $\{e_{j_1}, e_{j_2}, \ldots, e_{j_r}\} \subseteq H(S)$.*

Proof. For a fixed k such that $1 \leq k \leq r$, we have $x^{(j_k)} \neq 0$ by the assumption. If $h \in \mathrm{iso}(S)$, then $\{(x^{(1)}, x^{(2)}, \ldots, x^{(j_k-1)}, x^{(j_k)} \pm 1, x^{(j_k+1)}, \ldots, x^{(d)})\} \cap \mathbb{N}_0^d \subset S$. Then we obtain

$$\underbrace{(x^{(1)}, \ldots, x^{(d)})}_{\in H(S)} - \underbrace{(x^{(1)}, \ldots, x^{(j_k-1)}, x^{(j_k)} - 1, x^{(j_k+1)}, \ldots, x^{(d)})}_{\in S} = e_{j_k} \in H(S),$$

otherwise $h \notin H(S)$. The proof is completed. □

Definition 3.2. *Let $S \subseteq \mathbb{N}_0^d$ be a generalized numerical semigroup generated by A and let $S_k := \langle a_1^{(k)}, a_2^{(k)}, \ldots, a_r^{(k)} \rangle$, where $a_1^{(k)} e_k, \ldots, a_r^{(k)} e_k \in A$ and $\gcd(a_1^{(k)}, \ldots, a_r^{(k)}) = 1, k = 1, \ldots, d$. For each k, we define the set I_k as follows:*

$$I_k = \left\{ x = (x^{(1)}, \ldots, x^{(d-1)}) \in \mathbb{N}_0^{d-1} : \tilde{x} = (x^{(1)}, \ldots, x^{(k-1)}, z, x^{(k)}, \ldots, x^{(d-1)}) \in S, \textit{ for all } z \in S_k \right\}.$$

Proposition 3.12. *I_k is a generalized numerical semigroup on $\mathbb{N}_0^{d-1}$.*

Proof. $I_k \subset \mathbb{N}_0^{d-1}$. Since the operation "+" is associative on S, it is also associative on I_k. Let $x, y \in I_k$. Then for each $s \in S_k$, we will show that

$$\gamma := (x^{(1)} + y^{(1)}, \ldots, x^{(k-1)} + y^{(k-1)}, s, x^{(k)} + y^{(k)}, \ldots, x^{(d-1)} + y^{(d-1)}) \in S.$$

There exist $z, t \in S_k$ such that $s = z + t$. Hence, $\tilde{x} = (x^{(1)}, \ldots, x^{(k-1)}, z, x^{(k)}, \ldots, x^{(d-1)}) \in S$ and $\tilde{y} = (y^{(1)}, \ldots, y^{(k-1)}, t, y^{(k)}, \ldots, y^{(d-1)}) \in S$. Since S is a generalized numerical semigroup, we get $\tilde{x} + \tilde{y} = \gamma \in S$. Therefore, $x + y \in I_k$. If $\alpha = (z^{(1)}, z^{(2)}, \ldots, z^{(d-1)}) \in \mathbb{N}_0^{d-1} \backslash I_k$, then there exists $u \in S_k$ such that $\tilde{\alpha}$=$(z^{(1)}, \ldots, z^{(k-1)}, u, z^{(k)}, \ldots, z^{(d-1)}) \notin S$, where $\tilde{\alpha}^{(k)} = u$. Hence, $\tilde{\alpha} \in H(S)$ and we obtain that $H(I_k)$ is a finite set since $H(S)$ is finite. Then I_k is a generalized numerical semigroup on $\mathbb{N}_0^{d-1}$. □

Proposition 3.13. *Let $S \subseteq \mathbb{N}_0^d$ be a generalized numerical semigroup and $h = (x^{(1)}, \ldots, x^{(d)})$ be a gap of S. Then, h is an isolated gap of S if there exists an integer k, with $1 \leq k \leq d$, such that the following statements hold:*

1. $(x^{(1)}, \ldots, x^{(k-1)}, x^{(k+1)}, \ldots, x^{(d)}) \in I_k$,
2. $e_j + x^{(k)} e_k \in S$, *for each* $j = 1, 2, \ldots, d$.
3. $x^{(k)} \in \mathrm{iso}(S_k)$.

Proof. By the first condition, and since $h \in H(S)$, we have

$$h = (x^{(1)}, \ldots, x^{(d)}) = \underbrace{(x^{(1)}, \ldots, x^{(k-1)}, 0, x^{(k+1)} \ldots, x^{(d)})}_{\in S} + (0, \ldots, x^{(k)}, \ldots, 0).$$

Hence $x^{(k)} e_k \notin S$, otherwise $h \in S$. Then $x^{(k)} \notin S_k$. Since $x^{(k)} \in \mathrm{iso}(S_k)$, we get that $\left\{x^{(k)} \pm 1\right\} \cap \mathbb{N}_0 \subset S_k$. Therefore,

$$(x^{(1)}, \ldots, x^{(k)} \pm 1, \ldots, x^{(d)}) = (x^{(1)}, \ldots, x^{(k-1)}, 0, x^{(k)}, \ldots, x^{(d)}) + (0, \ldots, x^{(k)} \pm 1, \ldots, 0) \in S.$$

Similarly,

$$\{(x^{(1)} \pm 1, \ldots, x^{(k)}, \ldots, x^{(d)}), \ldots, (x^{(1)}, x^{(2)}, \ldots, x^{(k)}, \ldots, x^{(d)} \pm 1)\} \cap \mathbb{N}_0^d \subset S.$$

In fact, by the second condition, we get

$$(x^{(1)} \pm 1, \ldots, x^{(k)}, \ldots, x^{(d)}) = (x^{(1)}, \ldots, x^{(k-1)}, 0, x^{(k)}, \ldots, x^{(d)}) + (\pm 1, \ldots, x^{(k)}, \ldots, 0) \in S.$$

The assertion holds for the other elements, the proof follows from using the same argument. Thus, h is an isolated gap of S. □

Proposition 3.14. *Let $S \subseteq \mathbb{N}_0^d$ be a generalized numerical semigroup, $h = (x^{(1)}, \ldots, x^{(d)})$ be a gap of S. Then, h is an isolated gap of S if there exists an integer k, with $1 \le k \le d$, such that the following statements hold:*

1. $\left(x^{(1)}, \ldots, x^{(k-1)}, x^{(k+1)}, \ldots, x^{(d)}\right) \notin I_k$.
2. $\left\{x^{(k)} e_k, \left(x^{(k)} - 2\right) e_k\right\} \subset S$.
3. $(x^{(1)}, \ldots, x^{(k-1)}, 0, x^{(k+1)}, \ldots, x^{(d)}) \in \mathrm{iso}(S)$.

Proof. By the first condition, $\left(x^{(1)}, \ldots, x^{(k-1)}, x^{(k+1)}, \ldots, x^{(d)}\right) \in H(I_k)$ and there exists $t \in S_k$ such that $(x^{(1)}, \ldots, x^{(k-1)}, t, x^{(k+1)}, \ldots, x^{(d)}) \notin S$. Therefore, we get

$$\underbrace{\left(x^{(1)}, \ldots, x^{(k-1)}, t, x^{(k+1)}, \ldots, x^{(d)}\right)}_{\in H(S)} - \underbrace{te_k}_{\in S} = \left(x^{(1)}, \ldots, x^{(k-1)}, 0, x^{(k+1)}, \ldots, x^{(d)}\right) \in H(S).$$

Hence,

$$\{(x^{(1)} \pm 1, \ldots, x^{(k-1)}, 0, x^{(k+1)}, \ldots, x^{(d)}), (x^{(1)}, x^{(2)} \pm 1, \ldots, x^{(k-1)}, 0, x^{(k+1)}, \ldots, x^{(d)}), \ldots, (x^{(1)}, x^{(2)}, \ldots, x^{(k-1)}, 0, x^{(k+1)}, \ldots, x^{(d)} \pm 1), (x^{(1)}, \ldots, x^{(k-1)}, 1, x^{(k+1)}, \ldots, x^{(d)})\} \cap \mathbb{N}_0^d$$

is contained in S by the third condition, and we get the following statements

$$\begin{aligned}
&(x^{(1)} \pm 1, x^{(2)}, \ldots, x^{(k-1)}, 0, x^{(k+1)}, \ldots, x^{(d)}) + x^{(k)} e_k = (x^{(1)} \pm 1, x^{(2)}, \ldots, x^{(d)}) \in S,\\
&(x^{(1)}, x^{(2)} \pm 1, \ldots, x^{(k-1)}, 0, x^{(k+1)}, \ldots, x^{(d)}) + x^{(k)} e_k = (x^{(1)}, x^{(2)} \pm 1, \ldots, x^{(d)}) \in S\\
&\ldots\\
&(x^{(1)}, x^{(2)}, \ldots, x^{(k-1)}, 0, x^{(k+1)}, \ldots, x^{(d)} \pm 1) + x^{(k)} e_k = (x^{(1)}, \ldots, x^{(d-1)}, x^{(d)} \pm 1),\\
&(x^{(1)}, \ldots, x^{(k-1)}, 1, x^{(k+1)}, \ldots, x^{(d)}) + x^{(k)} e_k = (x^{(1)}, \ldots, x^{(k-1)}, x^{(k)} + 1, x^{(k+1)}, \ldots, x^{(d)}) \in S.
\end{aligned}$$

Since $\left(x^{(k)} - 2\right) e_k \in S$, similarly, we obtain

$$(x^{(1)}, \ldots, x^{(k-1)}, 1, x^{(k+1)}, \ldots, x^{(d)}) + \left(x^{(k)} - 2\right) e_k = (x^{(1)}, \ldots, x^{(k)} - 1, x^{(k+1)}, \ldots, x^{(d)}) \in S.$$

Thus, h is an isolated gap of S. □

Proposition 3.15. *Let $S_1, S_2 \subseteq \mathbb{N}_0^d$ be generalized numerical semigroups. Then, we have*

$$\mathrm{iso}(S_1 \cap S_2) \subseteq \mathrm{iso}(S_1) \cup \mathrm{iso}(S_2).$$

Proof. Let us assume that $h \in \mathrm{iso}(S_1 \cap S_2)$. Then the assertion can be proved as follows:

$$\begin{aligned}
h \in \mathrm{iso}(S_1 \cap S_2) \quad &\Rightarrow \quad h \in H(S_1 \cap S_2) \text{ and } \{h \pm e_j : 1 \le j \le d\} \cap \mathbb{N}_0^d \subset S_1 \cap S_2\\
&\Rightarrow \quad h \in H(S_1) \cup H(S_2) \text{ and } \{h \pm e_j : 1 \le j \le d\} \cap \mathbb{N}_0^d \subset S_i, \text{for } i = 1, 2.\\
&\Rightarrow \quad h \in \mathrm{iso}(S_1) \cup \mathrm{iso}(S_2).
\end{aligned}$$
□

Remark 3.16. *For two generalized numerical semigroups S_1 and S_2, it is known that $S_1 \cup S_2$ may not be a generalized numerical semigroup. If we consider S_1, S_2 as generalized numerical sets (subset of $\mathbb{N}_0^d$ that contains 0 and has a finite complement in $\mathbb{N}_0^d$), then we observe that $H(S_1 \cup S_2) = H(S_1) \cap H(S_2)$, and*

$$\begin{aligned} h \in \operatorname{iso}(S_1) \cap \operatorname{iso}(S_2) \ &\Rightarrow h \in H(S_1 \cup S_2) \cap \operatorname{iso}(S_1) \cap \operatorname{iso}(S_2) \\ &\Rightarrow \{h \pm e_j : 1 \leq j \leq d\} \cap \mathbb{N}_0^d \subset S_i,\ i = 1, 2 \ \ \Rightarrow h \in \operatorname{iso}(S_1 \cup S_2). \end{aligned}$$

Therefore, we have $\operatorname{iso}(S_1) \cap \operatorname{iso}(S_2) \subseteq \operatorname{iso}(S_1 \cup S_2)$.

Theorem 3.17. *If $S_1, S_2 \subseteq \mathbb{N}_0^d$ are generalized perfect numerical semigroups, then $S_1 \cap S_2$ is also generalized perfect numerical semigroup.*

Proof. It is known that the intersection of two generalized numerical semigroups is again generalized numerical semigroup. The rest of the proof follows from Proposition 3.15. □

The opposite side of the theorem is not true in $\mathbb{N}_0^d$, see Example 3.18.

Example 3.18. *It is shown that $S_1 \subseteq \mathbb{N}_0^2$ with gap set $H(S_1) = \{(0,1), (1,0), (1,1), (1,2), (1,3), (1,5), (2,1), (3,1), (5,1)\}$ is not a generalized perfect numerical semigroup, since $(1,5)$, $(5,1)$ are the isolated gaps of S_1.*

Let $S_2 \subseteq \mathbb{N}_0^2$ be a generalized numerical semigroup with the gap set $H(S_2) = \{(1,0), (2,0), (5,0), (0,1), (1,1), (4,1), (0,2), (2,2), (3,2), (2,3), (1,4), (0,5)\}$. In this case, $(5,0)$, $(0,5)$, $(1,4)$, $(4,1) \in H(S_2)$ are the isolated gaps of S_2. Therefore, S_2 is not a generalized perfect numerical semigroup. Now, it is not difficult to show that none of the gaps of $S_1 \cap S_2$ are isolated. Hence, $S_1 \cap S_2$ is a generalized perfect numerical semigroup.

4 Generalized Weierstrass semigroup and perfectness

In this section, we consider the Weierstrass semigroup of d rational points which is an important family because of its applications. It is a generalized numerical semigroup and we want to investigate the behavior of the perfectness condition.

Let X be a smooth, projective, and absolutely irreducible curve over a finite field F. Let d be a positive integer and suppose $|F| \geq d$. For a function $f \in F(X)$, the zero divisor and the pole divisor of f are denoted by $(f)_0$ and $(f)_\infty$, respectively. Let $L(D)$ denote the set of rational functions f on X with $(f) \geq -D$ together with the zero function, where D is a divisor on X.

Given d distinct rational points $Q_1, \ldots, Q_d$ on X, the Weierstrass semigroup of $Q_1, \ldots, Q_d$ is defined by

$$S(Q_1, \ldots, Q_d) := \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d : \text{there exists a function } f,\ (f)_\infty = \sum_{i=1}^{d} \alpha_i Q_i\} \tag{2}$$

and the gap set of $Q_1, \ldots, Q_d$ is defined by $H(Q_1, \ldots, Q_d) = \mathbb{N}_0^d \backslash S(Q_1, \ldots, Q_d)$, $d \geq 1$.

The structure of $S(Q_1, \ldots, Q_d)$ has been studied from various aspects, for instance it is studied the minimal generator system and the genus, for details see [2, 7–9].

From now on, $Q_1, \ldots, Q_d$ will be rational points on the curve X.

Proposition 4.1. ([2], Lemma 2.8) *For* $(a_1,\dots,a_d),(b_1,\dots,b_d) \in S(Q_1,\dots,Q_d)$*, let* $q_i := \max(a_i,b_i)$*, for* $i=1,\dots,d$*. Then* $(q_1,\dots,q_d)\in S(Q_1,\dots,Q_d)$*.*

Theorem 4.2. *For each* $i=1,\dots,d$*, if* $S(Q_i)$ *is perfect numerical semigroup, then* $S(Q_1,\dots,Q_d)$ *is a generalized perfect numerical semigroup.*

Proof. Let $\mathbb{L}$ be the set of isolated gaps of $S(Q_1,\dots,Q_d)$ which have at least two nonzero components. First, we observe that $\operatorname{iso}(S(Q_1,\dots,Q_d)) \subseteq \cup_{i=1}^{d}\operatorname{iso}(S(Q_i))e_i \cup \mathbb{L}$. Since $S(Q_i)$ is a perfect numerical semigroup for each $i=1,\dots,d$, we get $\operatorname{iso}(S(Q_1,\dots,Q_d))=\mathbb{L}$.

Take $h=(h_1,\dots,h_d)\in\mathbb{L}$ with $h^{(i_1)}\cdots h^{(i_r)}\neq 0$, $2<r\le d$. Hence, $h-e_{i_1},\dots,h-e_{i_r}\in S(Q_1,\dots,Q_d)$, by Proposition 4.1, we obtain $h\in S(Q_1,\dots,Q_d)$, but this is a contradiction. □

The next corollary follows from the proof of the Theorem 4.2.

Corollary 4.1. $\operatorname{iso}(S(Q_1,\dots,Q_d)) \subseteq \cup_{i=1}^{d}\operatorname{iso}(S(Q_i))e_i$.

Proposition 4.3. *Let* $Q_1,\dots,Q_d$ *be rational points on a Hermitian curve defined by* $y^q+y=x^{q+1}$ *over* F_{q^2}*, where* F_{q^2} *denotes the finite field with* q^2 *elements. Isolated gaps of the Weierstrass semigroup* $S(Q_1,\dots,Q_d)$ *constitute the set* $\{F(S(Q_i))e_i : i=1,\dots,d\}$*, where* $F(S(Q_i))=(q-2)(q+1)+1$ *is the Frobenius number of* $S(Q_i)$*.*

Proof. It is well-known that the genus of Hermitian curve is $g=\frac{q(q-1)}{2}$, and the curve has q^3+1 rational points. The group of automorphisms of the Hermitian curve acts doubly transitive on the set of rational points. Therefore, the Weierstrass semigroup of two distinct rational points on the curve is independent of choosing of points. Let $S(Q_i)$ be the Weierstrass semigroup of Q_i, $i=1,\dots,d$. Then, $S(Q_i)=\langle q,q+1\rangle$, and the Weierstrass gap set $H(Q_i)$ consists of the following numbers:

$$\begin{array}{ccccc}
1 & 2 & \cdots & q-2 & q-1\\
(q+1)+1 & (q+1)+2 & & (q+1)+(q-2) & \\
\vdots & \vdots & & & \\
(q-3)(q+1)+1 & (q-3)(q+1)+2 & & & \\
(q-2)(q+1)+1 & & & &
\end{array}$$

Hence, the Frobenius number $F(S(Q_i))=(q-2)(q+1)+1$. By Corollary 3.5 in [14], a numerical semigroup generated by two consecutive integers has exactly one isolated gap and this is the Frobenius number of $S(Q_i)$. Since $F(S(Q_i))+1=2g$, by Theorem 1.5.17 in [15], we obtain that

$$L((F(S(Q_i))-1)Q_i+Q_j)\neq L(F(S(Q_i))Q_i+Q_j)\neq L(F(S(Q_i))Q_i).$$

Then $F(S(Q_i))e_i+e_j\in S(Q_1,\dots,Q_d)$, for all j. □

Proposition 4.4. *The Suzuki curve* $\mathcal{S}$ *over the field* F_q *defined by the equation*

$$y^q-y=x^{q_0}(x^q-x),$$

where $q_0=2^t, q=2^{2t+1}$ *for some positive integer* t*, and* F_q *is the finite field with* q *elements. Isolated gaps of the Weierstrass semigroup* $S(Q_1,Q_2)$ *are* $(2q_0(q-1)-1,0),(0,2q_0(q-1)-1)$*, where* $2q_0(q-1)-1$ *is the Frobenius number of* $S(Q_i), i=1,2$*.*

Proof. The curve has exactly $q^2 + 1$ rational points, and the genus of the curve is $g = q_0(q-1)$. The group of automorphisms of the Suzuki curve acts doubly transitive on the set of rational points. Therefore, the Weierstrass semigroups of two distinct rational points on the Suzuki curve is independent of choosing of points. Let Q_1, Q_2 be rational points of the Suzuki curve. The Weierstrass semigroup of Q_i is $S(Q_i) = \langle q, q + q_0, q + 2q_0, q + 2q_0 + 1\rangle$, $i = 1, 2$, and $F(S(Q_i)) = 2g - 1$, see Lemma 3.1 in [10]. Hence, $S(Q_i)$ is a symmetric numerical semigroup and $F(S(Q_i)) - 1 \in S(Q_i)$, by Proposition 4.4 and Corollary 4.5 in [13]. Clearly, $2g \in S(Q_i)$. Then, $F(S(Q_i))$ is an isolated point of $S(Q_i)$. By Theorem 3.3 in [10], we see that $(1, 2q_0(q-1)-1) \in S(Q_1, Q_2)$. Note that $2q_0(q-1)-1 = min\{j : (1, j) \in S(Q_1, Q_2)\}$. Similiarly, we have $(2q_0(q-1)-1, 1) \in S(Q_1, Q_2)$. Hence, $(2q_0(q-1)-1, 0), (0, 2q_0(q-1)-1)$ are isolated gaps of $S(Q_1, Q_2)$. □

Example 4.5. *Let $\mathcal{S}$ be Suzuki curve over the field F_8 defined by $y^8 - y = x^2(x^8 - x)$. Let Q_1, Q_2 be rational points on $\mathcal{S}$. Then $S(Q_i) = \langle 8, 10, 12, 13\rangle$ and $\mathrm{iso}(S(Q_i)) = \{9, 11, 17, 19, 27\}$, for $i = 1, 2$. By Theorem 3.3 and Example 3.4 in [10], the generators of the Weierstrass semigroup $S(Q_1, Q_2)$ is $\Gamma \cup (\langle 8, 10, 12, 13\rangle \times \{0\}) \cup (\{0\} \times \langle 8, 10, 12, 13\rangle)$, where*

$$\Gamma := \left\{ \begin{array}{l} (1,27),(2,19),(3,11),(4,17),(5,9),(6,15),(7,7), \\ (27,1),(19,2),(11,3),(17,4),(9,5),(15,6),(14,14) \end{array} \right\}.$$

Hence, we have $\mathrm{iso}(S(Q_1, Q_2)) = \{(0, 27), (27, 0)\}$. For details on $S(Q_1, Q_2)$, we refer to [1,10].

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