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On the characterization of rectangular duals

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Abstract: A rectangular partition is a partition of a rectangle into a finite number of rectangles. A rectangular partition is generic if no four of its rectangles meet at the same point. A plane graph G is called a rectangularly dualizable graph if G can be represented as a rectangular partition such that each vertex is represented by a rectangle in the partition and each edge is represented by a common boundary segment shared by the corresponding rectangles. Then the rectangular partition is called a rectangular dual of the RDG. In this paper, we have found a minor error in a characterization for rectangular duals given by Koźmiński and Kinnen in 1985 without formal proof, and we fix this characterization with formal proof.

Keywords: Planar graph, Rectangularly dualizable graph, Rectangular partition, Rectangular dual.
2020 Mathematics Subject Classification: 68U05.

1 Introduction

A *rectangular partition* is a partition of a rectangle into a finite number of rectangles. A rectangular partition is *generic* if no four of its rectangles meet at the same point. A plane graph G is called a *rectangularly dualizable graph* (RDG) if G can be represented as a rectangular partition such that each vertex is represented by a rectangle in the partition, and each edge is represented by a common boundary segment shared by the corresponding rectangles. Then the rectangular partition is called a *rectangular dual* of the RDG.

In this paper, we consider simple connected planar graphs. A graph is *simple* if it is free of multiple edges (also known as parallel edges) as well as loops. A graph G is called *planar* if G can be drawn in the plane with a node-link diagram such that no two edges cross except at a common endpoint. A *plane graph* is a planar graph with a planar embedding specified. A planar drawing splits the Euclidean plane into connected regions called *faces*; the unbounded region is the *exterior face* (the outer face) and all other faces are interior faces. The vertices lying on the exterior face are exterior vertices and all other vertices are *interior vertices*. A vertex v_c of a graph is called a *cut-vertex* if the removal of v_c from the graph disconnects the graph. A graph G is said to be k-connected if G has at least k vertices and the removal of fever than k vertices does not disconnect the graph G. If a connected graph G has a cut-vertex, then G is called a *separable graph*; otherwise G is called a *non-separable graph*. We only consider connected floorplans. So, we consider non-separable (biconnected) and separable connected graphs in this paper. A maximal connected subgraph of a graph without a cut-vertex vertex is called a block. A *plane block* of a plane graph is a biconnected graph. A maximal block of a graph G is a maximal biconnected subgraph. A plane graph G is called a *plane triangulated graph* (PTG) if G has all faces, except may be the outer face, triangles. If the exterior face is also triangular, then the PTG is called a *plane triangulation.* If all faces except the exterior face of a plane graph G are triangular, then G is said to be an internally triangulated plane graph. A graph H is called *dual* of a plane graph G if there is one-to-one correspondence between the vertices of G and the regions of G, and two vertices of G are adjacent if and only if the corresponding faces of H have a common boundary segment. A vertex-weighted graph is a graph that carries a weight assigned to each of its vertices. In our case, the assigned weights are the areas of rectangles.

An extended graph $E(G)$ of a plane graph G can be obtained by inserting a cycle of length 4 at the exterior of $\mathcal{G}$ and then connecting the vertices of the cycle to the exterior vertices of $\mathcal{G}$ such that the resulting graph $E(G)$ is internally triangulated. These four vertices of the cycle are known as the poles of $E(G)$ and are labeled l, u, r and b in clockwise order. The vertices of the original graph G are its interior vertices.

Not every plane graph can be rectangularly dualized [13, 18, 20, 23]. The study of graphs that admit a rectangular dual was first studied by Koźmiński and Kinnen in 1985. Koźmiński and Kinnen [13] derived a necessary and sufficient condition for a PTG to be an RDG and showed that this theory can be implemented in quadratic time [13]. Thereafter in 1987, Bhasker and Sahni [3] reduced the complexity from quadratic to linear time.

In 1990, Lai and Leinward [20] showed that solving a rectangular partition problem of a planar graph is equivalent to a matching problem of a bipartite graph derived from the given plane graph. This theory relies on the assigned regions to vertices of a graph. This theory is not easy to implement. From the point of practicality, many constructive algorithms were given by various authors [4, 5, 7–9, 12, 14, 19] that are based on the characterizations of rectangular duals.

Floorplanning benefits greatly from an understanding of the theory of rectangularly dualizable graphs, especially when working on large-scale projects like VLSI circuit design. It provides us information at early stage to decide whether a graph modeling VLSI system, can be realized by a rectangular partition (floorplan).

A series of papers studies [15–17, 24] transformations among rectangular partitions with graph notion. Rectangular partitions have been studied without graph notion also. Various authors provided [1, 10, 21, 22, 25] constructive methods of rectangular partitions of a rectangles into n-rectangles that provide all rectangular partitions with all possible adjacencies among n-rectangles without using graph notion and hence these methods produce a large solution set. However, all rectangular partitions corresponding to an RDG preserve the adjacencies among rectangles. It is highly difficult and time consuming to locate an optimal solution in such a large solution set. These methods also focus on block-packing into the smallest possible rectangular region. The other important issues, like the length of the connecting line, are not keeping up.

A VLSI system structure is described by a graph where vertices correspond to component modules and edges correspond to required interconnections. For a given graph structure of a VLSI circuit, floorplanning is concerned with allocating space to component modules and their interconnections. Due to the advancement of VLSI technology,floorplans are now-a-days designed with rectilinear modules other than rectangular modules also.

There has been some work on floorplans using rectilinear modules (concave modules) in light of the renewed interest in floorplanning [2, 6, 11, 26, 27]. A concave rectilinear module has more design complexity than a rectangular module (convex) because it is composed of multiple rectangles. The quality of a floorplan may suffer if concave rectilinear modules are used in its construction.

2 Oversight of rectangular duals

In 1985, Koźmiński and Kinnen [13] studied rectangular partitions in terms of graphs. In fact, they derived necessary and sufficient conditions for a graph to be a *rectangularly dualizable graph*. To understand their results precisely, we first describe the following terms.

Definition 2.1 ([13])**.** A maximal block of a graph G is a biconnected subgraph of G which is not contained in any other block. The block neighborhood graph (BNG) of a plane graph G is a graph where vertices are represented by biconnected components of G such that there is an edge between two vertices if and only if the two biconnected components they represent, have a vertex in common.

Definition 2.2 ([13])**.** A shortcut in a plane block G is an edge that is incident to two vertices on the outermost cycle C of G and is not a part of C. A corner implying path (CIP) in G is a v_1—v_k path on the outermost cycle of G such that it does not contain any vertex of a shortcut other than v_1 and v_k. Then the shortcut (v_1, v_k) is called a critical shortcut. A critical CIP in a biconnected component H_k of a separable plane graph G is a CIP of H_k that does not contain any cut-vertex of G in its interior.

For a better understanding of Definition 2.2, consider the graph shown in Figure 1. Edges (v_1, v_4), (v_4, v_6) and (v_9, v_{10}) are shortcuts. Path $v_9v_5v_{10}$ is a CIP while the path $v_4v_9v_5v_{10}v_6$ is not a CIP since it contains the endpoints of another shortcut (v_9, v_{10}) and hence (v_4, v_6) is not a critical shortcut. Path $v_1v_2v_3v_4$ is not a critical CIP because the CIP has a cut-vertex v_2 of the graph in its interior. On the other hand, there is a CIP $v_9v_5v_{10}$ in the graph that is critical because it has one interior vertex v_5 only that is not a cut-vertex of the graph.

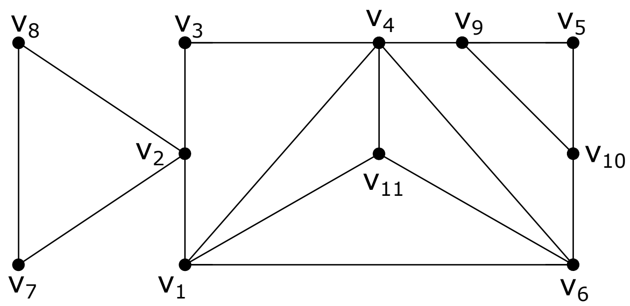

Figure 1. The presence of a CIP $v_9v_5v_{10}$, a non-critical CIP $v_1v_2v_3v_4$, and a critical CIP $v_9v_5v_{10}$.

Definition 2.3. A *separating cycle* is a cycle in a plane graph G that contains vertices in its interior and exterior. A separating cycle of length 3 is called a *separating triangle* or a complex triangle. For instance, in Figure 1, the cycle $v_1v_4v_6v_1$ is a separating triangle.

Definition 2.4. A plane graph G is a properly triangulated plane (PTP) graph if

1. G is internally triangulated,
2. G has no separating triangle.

To construct an extended PTP graph from a given internally plane triangulated graph G, we need to assign 4 labels v_{br}, v_{bl}, v_{ul}, v_{ur} to clockwise ordered at most four vertices of the outside face of G. If this assignment later results in an extended PTP graph, then a rectangular dual of G corresponding to this extended PTP graph is found. The labeled vertices correspond to the corner rectangles of the rectangular dual. Therefore, the labeled vertices are called corner vertices. These corner vertices divide the cycle bounding the outer face into less than four edge-disjoint paths called outer paths. In order to construct an extended PTP graph, connect each vertex on the path v_{br}—v_{bl} to b, each vertex on the path v_{bl}—v_{ul} to I, each vertex on the path v_{ul}—v_{ul}, to u and each vertex on the path v_{ur}—v_{br} to r.

Theorem 2.1 ([13, Theorem 2]). *An internally triangulated plane graph G admits a rectangular dual if and only if its extended graph $E(G)$ is a PTP graph.*

Theorem 2.2 ([13, Theorem 3]). *Suppose that G is a non-separable connected plane graph that is internally triangulated. Then G is an RDG if and only if it has at most 4 CIPs and has no separating triangle.*

Theorem 2.3 ([13, Theorem 5]). *Suppose that G is a separable connected plane graph that is internally triangulated. Then G is an RDG if and only if:*

1. *G has no separating triangle;*
2. *the BNG of G is a path;*
3. *each maximal block corresponding to the endpoints of the BNG contains at most 2 critical CIPs;*
4. *no other maximal blocks contain a critical CIP.*

Now, we present a counterexample for which this theorem fails. Consider the graph G shown in Figure 2.

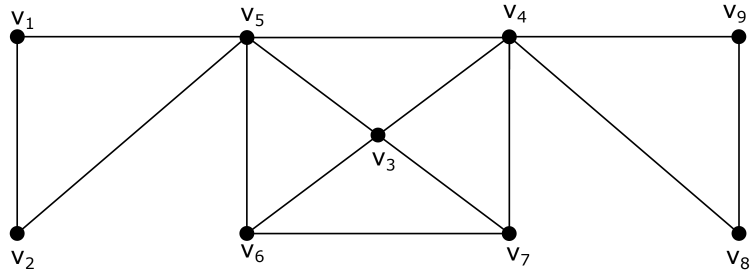

Figure 2. A counterexample that invalidates Theorem 2.3.

From the plane embedding in Figure 2, we found that there is no cycle of length 3 that contains vertices inside and outside, i.e., the graph contains no separating triangles. G has three blocks and hence the BNG of G is a path of three vertices. One of the maximal block corresponding to endpoints of the BNG of G has the CIP $v_2v_1v_5$ and the other maximal block has the CIP $v_4v_9v_8$ only. There is no CIP in the middle block of G. Thus, G satisfies all the conditions given in Theorem 2.3.

Now we show that G is not an RDG. With the aid of the algorithm in [4], one can find a rectangular partition for each block of an RDG and then, a rectangular partition for the RDG can be constructed by gluing them in a rectangular area. In our case, it is not possible to glue rectangular partitions for these three blocks of G in a rectangular area because of the existing adjacency between cut-vertices v_4 and v_5. In fact, a cut-vertex of an RDG is dualized to a through rectangle in the corresponding rectangular partition. But in Figure 2, the cut-vertices are adjacent. Therefore, it is not possible to maintain rectangular enclosure while keeping R_4 and R_5 as through rectangles.

Thus G satisfies all the conditions given in Theorem 2.3, but it is not an RDG. As far as the authors know, there is no theorem except Theorem 2.3 to check whether a separable connected planar graph admits a rectangular dual.

3 Fixing the characterization of rectangular duals

Motivated by the counterexample given in Section 2, we fix the minor oversight in a characterization of rectangular duals provided for separable connected plane graphs by Koźmiński and Kinnen [13, Theorem 5].

Theorem 3.1. *Let G be a separable connected plane graph that is internally triangulated. A necessary and sufficient condition for G to be an RDG is that:*

1. *G has no separating triangle;*
2. *BNG of G is a path;*
3. *both endpoints of an exterior cut-edge of G can not be cut-vertices unless the edge is a bridge;*

4. *each maximal block corresponding to the endpoints of the BNG contains at most* 2 *critical CIPs;*
5. *other maximal blocks contain no CIP.*

Proof. Necessary condition. Assume that G is an RDG that admits a rectangular dual F. We prove the first condition. If any of blocks of G has exactly two vertices, trivially the block has no separating triangle. If any block of G has more than two vertices, then the block is non-separable. Since G is an RDG, each of its blocks is an RDG. By Theorem 2.2, the block has no separating triangle and hence, G has no separating triangle.

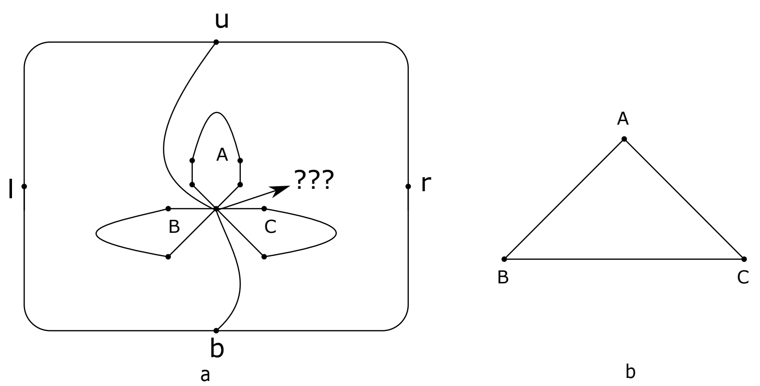

Figure 3. (a) A separable connected graph that is composed of three blocks A, B and C, and (b) its BNG. Here only the outermost cycles of the blocks are shown.

Now we prove the second condition. The BNG of G has the following possibilities:

1. it can be a cycle of length ≥ 3;
2. it can be a tree;
3. it can be a path.

By the definition of BNG, a cycle of length one or two is not possible. Suppose, if possible, that the BNG of G is a cycle of length ≥ 3. This implies that at least three blocks share some cut-vertex v_c of G. Therefore, the construction of any extended graph $E(G)$ creates at least one separating triangle passing through v_c, two adjacent exterior vertices of $E(G)$. This situation is depicted in Figure 3a. One of the endpoints of the arrow edge is incident to a cut-vertex and to augment the graph to an extended graph, the other endpoint of the arrow edge is made incident to exterior vertices, but every choice creates a separating triangle in the extended graph. This ceases the extended graph to be a PTP graph. This means we can not turn $E(G)$ into a PTP graph and hence, by Theorem 2.1, G is not an RDG. A similar argument can be applied in case when the BNG of G is a tree. This situation is depicted in Figure 4a. Thus, one last option for the BNG is a path.

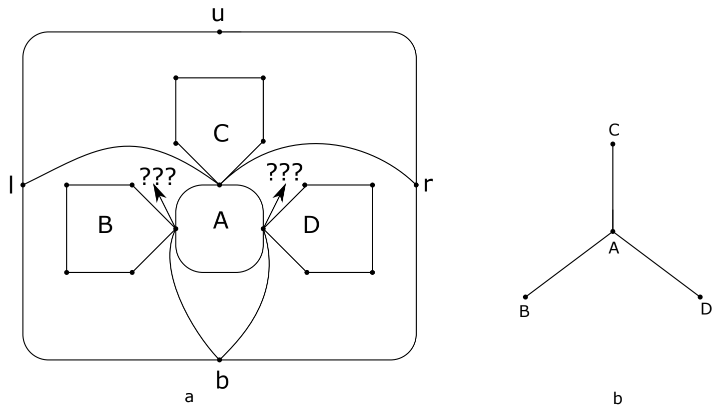

Figure 4. A separable connected graph that is composed of four blocks A, B, C and D, and (b) its BNG. Here only the outermost cycles of the blocks are shown.

In contrast to the third requirement, assume that both endpoints of an exterior cut-edge (v_i, v_j) of G that is not a bridge, are cut-vertices. Then any extended graph $E(G)$ has at least one separating triangles that cross v_i, v_j, and one of the exterior vertices of $E(G)$. This means it is impossible for extended graph $E(G)$ to be a PTP graph. This contradicts Theorem 2.1. Hence, both endpoints of an exterior cut-edge of G can not be cut-vertices unless the edge is a bridge. This situation is demonstrated in Figure 5. One of endpoints of each of arrow edges is incident to a cut-vertex and to augment the graph as an extended graph, the other endpoints of the arrow edges are made incident to exterior vertices, but every choice creates a separating triangle in the extended graph. This ceases the extended graph to be a PTP graph.

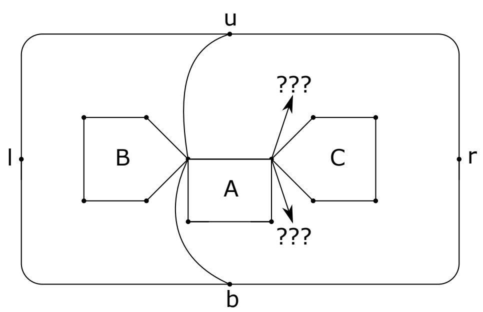

Figure 5. A separable connected graph that is composed of three blocks A, B and C. Here only the outer cycles of the blocks are shown.

Consider M_i to be a maximal block that associates with one of the endpoints of the BNG of G. Since G is an RDG, each of its blocks is an RDG. Suppose that M_i is an RDG that admits a rectangular dual F_i. It is evident that at most two of the corner rectangles of F_i can be corner rectangles of F. By the discussion of the construction of an extended PTP graph in Section 2, for a corner rectangle of F_i, there is a critical CIP in M_i. Therefore, M_i has at most two critical CIPs. It follows from this that the fourth requirement is met.

If any other maximal block of the BNG of G shares a CIP, then there are at most five critical CIPs because there are at most four critical CIPs corresponding to the maximal blocks corresponding to the endpoints of the BNG of G. Then the outer face of G divides into at most five edge-disjoint paths. As discussed in Section 2. we can divide the outer face of G into at most four edge-disjoint paths. Therefore, no other maximal block contains a CIP.

Sufficient condition. Assume that the given conditions hold. By Theorem 2.2, the construction of any extended graph $E(G)$ is a PTP graph and hence, by Theorem 2.1, G is an RDG. This concludes the proof. $\square$

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