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On certain arithmetical products involving the divisors of an integer

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Abstract: We study the arithmetical products $\prod d^d$, $\prod d^{\frac{1}{d}}$ and $\prod d^{\log d}$, where d runs through the divisors of an integer $n > 1$.

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1 Introduction

Let $n > 1$ be a positive integer. Define the arithmetical products

$$P_1(n) = \prod_{d|n} d^d, P_2(n) = \prod_{d|n} d^{1/d}, P_3(n) = \prod_{d|n} d^{\log d} \tag{1}$$

where $d \mid n$ means that d runs through all distinct and positive divisors of $n > 1$.

The aim of this paper is to study some properties of these arithmetical products. In what follows, we will use also the notation

$$\sigma_k(n) = \sum_{d|n} d^k, \tag{2}$$

which denotes the sum of k-th powers of divisors of n.

Particularly, $\sigma_1(n) = \sigma(n)$ equals the sum of divisors of n, $\sigma_0(n) = d(n)$ equals the number of divisors of n. We will need also $\sigma_2(n)$ to equal the sum of squares of divisors of n, or $\sigma_{3/2}(n)$, the value of $\sigma_k(n)$ for $k = \frac{3}{2}$.

We will need also some algebraic and analytic inequalities, which will be mentioned in the contexts of the proofs.

2 Main results

Theorem 1. *One has the identity*

$$P_1(n) = \frac{n^{\sigma(n)}}{(P_2(n))^n}. \tag{3}$$

Proof. Let $d_1, d_2, \ldots, d_r$ be the distinct divisors of n. Remark that $\frac{n}{d_1}, \frac{n}{d_2}, \ldots, \frac{n}{d_r}$ are also the divisors of n, and $n^{\frac{n}{d_1}} n^{\frac{n}{d_2}} \cdots n^{\frac{n}{d_r}} = n^{\sigma(n)}$. On the other hand,

$$\begin{aligned} P_2(n)^n / n^{\sigma(n)} &= \left(\frac{d_1}{n}\right)^{\frac{n}{d_1}} \cdots \left(\frac{d_1}{n}\right)^{\frac{n}{d_r}} = \left(\frac{1}{\frac{n}{d_1}}\right)^{\frac{n}{d_1}} \cdots \left(\frac{1}{\frac{n}{d_1}}\right)^{\frac{n}{d_r}} \\ &= \prod \left(\frac{1}{\frac{n}{d}}\right)^{\frac{n}{d}} = \prod \left(\frac{1}{d}\right)^{d} = \frac{1}{\prod d^d}, \end{aligned}$$

so the equality (3) follows. $\square$

Theorem 2. *One has the inequality*

$$n^n \leq P_1(n) \leq n^{\sigma(n)-1}, \tag{4}$$

for $n > 1$, with equality in both sides only for n being prime.

Proof. The left side of (4) follows from the definition of $P_1(n)$, as $\prod d^d \geq n^n$ with equality only if $n > 1$ has only one divisor, i.e., if n is prime. The right side of (4) follows by identity (3), as $P_2(n) \geq n^{\frac{1}{n}}$, with equality if and only if $n > 1$ is prime. $\square$

A refinement of left side of (4) is given in the following remark.

Remark 1. *The left side of* (4) *can be improved as follows:*

$$n^{\frac{\sigma(n)}{2}} \leq P_1(n). \tag{5}$$

Indeed, apply the Chebyshev sum inequality (see [3])

$$r \sum_{i=1}^{r} a_i b_i \geq \left(\sum_{i=1}^{r} a_i\right)\left(\sum_{i=1}^{r} b_i\right) \tag{6}$$

(where $(a_i), (b_i)$ have the same type of monotonicity) for the sequences $a_i = d_i$, $b_i = \log d_i$, and $d_1 < \cdots < d_r$ are the divisors of n.

Now, it is well known that

$$\prod_{d|n} d = n^{\frac{d(n)}{2}} \tag{7}$$

so we get from (6) that

$$d(n)\sum_{d|n} d\log d \geq \left(\sum_{d|n} d\right)\left(\sum_{d|n}\log d\right) = \sigma(n)\cdot\frac{d(n)\log n}{2}.$$

This gives $\sum\limits_{d|n}\log d \geq \frac{\sigma(n)\log n}{2}$, so (5) follows.

However, an even stronger relation will be contained in the left side of the following theorem.

Theorem 3.

$$\left(\frac{\sigma(n)}{d(n)}\right)^{\sigma(n)} \leq P_1(n) \leq \left(\frac{\sigma_{2(n)}}{\sigma(n)}\right)^{\sigma(n)}. \tag{8}$$

Proof. The function $f(x) = x\log x$ $(x > 0)$ is strictly convex, as $f''(x) = \frac{1}{x} > 0$, so by Jensen's inequality we can write

$$f\left(\frac{x_1 + \cdots + x_r}{r}\right) \leq \frac{f(x_1) + \cdots + f(x_r)}{r}, \tag{9}$$

for any $x_i > 0$. Applying (9) for $x_i = d_i$ (divisors of n) and remarking that $\frac{d_1 + \cdots + d_r}{r} = \frac{\sigma(n)}{d(n)}$, the left side of (8) follows.

Apply now the weighted arithmetic mean-geometric mean inequality

$$p_1x_1 + \cdots + p_rx_r \geq x_1^{p_1}\cdots\, x_r^{p_i}, \tag{10}$$

where $p_i > 0$, $\sum\limits_{i=1}^{r} p_i = 1$ and $x_i > 0(i = \overline{1,r})$ for $p_i = \frac{d_i}{\sigma(n)}$, $x_i = d_i$. As $\frac{d_1}{\sigma(n)} + \cdots + \frac{d_r}{\sigma(n)} = 1$, we can apply (10), so the right side of (8) follows. □

Remark 2. *By the known inequality* $\frac{\sigma(n)}{d(n)} \geq \sqrt{n}$ (*see e.g.* [4]) *we get that the left side of* (8) *is stronger than inequality* (5).

Remark 3. *Applying* (10) *to* $x_i = \sqrt{d_i}$, $p_i = \frac{d_i}{\sigma(n)}$ *after some computations, we get*

$$P_1(n) \leq \left(\frac{\sigma_{\frac{3}{2}(n)}}{\sigma(n)}\right)^{2\sigma(n)}. \tag{11}$$

This is slightly stronger than the right side of (8)*, which follows from the inequality*

$$\sigma_{\frac{3}{2}}^2(n) \leq \sigma(n)\cdot\sigma_2(n), \tag{12}$$

i.e., $\left(d_1^{\frac{3}{2}} + \cdots + d_r^{\frac{3}{2}}\right)^2 \leq (d_1 + \cdots + d_r)\cdot(d_1^2 + \cdots + d_r^2)$. *This follows by the Cauchy–Schwarz inequality, as*

$$\left(\sum_{i=1}^{r} d_i^{\frac{1}{2}} d_i\right)^2 \leq \left(\sum_{i=1}^{r} d_i\right)\left(\sum_{i=1}^{r} d_i^2\right),$$

for application of $\left(\sum a_i.b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$ *to* $a_i = d_i^{\frac{1}{2}}$, $b_i = d_i$.

Theorem 4. *One has*

$$\frac{A(n)}{n} \le \frac{\sigma_2(n)}{\sigma(n)} - P_1(n)^{\frac{1}{\sigma(n)}} \le A(n), \tag{13}$$

where

$$A(n) = \frac{1}{2}\left[\frac{\sigma_3(n)}{\sigma(n)} - \frac{\sigma_2^2(n)}{\sigma^2(n)}\right] \ge 0.$$

Proof. Apply the Cartwright–Field inequality (see [1]):

$$\frac{1}{2M}\sum_{i=1}^{r} p_i\left(x_i - \sum_{i=1}^{r} p_k x_k\right)^2 \le \sum_{i=1}^{r} p_i x_i - \prod_{i=1}^{r} x_i^{p_i} \le \frac{1}{2m}\sum_{i=1}^{r} p_i\left(x_i - \sum_{i=1}^{r} p_k x_k\right), \tag{14}$$

where p_i, x_i are as in (10), but $0 < m \le x_i \le M\cdot$

Let $p_i = \frac{d_i}{\sigma(n)}$, $x_i = d_i$, $r = d(n)$. Then clearly $m = 1$ and $M = n$. Remark that one has

$$\sum_{d|n} d \cdot \left(d - \frac{\sigma_2(n)}{\sigma(n)}\right)^2 = \sum_{d|n}\left(d^3 - 2d^2 \cdot \frac{\sigma_2(n)}{\sigma(n)} + d \cdot \frac{\sigma_2^2(n)}{\sigma^2(n)}\right)$$
$$= \sigma_3(n) - 2\frac{\sigma_2^2(n)}{\sigma(n)} + \frac{\sigma_2^2(n)}{\sigma(n)} = \sigma_3(n) - \frac{\sigma_2^2(n)}{\sigma(n)} \ge 0,$$

and the inequalities (13) follow. □

Theorem 5. *One has*

$$n^{\log n} \le P_3(n) \le \frac{1}{t_n}(\log P_1(n))^{t_n}, \tag{15}$$

where $t_n = \frac{d(n)\cdot \log n}{2}$.

Proof. The left side of (15) is obvious from the definition of $P_3(n)$, with equality only if $n > 1$ is prime. For the right side of (15), apply (10) for $x_i = d_i$, $p_i = \frac{\log d_i}{t_n}$. One has indeed $\sum_{d|n} \log d = \log \prod_{d|n} d = \frac{d(n)\log n}{2}$, by (7), so $\frac{\sum \log d_i}{t_n} = 1$. □

By other methods, we can deduce another result on $P_3(n)$, namely Theorem 6.

Theorem 6.

$$\frac{1}{4}d(n)\log^2(n) < \log P_3(n) \le \frac{1}{2}d(n)\log^2 n \tag{16}$$

for $n > 1$.

Proof. Remark that $\log P_3(n) = \sum_{\frac{d}{n}} \log^2 d$. Now, apply the classical inequality

$$\frac{x_1^2 + \cdots + x_r^2}{r} \ge \left(\frac{x_1 + \cdots + x_r}{r}\right)^2 \tag{17}$$

to $x_i = \log d_i$. As $\sum_{i=1}^{r} \log d_i = \log(\prod_{i=1}^{r} d_i) = \frac{(\log n)d(n)}{2}$, we get from (17) the left side of (16). As there is equality in (17) only for $r = 1$, for $n > 1$ there is strict inequality in the left side of (16). For the right side we can remark that

$$\sum_{d|n} \log^2 d = \sum_{d|n} \log d \cdot \log d \le (\log n) \cdot \sum_{d|n} \log d = (\log n)\left(\frac{1}{2}d(n)\log n\right),$$

so the inequality follows. □

As for any prime p one has $\log P_3(p) = \frac{1}{2} d(p) \log^2 p$, we can state the following:

Remark 4.

$$\limsup_{n\to\infty} \frac{\log P_3(n)}{d(n)\log^2 n} = \frac{1}{2}. \tag{18}$$

Theorem 7. *One has*

$$\liminf_{n\to\infty} \frac{\sigma(n)\log n - \log P_1(n)}{n(\log\log n)^2} = 0 \tag{19}$$

and

$$\limsup_{n\to\infty} \frac{\sigma(n)\log n - \log P_1(n)}{n(\log\log n)^2} = e^{\gamma}, \tag{20}$$

where e *and* γ *are the classical Euler constants.*

Proof. As $\log P_1(p) = p\log p$ for a prime p, and $\sigma(p) = p + 1$ by,

$$\frac{(p+1)\log p - p\log p}{p(\log\log p)^2} = \frac{\log p}{p(\log\log p)^2} \to 0,$$

as $p \to \infty$ relation (19) follows.

For the proof of (20), remark that $\log P_1(n) = \sigma(n)\log n - n\log P_2(n)$, thus

$$\sigma(n)\log n - \log P_1(n) = n\log P_2(n) = n\sum_{d|n} \frac{\log d}{d}.$$

Now, a result of Erdős and Zaremba [2] states that

$$\limsup_{n\to\infty} \frac{F(n)}{(\log\log n)^2} = \epsilon^{\gamma}, \tag{21}$$

where $F(n) = \sum_{d|n} \frac{\log d}{d}$. By using the above, from (21) we deduce relation (20). $\square$

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