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Some new properties of hyperbolic k-Fibonacci and k-Lucas octonions

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Abstract: The aim of this paper is to establish some novel identities for hyperbolic k-Fibonacci octonions and k-Lucas octonions. We prove these properties using the identities of k-Fibonacci and k-Lucas numbers, which we determined previously.
Keywords: Fibonacci number, Lucas number, k-Fibonacci number, k-Lucas number.
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1 Introduction

Hamilton invented quaternions in 1843 and showed that they form a 4-dimensional non-commutative division ring under multiplication [13,14]. Quaternions can be used to show rotations in three-dimensional space, and have applications in computer graphics, robotics, and aerospace engineering. They are also closely related to the more general Clifford algebras. Quaternions are also used in various branches of mathematics, such as differential geometry and number theory. Quaternion calculus is a dominant tool for solving problems related to 3-dimensional rotations. Quaternions have also been used in artificial intelligence and image processing applications. They are also used in robotics for controlling the movement of robotic arms.

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Definition 1.1. (Horadam [15]) *A quaternion ρ is an element of the form $\rho = \rho_0+\rho_1 i+\rho_2 j+\rho_3 k$, where ρ_0, ρ_1, ρ_2, ρ_3 are real components and 1, i, j, k are basis elements satisfying the properties $i^2 = j^2 = k^2 = ijk = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$.*

In 1963, Horadam [15] introduced the Fibonacci and Lucas quaternions and investigated some of their properties. In recent years, these quaternions were studied by many authors.

The concept of k-Fibonacci and k-Lucas quaternions was introduced and explored in 2015 by Ramirez [21]. These quaternions are a generalization of the Fibonacci and Lucas quaternions. They have applications across a wide range of branches like the usual quaternions. Numerous researchers have extensively studied these quaternions since their introduction. See [19, 20] for more details.

There has been a lot of research done on the different quaternions in recent years and their generalizations have been examined by several authors. The hyperbolic quaternions were discovered by Macfarlane [17] in 1900. Macfarlane's work was further expanded to the Cayley–Dickson algebras by Hurwitz. These algebras are now expressed as Hurwitz algebras.

Definition 1.2. (Macfarlane [17]) *The hyperbolic quaternion $\eth$ is an element of the form $\eth = \eth_1 + \eth_2\epsilon_1 + \eth_3\epsilon_2 + \eth_4\epsilon_3 = \big(\eth_1, \eth_2, \eth_3, \eth_4\big)$, with real components $\eth_1$, $\eth_2$, $\eth_3$, h_4 and $1, \epsilon_1, \epsilon_2, \epsilon_3$ are hyperbolic quaternion units that satisfy the non-commutative multiplication rules*

$$\epsilon_1{}^2 = \epsilon_2{}^2 = \epsilon_3{}^2 = \epsilon_1\epsilon_2\epsilon_3 = 1, \tag{1}$$

$$\epsilon_1\epsilon_2 = \epsilon_3 = -\epsilon_2\epsilon_1, \quad \epsilon_2\epsilon_3 = \epsilon_1 = -\epsilon_3\epsilon_2, \quad \epsilon_3\epsilon_1 = \epsilon_2 = -\epsilon_1\epsilon_3. \tag{2}$$

The hyperbolic quaternion is neither a commutative nor an associative algebraic structure. In the classical quaternion, every imaginary basis element has the property $e_n^2 = -1$, while in the hyperbolic quaternion all basis elements satisfy $\epsilon_m^2 = +1$. For more details, see [17]. Hyperbolic k-Fibonacci quaternion and hyperbolic k-Lucas quaternion first defined by Godase [9, 10, 12].

Definition 1.3. (Godase [9, 10, 12]) *The hyperbolic k-Fibonacci quaternion $\eth^{\phi}{}_{k,n}$ is an element of the form $\eth^{\phi}{}_{k,n} = \phi_{k,n} + \phi_{k,n+1}\epsilon_1 + \phi_{k,n+2}\epsilon_2 + \phi_{k,n+3}\epsilon_3$, and the hyperbolic k-Lucas quaternion $\eth^{\psi}{}_{k,n}$ is an element of the form $\eth^{\psi}{}_{k,n} = \psi_{k,n} + \psi_{k,n+1}\epsilon_1 + \psi_{k,n+2}\epsilon_2 + \psi_{k,n+3}\epsilon_3$. The hyperbolic quaternion units 1, ϵ_1, ϵ_2 and ϵ_3 satisfy the multiplication rules defined in the Definition 1.2 and $\phi_{k,n}$ & $\psi_{k,n}$ are k-Fibonacci and k-Lucas numbers.*

The octonions were formed by A. Cayley [5] in 1845. Octonions are also called Cayley numbers. In 2014, O. Kecilioglu and I. Akkus [16] defined Fibonacci and Lucas octonions and explored some basic properties of octonions. In 1988, K. Carmody [4] discovered hyperbolic octonions and inspected the different properties of hyperbolic octonions. Furthermore, he demonstrated that the process of multiplying hyperbolic octonions does not follow a commutative or associative order.

Definition 1.4. (Carmody [4]) *The hyperbolic octonion ϱ is an element of the form $\varrho = \varrho_0 + \varrho_1 i_1 + \varrho_2 i_2 + \varrho_3 i_3 + \varrho_4\epsilon_4 + \varrho_5\epsilon_5 + \varrho_6\epsilon_6 + \varrho_7\epsilon_7$, where ϱ_0, ϱ_1, ϱ_2, ϱ_3, ϱ_4, ϱ_5, ϱ_6, ϱ_7 are real components and 1, i_1, i_2, i_3 are quaternion units. Besides, $\epsilon_4(\epsilon_4{}^2 = 1)$ is a counter imaginary unit and the bases ϵ_5, ϵ_6 and ϵ_7 of hyperbolic octonion are defined as follows:*

$$i_1\epsilon_4 = \epsilon_5, \quad i_2\epsilon_4 = \epsilon_6, \quad i_3\epsilon_4 = \epsilon_7,$$
$$\epsilon_4{}^2 = \epsilon_5{}^2 = \epsilon_6{}^2 = \epsilon_7{}^2 = 1.$$

The bases of the hyperbolic octonion ϱ satisfy the multiplication rule given in the Table 1 (Carmody [4]) below. The algebraic properties of the hyperbolic octonion can be constructed using the multiplication rule. A. Cariow, G. Cariowa and J. Knapinski [2, 3] established the low multiplicative complexity algorithm for multiplying two hyperbolic octonions.

Table 1. Multiplication rule for hyperbolic octonions.

$\cdot$	i_1	i_2	i_3	ϵ_4	ϵ_5	ϵ_6	ϵ_7
i_1	-1	i_3	$-i_2$	ϵ_5	$-\epsilon_4$	$-\epsilon_7$	ϵ_6
i_2	$-i_3$	-1	i_1	ϵ_6	ϵ_7	$-\epsilon_4$	$-\epsilon_5$
i_3	i_2	$-i_1$	-1	ϵ_7	$-\epsilon_6$	ϵ_5	$-\epsilon_4$
ϵ_4	$-\epsilon_5$	$-\epsilon_6$	$-\epsilon_7$	1	$-i_1$	$-i_2$	$-i_3$
ϵ_5	ϵ_4	$-\epsilon_7$	ϵ_6	i_1	1	i_3	$-i_2$
ϵ_6	ϵ_7	ϵ_4	$-\epsilon_5$	i_2	$-i_3$	1	i_1
ϵ_7	$-\epsilon_6$	ϵ_5	ϵ_4	i_3	i_2	$-i_1$	1

The hyperbolic octonion forms an 8-dimensional non-commutative algebraic structure. Hyperbolic octonions are distinct from classical octonions. In classical octonions, all imaginary basis elements have the property $e_n^2 = -1$; while in hyperbolic octonions, hyperbolic basis elements satisfy the property $\epsilon_m^2 = +1$. As classical octonions, hyperbolic octonions are also non-associative, and they further satisfy the property $(\epsilon_c e_d)\epsilon_p = -\epsilon_c(e_d\epsilon_p)$, for $c \neq d$, $d \neq p$ and $p \neq c$. Details can be found in [6–8, 18, 22, 23].

In 2019, Godase A. D. [9, 10] expanded the ideas of hyperbolic k-Fibonacci octonion and hyperbolic k-Lucas octonion from the ideas of hyperbolic k-Fibonacci quaternion and hyperbolic k-Lucas quaternion.

Definition 1.5. (Godase [9, 10]) *The hyperbolic k-Fibonacci and k-Lucas octonions ${\varrho^{\phi}}_{k,n}$ and ${\varrho^{\psi}}_{k,n}$ are defined by the expressions ${\varrho^{\phi}}_{k,n} = \phi_{k,n} + \phi_{k,n+1}i_1 + \phi_{k,n+2}i_2 + \phi_{k,n+3}i_3 + \phi_{k,n+4}\epsilon_4 + \phi_{k,n+5}\epsilon_5 + \phi_{k,n+6}\epsilon_6 + \phi_{k,n+7}\epsilon_7$, and ${\varrho^{\psi}}_{k,n} = \psi_{k,n} + \psi_{k,n+1}i_1 + \psi_{k,n+2}i_2 + \psi_{k,n+3}i_3 + \psi_{k,n+4}\epsilon_4 + \psi_{k,n+5}\epsilon_5 + \psi_{k,n+6}\epsilon_6 + \psi_{k,n+7}\epsilon_7$, where i_1, i_2, i_3, ϵ_4, ϵ_5, ϵ_6, ϵ_7 are hyperbolic octonion units given as in the Definition 1.4.*

2 Properties of hyperbolic k-Fibonacci and k-Lucas octonions

Lemmas 2.1 to 2.4 are essential for proving the main theorems in this section, namely Theorems 2.1 to 2.7. The proofs of the Lemmas 2.1 to 2.4 can be found in [11]. In this section, different properties containing hyperbolic k-Fibonacci and k-Lucas octonions are established. Furthermore, we explore binomial sums for these octonions.

Lemma 2.1. *In the case where $u = \mu_1$ or $u = \mu_2$ are the roots of the characteristic equation $r^2 - kr - 1 = 0$, the following results hold:*

$$(a) \quad u^2 = ku + 1, \tag{3}$$

$$(b) \quad u^n = u\phi_{k,n} + \phi_{k,n-1}, \tag{4}$$

$$(c) \quad u^{2n} = u^n \psi_{k,n} - (-1)^n, \tag{5}$$

$$(d) \quad u^{tn} = u^n \frac{\phi_{k,tn}}{\phi_{k,n}} - (-1)^n \frac{\phi_{k,(t-1)n}}{\phi_{k,n}}, \tag{6}$$

$$(e) \quad u^{sn}\phi_{k,rn} - u^{rn}\phi_{k,sn} = (-1)^{sn}\phi_{k,(r-s)n}. \tag{7}$$

Proof. (b) In order to prove this part, we apply the principle of mathematical induction to n. For $n = 2$, we have from (3)

$$\mu_1^2 = \mu_1\phi_{k,2} + \phi_{k,1},$$
$$\mu_2^2 = \mu_2\phi_{k,2} + \phi_{k,1}.$$

The result is assumed to be true for n. As a result, we have

$$\mu_1^n = \mu_1\phi_{k,n} + \phi_{k,n-1}, \tag{8}$$

$$\mu_2^n = \mu_2\phi_{k,n} + \phi_{k,n-1}. \tag{9}$$

Using (3) and (8), we obtain

$$\begin{aligned}
\mu_1^{n+1} &= \mu_1 {\mu_1}^n \\
&= \mu_1\big(\mu_1\phi_{k,n} + \phi_{k,n-1}\big) \\
&= {\mu_1}^2\phi_{k,n} + \mu_1\phi_{k,n-1} \\
&= (k\mu_1 + 1)\phi_{k,n} + \mu_1\phi_{k,n-1} \\
&= (k\phi_{k,n} + \phi_{k,n-1})\mu_1 + \phi_{k,n} \\
&= \phi_{k,n+1}\mu_1 + \phi_{k,n}.
\end{aligned}$$

In a similar manner, we can prove that

$$\mu_2^{n+1} = \mu_2\phi_{k,n+1} + \phi_{k,n}.$$ □

Lemma 2.2. *If $u = \mu_1$ or μ_2, then*

$$1 + ku + u^{2(2^{n+1}+1)} = \psi_{k,2^{n+1}} u^{2(2^n+1)}. \tag{10}$$

Proof. Let $u = \mu_1$. As a result of using the Binet formula, we have

$$\begin{aligned}
\psi_{k,2^{n+1}}\mu_1^{2(2^n+1)} &= \left(\mu_1^{2^{n+1}} + \mu_2^{2^{n+1}}\right)\mu_1^{2(2^n+1)} \\
&= \left(\mu_1^{2^{n+1}}\mu_1^{2^{n+1}}\right)\mu_1^2 + \left(\mu_2^{2^{n+1}}\mu_1^{2^{n+1}}\right)\mu_1^2
\end{aligned}$$

$$
\begin{aligned}
&=\mu_1^2\left(\mu_1^{2^{n+1}+2^{n+1}}+(-1)^{2^{n+1}}\right)\\
&=\mu_1^2\left(\mu_1^{2^{n+2}}+1\right)\\
&=\left(\mu_1^2+\mu_1^2\mu_1^{2^{n+2}}\right)\\
&=\left(\mu_1^2+\mu_1^{2+2^{n+2}}\right)\\
&=1+k\mu_1+\mu_1^{2(2^{n+1}+1)}.
\end{aligned}
$$

It is also possible to prove the result for $u=\mu_2$. □

Lemma 2.3. *If $t\in\mathbb{Z}^+$ with $t\geq 1$, the following results hold*

$$
\begin{aligned}
&(1)\quad \mu_1^{2t}=\frac{\phi_{k,2t}}{k}\mu_1\sqrt{\Delta}-\frac{\psi_{k,2t-1}}{k}, &(11)\\
&(2)\quad \mu_2^{2t}=-\frac{\phi_{k,2t}}{k}\mu_2\sqrt{\Delta}-\frac{\psi_{k,2t-1}}{k}. &(12)
\end{aligned}
$$

Lemma 2.4. *The following results hold for $t\in\mathbb{Z}^+$ with $t\geq 1$*

$$
\begin{aligned}
&(1)\quad \mu_1^{2t+1}=\frac{\psi_{k,2t+1}}{k}\mu_1-\frac{\phi_{k,2t}}{k}\sqrt{\Delta}, &(13)\\
&(2)\quad \mu_2^{2t+1}=\frac{\psi_{k,2t+1}}{k}\mu_2+\frac{\phi_{k,2t}}{k}\sqrt{\Delta}. &(14)
\end{aligned}
$$

Theorem 2.1. *For $n,m,r,s,t\in\mathbb{Z}^+$, we have*

$$\phi_{k,rn}{\varrho^{\phi}}_{k,sn+mt}=\phi_{k,sn}{\varrho^{\phi}}_{k,rn+mt}+(-1)^{sn}\phi_{k,(r-s)n}{\varrho^{\phi}}_{k,mt}. \tag{15}$$

Proof. From the Lemma 2.1(e), we can write

$$
\begin{aligned}
\mu_1^{sn}\phi_{k,rn}-\mu_1^{rn}\phi_{k,sn}&=(-1)^{sn}\phi_{k,(r-s)n}, &(16)\\
\mu_2^{sn}\phi_{k,rn}-\mu_2^{rn}\phi_{k,sn}&=(-1)^{sn}\phi_{k,(r-s)n}. &(17)
\end{aligned}
$$

By multiplying Equation (16) by $\dfrac{\hat{\mu_1}\mu_1^{mt}}{\mu_1-\mu_2}$ and (17) by $\dfrac{\hat{\mu_2}\mu_2^{mt}}{\mu_1-\mu_2}$ and subtracting, we obtain

$$
\begin{aligned}
\phi_{k,rn}\left(\frac{\hat{\mu_1}\mu_1^{sn+mt}-\hat{\mu_2}\mu_2^{sn+mt}}{\mu_1-\mu_2}\right)=\phi_{k,sn}&\left(\frac{\hat{\mu_1}\mu_1^{rn+mt}-\hat{\mu_2}\mu_2^{rn+mt}}{\mu_1-\mu_2}\right)\\
&+(-1)^{sn}\phi_{k,(r-s)n}\left(\frac{\hat{\mu_1}\mu_1^{mt}-\hat{\mu_2}\mu_2^{mt}}{\mu_1-\mu_2}\right),
\end{aligned}
$$

i.e.,

$$\phi_{k,rn}{\varrho^{\phi}}_{k,sn+mt}=\phi_{k,sn}{\varrho^{\phi}}_{k,rn+mt}+(-1)^{sn}\phi_{k,(r-s)n}{\varrho^{\phi}}_{k,mt}.$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $n,m,t\in\mathbb{Z}^+$. Then prove that*

$$\psi_{k,2^{n+1}}{\varrho^{\phi}}_{k,mt+2^{n+1}+2}={\varrho^{\phi}}_{k,mt+2}+{\varrho^{\phi}}_{k,mt+2^{n+2}+2}. \tag{18}$$

Proof. Applying the Lemma 2.2, we write

$$\psi_{k,2^{n+1}}\mu_1^{(2^{n+1}+2)} = \mu_1^2 + \mu_1^{(2^{n+2}+2)}, \tag{19}$$

$$\psi_{k,2^{n+1}}\mu_2^{(2^{n+1}+2)} = \mu_2^2 + \mu_2^{(2^{n+2}+2)}. \tag{20}$$

Now, by multiplying Equation (19) by $\dfrac{\hat{\mu}_1\mu_1^{mt}}{\mu_1-\mu_2}$ and Equation (20) by $\dfrac{\hat{\mu}_2\mu_2^{mt}}{\mu_1-\mu_2}$ and subtracting, we obtain

$$\psi_{k,2^{n+1}}\left(\frac{\hat{\mu}_1\mu_1^{mt+2^{n+1}+2}-\hat{\mu}_2\mu_2^{mt+2^{n+1}+2}}{\mu_1-\mu_2}\right)$$
$$=\left(\frac{\hat{\mu}_1\mu_1^{mt+2}-\hat{\mu}_2\mu_2^{mt+2}}{\mu_1-\mu_2}\right)+\left(\frac{\hat{\mu}_1\mu_1^{mt+2^{n+2}+2}-\hat{\mu}_2\mu_2^{mt+2^{n+2}+2}}{\mu_1-\mu_2}\right),$$

which implies

$$\psi_{k,2^{n+1}}{\varrho^{\phi}}_{k,mt+2^{n+1}+2} = {\varrho^{\phi}}_{k,mt+2} + {\varrho^{\phi}}_{k,mt+2^{n+2}+2}.$$

Thus the proof of Theorem 2.2. □

Theorem 2.3. *Let $n,m,r,s,t \in \mathbb{Z}^+$. Then show that*

$$(i) \quad k{\varrho^{\phi}}_{k,mr+2t} = \phi_{k,2t}{\varrho^{\psi}}_{k,mr+1} - \psi_{k,2t-1}{\varrho^{\phi}}_{k,mr}, \tag{21}$$

$$(ii) \quad k{\varrho^{\psi}}_{k,mr+2t} = \Delta\phi_{k,2t}{\varrho^{\phi}}_{k,mr+1} - \psi_{k,2t-1}{\varrho^{\psi}}_{k,mr}. \tag{22}$$

Proof. (i) By making use of the Lemma 2.3, we have

$$k\mu_1^{2t} = \phi_{k,2t}\mu_1\sqrt{\Delta} - \psi_{k,2t-1}, \tag{23}$$

$$k\mu_2^{2t} = -\phi_{k,2t}\mu_2\sqrt{\Delta} - \psi_{k,2t-1}. \tag{24}$$

By multiplying Equation (23) by $\dfrac{\hat{\mu}_1\mu_1^{mr}}{\mu_1-\mu_2}$ and Equation (24) by $\dfrac{\hat{\mu}_2\mu_2^{mr}}{\mu_1-\mu_2}$ and subtracting, we obtain

$$k{\varrho^{\phi}}_{k,mr+2t} = \phi_{k,2t}{\varrho^{\psi}}_{k,mr+1} - \psi_{k,2t-1}{\varrho^{\phi}}_{k,mr}.$$

(ii) Again, multiplying the Equation (23) by $\hat{\mu}_1\mu_1^{mr}$ and (24) by $\hat{\mu}_2\mu_2^{mr}$ and adding, we get

$$k{\varrho^{\psi}}_{k,mr+2t} = \Delta\phi_{k,2t}{\varrho^{\phi}}_{k,mr+1} - \psi_{k,2t-1}{\varrho^{\psi}}_{k,mr},$$

which completes the proof of Theorem 2.3. □

Theorem 2.4. *Let $n,m,r,s,t \in \mathbb{Z}^+$. Then show that*

$$(i) \quad k{\varrho^{\phi}}_{k,mr+2t+1} = \psi_{k,2t+1}{\varrho^{\phi}}_{k,mr+1} - \phi_{k,2t}{\varrho^{\psi}}_{k,mr}, \tag{25}$$

$$(ii) \quad k{\varrho^{\psi}}_{k,mr+2t+1} = \psi_{k,2t+1}{\varrho^{\psi}}_{k,mr+1} - \Delta\phi_{k,2t}k{\varrho^{\phi}}_{k,mr}. \tag{26}$$

Proof. From the Lemma 2.4, we have

$$k\mu_1^{2t+1} = \psi_{k,2t+1}\mu_1 - \sqrt{\Delta}\phi_{k,2t}, \tag{27}$$

$$k\mu_2^{2t+1} = \psi_{k,2t+1}\mu_2 + \sqrt{\Delta}\phi_{k,2t}. \tag{28}$$

By multiplying Equation (27) by $\dfrac{\hat{\mu_1}\mu_1^{mr}}{\mu_1-\mu_2}$ and Equation (28) by $\dfrac{\hat{\mu_2}\mu_2^{mr}}{\mu_1-\mu_2}$ and subtracting, we obtain

$$k{\varrho^{\phi}}_{k,mr+2t+1} = \psi_{k,2t+1}{\varrho^{\phi}}_{k,mr+1} - \phi_{k,2t}{\varrho^{\psi}}_{k,mr}.$$

(ii) Furthermore, multiplying the equation (27) by $\hat{\mu_1}\mu_1^{mr}$ and (28) by $\hat{\mu_2}\mu_2^{mr}$ and adding, we get

$$k{\varrho^{\psi}}_{k,mr+2t+1} = \psi_{k,2t+1}{\varrho^{\psi}}_{k,mr+1} - \Delta\phi_{k,2t}{\varrho^{\phi}}_{k,mr}.$$

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let $n,m,r,s,t,l \in \mathbb{Z}^+$. Then prove that*

$$(-1)^{smn}\phi^n_{k,(r-s)m}{\varrho^{\phi}}_{k,lt} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\phi^i_{k,rm}\phi^{n-i}_{k,sm}{\varrho^{\phi}}_{k,smi+rm(n-i)+lt}. \tag{29}$$

Proof. Applying the Lemma 2.1(e), we have

$$\begin{aligned}\mu_1^{sm}\phi_{k,rm} - \mu_1^{rm}\phi_{k,sm} &= (-1)^{sm}\phi_{k,(r-s)m},\\ \mu_2^{sm}\phi_{k,rm} - \mu_2^{rm}\phi_{k,sm} &= (-1)^{sm}\phi_{k,(r-s)m}.\end{aligned}$$

By employing the binomial theorem, we get

$$(-1)^{smn}\phi^n_{k,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\phi^i_{k,rm}\phi^{n-i}_{k,sm}\mu_1^{smi+rm(n-i)}, \tag{30}$$

$$(-1)^{smn}\phi^n_{k,(r-s)m} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\phi^i_{k,rm}\phi^{n-i}_{k,sm}\mu_2^{smi+rm(n-i)}. \tag{31}$$

Now, by multiplying Equation (30) by $\dfrac{\hat{\mu_1}\mu_1^{lt}}{\mu_1-\mu_2}$ and equation (31) by $\dfrac{\hat{\mu_2}\mu_2^{lt}}{\mu_1-\mu_2}$ and subtracting, we obtain

$$\begin{aligned}&(-1)^{smn}\phi^n_{k,(r-s)m}\left(\frac{\hat{\mu_1}\mu_1^{lt} - \hat{\mu_2}\mu_2^{lt}}{\mu_1-\mu_2}\right)\\ &= \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\phi^i_{k,rm}\phi^{n-i}_{k,sm}\left(\frac{\hat{\mu_1}\mu_1^{smi+rm(n-i)+lt} - \hat{\mu_2}\mu_2^{smi+rm(n-i)+lt}}{\mu_1-\mu_2}\right),\end{aligned}$$

i.e.,

$$(-1)^{smn}\phi^n_{k,(r-s)m}{\varrho^{\phi}}_{k,lt} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\phi^i_{k,rm}\phi^{n-i}_{k,sm}{\varrho^{\phi}}_{k,smi+rm(n-i)+lt},$$

which completes the proof of Theorem 2.5. □

Theorem 2.6. *Let $n, m, r, s, t \in \mathbb{Z}^+$. Then show that*

$$(i) \quad \psi^n_{k,2^{r+1}} {\varrho^{\phi}}_{k,(2^{r+1}+2)n+mt} = \sum_{i=0}^{n} \binom{n}{i} {\varrho^{\phi}}_{k,(2^{r+2})i+2n+mt}, \tag{32}$$

$$(ii) \quad \psi^n_{k,2^{r+1}} {\varrho^{\psi}}_{k,(2^{r+1}+2)n+mt} = \sum_{i=0}^{n} \binom{n}{i} {\varrho^{\psi}}_{k,(2^{r+2})i+2n+mt}. \tag{33}$$

Proof. By making the use of Lemma 2.2 and Lemma 2.1 (a), we write

$$\begin{aligned} \psi_{k,2^{r+1}} \mu_1^{(2^{r+1}+2)} &= \mu_1^2 + \mu_1^{(2^{r+2}+2)}, \\ \psi_{k,2^{r+1}} \mu_2^{(2^{r+1}+2)} &= \mu_2^2 + \mu_2^{(2^{r+2}+2)}. \end{aligned}$$

Thanks to Binomial theorem. By using it, we obtain

$$\psi^n_{k,2^{r+1}} \mu_1^{(2^{r+1}+2)n} = \sum_{i=0}^{n} \binom{n}{i} \mu_1^{(2^{r+2})i+2n}, \tag{34}$$

$$\psi^n_{k,2^{r+1}} \mu_2^{(2^{r+1}+2)n} = \sum_{i=0}^{n} \binom{n}{i} \mu_2^{(2^{r+2})i+2n}. \tag{35}$$

By multiplying Equation (34) by $\dfrac{\hat{\mu_1}\mu_1^{mt}}{\mu_1 - \mu_2}$ and Equation (35) by $\dfrac{\hat{\mu_2}\mu_2^{mt}}{\mu_1 - \mu_2}$ and subtracting, we get

$$\psi^n_{k,2^{r+1}} {\varrho^{\phi}}_{k,(2^{r+1}+2)n+mt} = \sum_{i=0}^{n} \binom{n}{i} {\varrho^{\phi}}_{k,(2^{r+2})i+2n+mt}.$$

Moreover, by multiplying Equation (34) by $\hat{\mu_1}\mu_1^{mt}$ and Equation (35) by $\hat{\mu_2}\mu_2^{mt}$ and adding, we obtain

$$\psi^n_{k,2^{r+1}} {\varrho^{\psi}}_{k,(2^{r+1}+2)n+mt} = \sum_{i=0}^{n} \binom{n}{i} {\varrho^{\psi}}_{k,(2^{r+2})i+2n+mt}.$$

This proves the Theorem 2.6. □

Theorem 2.7. *Let $n, m, r, s, t, q \in \mathbb{Z}^+$. Then prove that*

$$(i) \quad \sum_{i=0}^{n} \binom{n}{i} k^{(i-n)} \psi^{(n-i)}_{k,2q-1} {\varrho^{\phi}}_{k,2qi+mt} = \begin{cases} k^{-n}(\phi_{k,2q})^n \Delta^{\frac{n}{2}} {\varrho^{\phi}}_{k,n+mt}, \\ \text{if } n \text{ is even;} \\ \\ k^{-n}(\phi_{k,2q})^n \Delta^{\frac{n-1}{2}} {\varrho^{\psi}}_{k,n+mt}, \\ \text{if } n \text{ is odd,} \end{cases} \tag{36}$$

$$(ii) \quad \sum_{i=0}^{n} \binom{n}{i} k^{(i-n)} (\psi^{(n-i)}_{k,2q-1}) {\varrho^{\psi}}_{k,2qi+mt} = \begin{cases} k^{-n}(\phi_{k,2q})^n \Delta^{\frac{n}{2}} {\varrho^{\psi}}_{k,n+mt}, \\ \text{if } n \text{ is even;} \\ \\ k^{-n}(\phi_{k,2q})^n \Delta^{\frac{n+1}{2}} {\varrho^{\phi}}_{k,n+mt}, \\ \text{if } n \text{ is odd.} \end{cases} \tag{37}$$

Proof. (i) By applying the Lemma 2.3, we have

$$\mu_1^{2q} + \frac{\psi_{k,2q-1}}{k} = \frac{\phi_{k,2q}}{k}\mu_1\sqrt{\Delta},$$
$$\mu_2^{2q} + \frac{\psi_{k,2q-1}}{k} = -\frac{\phi_{k,2q}}{k}\mu_2\sqrt{\Delta}.$$

Now, by employing the binomial theorem, we achieve

$$\sum_{i=0}^{n}\binom{n}{i}k^{i-n}\psi_{k,2q-1}^{(n-i)}(\mu_1^{2qi}) = k^{-n}\phi_{k,2q}^{n}\Delta^{\frac{n}{2}}(\mu_1^{n}), \tag{38}$$

$$\sum_{i=0}^{n}\binom{n}{i}k^{i-n}\psi_{k,2q-1}^{(n-i)}(\mu_2^{2qi}) = (-1)^n k^{-n}\phi_{k,2q}^{n}\Delta^{\frac{n}{2}}(\mu_2^{n}). \tag{39}$$

Multiplying Equation (38) by $\frac{\hat{\mu}_1\mu_1^{mt}}{\mu_1-\mu_2}$ and Equation (39) by $\frac{\hat{\mu}_2\mu_2^{mt}}{\mu_1-\mu_2}$ and subtracting, we get

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\psi_{k,2q-1}^{(n-i)})\left(\frac{\hat{\mu}_1\mu_1^{2qi+mt}-\hat{\mu}_2\mu_2^{2qi+mt}}{\mu_1-\mu_2}\right)$$
$$=\begin{cases} k^{-n}(\phi_{k,2q})^n\Delta^{\frac{n}{2}}\big(\frac{\hat{\mu}_1\mu_1^{n+mt}-\hat{\mu}_2\mu_2^{n+mt}}{\mu_1-\mu_2}\big), & \text{if } n \text{ is even,}\\ k^{-n}(\phi_{k,2q})^n\Delta^{\frac{n-1}{2}}\big(\hat{\mu}_1\mu_1^{n+mt}+\hat{\mu}_2\mu_2^{n+mt}\big), & \text{if } n \text{ is odd,}\end{cases}$$

i.e.,

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\psi_{k,2q-1}^{(n-i)}\varrho^{\phi}{}_{k,2qi+mt} = \begin{cases} k^{-n}(\phi_{k,2q})^n\Delta^{\frac{n}{2}}\varrho^{\phi}{}_{k,n+mt}, & \text{if } n \text{ is even,}\\ k^{-n}(\phi_{k,2q})^n\Delta^{\frac{n-1}{2}}\varrho^{\psi}{}_{k,n+mt}, & \text{if } n \text{ is odd.}\end{cases}$$

(ii) Again, by multiplying Equation (38) by $\hat{\mu}_1\mu_1^{mt}$ and Equation (39) by $\hat{\mu}_2\mu_2^{mt}$ and adding, we get

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\psi_{k,2q-1}^{(n-i))}\left(\hat{\mu}_1\mu_1^{2qi+mt}+\hat{\mu}_2\mu_2^{2qi+mt}\right)$$
$$=\begin{cases} k^{-n}(\phi_{k,2q})^n\Delta^{\frac{n}{2}}\big(\hat{\mu}_1\mu_1^{n+mt}+\hat{\mu}_2\mu_2^{n+mt}\big), & \text{if } n \text{ is even;}\\ k^{-n}(\phi_{k,2q})^n\Delta^{\frac{n+1}{2}}\big(\frac{\hat{\mu}_1\mu_1^{n+mt}-\hat{\mu}_2\mu_2^{n+mt}}{\mu_1-\mu_2}\big), & \text{if } n \text{ is odd,}\end{cases}$$

which implies

$$\sum_{i=0}^{n}\binom{n}{i}k^{(i-n)}(\psi_{k,2q-1}^{(n-i)})\varrho^{\psi}{}_{k,2qi+mt} = \begin{cases} k^{-n}(\phi_{k,2q})^n\Delta^{\frac{n}{2}}\varrho^{\psi}{}_{k,n+mt}, & \text{if } n \text{ is even;}\\ k^{-n}(\phi_{k,2q})^n\Delta^{\frac{n+1}{2}}\varrho^{\phi}{}_{k,n+mt}, & \text{if } n \text{ is odd.}\end{cases}$$

This completes the proof of Theorem 2.7. □

Conclusions

Our work establishes new identities for hyperbolic k-Fibonacci octonions and k-Lucas octonions. In our earlier research, we developed some properties of k-Fibonacci and k-Lucas numbers which allowed us to establish the identities of hyperbolic k-Fibonacci and k-Lucas octonions. These identities can be used to solve various problems related to these octonions, and can be applied to many other areas of mathematics. We believe that this work can open up new research avenues in the field of octonions.

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