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Generalization of the 2-Fibonacci sequences and their Binet formula

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Abstract: We will explore the generalization of the four different 2-Fibonacci sequences defined by Atanassov. In particular, we will define recurrence relations to generate each part of a 2-Fibonacci sequence, discuss the generating function and Binet formula of each of these sequences, and provide the necessary and sufficient conditions to obtain each type of Binet formula.

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1 Introduction

The Fibonacci sequence $\{F_n\}$ is a sequence that satisfies the recurrence relation $F_n = F_{n-1} + F_{n-2}$ and has the initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence and its generalizations have been studied extensively by various researchers [7, 10, 12, 13, 15, 16, 18, 20–22, 27, 28]. Authors have generalized the sequence by altering the initial conditions [10,15,16,18,20], altering the coefficients in the recurrence relation [7, 12, 13, 21, 22] or both [3, 7]. In 2009, Edson and Yayenie introduced the concept of a bi-periodic Fibonacci sequence $q_n = a_{n-1}q_{n-1} + a_{n-2}q_{n-2}$ where the coefficients are periodic [12]. Vernon, Arora, and Unnithan, in 2020, proved that a linear recurrence relation with periodic coefficients can sometimes be split into relations with constant coefficients [25]. In another paper, Vernon, Arora, and Unnithan proved that all such linear recurrence relations with periodic coefficients can be split, regardless of the coefficients, and provided a simple formula to compute the new constant coefficients [26].

Atanassov introduced four different ways of constructing pairs of sequences $\{\alpha_n\}$ and $\{\beta_n\}$ and called these pairs 2-Fibonacci sequences, or 2-F sequences [8]. These pairs and their generalizations have been studied by different researchers [1, 4, 5, 14].

In Section 2, we will explore a generalization of the four 2-Fibonacci sequences that Atanassov defined but with arbitrary initial conditions and coefficients. In addition, we will define new recurrence relations to construct these generalized 2-Fibonacci sequences, discuss the generating function of each of these sequences, obtain the Binet formula for each case, and provide necessary and sufficient conditions for each type of Binet formula obtained. In Section 3, we explore the effects of using period-p coefficients when constructing 2-Fibonacci sequences. In Section 4, we provide examples of using each type of Binet formula to define the first few terms of the sequences for each applicable case.

2 Main results

An *order-k linear recurrence relation* is an equation of the form $a_{n+k} = f(a_{n+k-1}, \ldots, a_n)$, where f is a linear function. We will define an *order-2 linear recurrence relation* in two variables as a pair of equations $a_{n+2} = f_1(a_{n+1}, b_{n+1}, a_n, b_n)$, $b_{n+2} = f_2(a_{n+1}, b_{n+1}, a_n, b_n)$ for some linear functions f_1, f_2. If f_1 and f_2 are as expressed below, we will refer to the relation as a *2-Fibonacci recurrence relation*. A pair of sequences $(\{a_n\}, \{b_n\})$ that satisfy a 2-Fibonacci recurrence relation will be referred to as a 2-Fibonacci sequence.

There are four 2-Fibonacci recurrence relations in consideration:

$$\text{Case 1: } a_{n+2} = \gamma_1 b_{n+1} + \gamma_2 a_n, b_{n+2} = \delta_1 a_{n+1} + \delta_2 b_n$$
$$\text{Case 2: } a_{n+2} = \gamma_1 a_{n+1} + \gamma_2 b_n, b_{n+2} = \delta_1 b_{n+1} + \delta_2 a_n$$
$$\text{Case 3: } a_{n+2} = \gamma_1 b_{n+1} + \gamma_2 b_n, b_{n+2} = \delta_1 a_{n+1} + \delta_2 a_n$$
$$\text{Case 4: } a_{n+2} = \gamma_1 a_{n+1} + \gamma_2 a_n, b_{n+2} = \delta_1 b_{n+1} + \delta_2 b_n$$

We will assume γ_i and δ_i are real constants for each i such that $\gamma_i \neq 0$ and $\delta_i \neq 0$. The initial conditions for a corresponding 2-Fibonacci sequence are a_0, a_1, b_0, and b_1. There has been extensive literature examining each of these cases where $\gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 1$, and $a_0 = b_0 = 0$, $a_1 = b_1 = 1$ [2, 3, 6, 7]. These results have also inspired discussion from others [9–11, 17, 19, 24]. The goal of this section will be to find the Binet formula for each case where all of the initial conditions and coefficients are arbitrary. We will omit the discussion of Case 4 since it is trivial.

2.1 Recurrence relation

In each of the cases of the 2-Fibonacci sequences, we can separate the recurrence relations into two new recurrence relations each using only one of the sequences and obtain the following:

Theorem 2.1. *If* $(\{a_n\}, \{b_n\})$ *is a* 2*-Fibonacci sequence, then there exists a single order-*4 *linear recurrence relation such that* $\{a_n\}$ *and* $\{b_n\}$ *are each solutions to the relation.*

Proof. We will prove the result for $\{a_n\}$ in Case 1. The proofs for Cases 2 and 3 and for $\{b_n\}$ in Case 1 are similar and will be omitted. Note that in each case, the relation we obtain for $\{b_n\}$ will have the same coefficients as the corresponding relation for $\{a_n\}$. Let $\gamma_1, \gamma_2, \delta_1, \delta_2$ be real numbers. Let $\{a_n\}, \{b_n\}$ be sequences such that $a_{n+2} = \gamma_1 b_{n+1} + \gamma_2 a_n$ and $b_{n+2} = \delta_1 a_{n+1} + \delta_2 b_n$ for all n. Then

$$\begin{aligned}
\gamma_1 b_{n+1} &= a_{n+2} - \gamma_2 a_n \\
b_{n+3} &= \delta_1 a_{n+2} + \delta_2 b_{n+1} \\
a_{n+4} &= \gamma_1 b_{n+3} + \gamma_2 a_{n+2} \\
&= \gamma_1(\delta_1 a_{n+2} + \delta_2 b_{n+1}) + \gamma_2 a_{n+2} \\
&= \gamma_1 \delta_1 a_{n+2} + \gamma_1 \delta_2 b_{n+1} + \gamma_2 a_{n+2} \\
&= \gamma_1 \delta_1 a_{n+2} + \delta_2(a_{n+2} - \gamma_2 a_n) + \gamma_2 a_{n+2} \\
&= (\gamma_1 \delta_1 + \gamma_2 + \delta_2) a_{n+2} - \gamma_2 \delta_2 a_n.
\end{aligned}$$

Using the same technique as above, for the cases given at the beginning of Section 2 we can obtain the following linear recurrence relations for which $\{a_n\}$ and $\{b_n\}$ are each solutions.

Case 1: $c_{n+4} = (\gamma_1\delta_1 + \gamma_2 + \delta_2)c_{n+2} - \gamma_2\delta_2 c_n$
Case 2: $c_{n+4} = (\gamma_1 + \delta_1)c_{n+3} - \gamma_1\delta_1 c_{n+2} + \gamma_2\delta_2 c_n$
Case 3: $c_{n+4} = (\gamma_1\delta_1)c_{n+2} + (\gamma_2\delta_1 + \gamma_1\delta_2)c_{n+1} + \gamma_2\delta_2 c_n.$ □

2.2 Generating function

In each of the cases of the 2-Fibonacci sequences, we can establish a generating function to obtain a closed form for a_n or b_n. By Theorem 2.1, in each case we have $\{a_n\}$ and $\{b_n\}$ satisfy the same recurrence relation, so the only difference in the generating functions will be the initial conditions. Therefore, we will simply focus on obtaining a generating function for $\{c_n\}$, where $\{c_n\}$ is equal to either $\{a_n\}$ or $\{b_n\}$. That is, define

$$g(x) = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n. \tag{1}$$

We can determine the rational function form of $g(x)$, decompose $g(x)$ into its partial fractions, and determine the coefficients of the partial fractions to establish a Binet formula for c_n.

Consider a sequence $\{c_n\}$ satisfying the order-4 linear recurrence relation:

$$c_4 = ac_3 + bc_2 + cc_1 + dc_0$$

.

This corresponds to the following generating function.

$$g(x) = \frac{c_0 + (c_1 - ac_0)x + (c_2 - ac_1 - bc_0)x^2 + (c_3 - ac_2 - bc_1 - cc_0)x^3}{1 - ax - bx^2 - cx^3 - dx^4}. \tag{2}$$

2.2.1 Case 1

From Theorem 2.1, $a = c = 0$, and we have that $c_{n+4} = bc_{n+2} + dc_n$, $b = \delta_1\gamma_1 + \gamma_2 + \delta_2$, and $d = -\gamma_2\delta_2$. Thus from (2), we have

$$g(x) = \frac{c_0 + c_1x + (c_2 - bc_0)x^2 + (c_3 - bc_1)x^3}{1 - bx^2 - dx^4}. \tag{3}$$

2.2.2 Case 2

From Theorem 2.1, we have that

$$c_{n+4} = ac_{n+3} + bc_{n+2} + dc_n,$$

where $a = \delta_1 + \gamma_1$, $b = -\delta_1\gamma_1$, $c = 0$, $d = \gamma_2\delta_2$. Thus from (2), can write $g(x)$ with these values.

2.2.3 Case 3

From Theorem 2.1, we have that

$$c_{n+4} = bc_{n+2} + cc_{n+1} + dc_n,$$

where $a = 0$, $b = \delta_1\gamma_1$, $c = \delta_1\gamma_2 + \delta_2\gamma_1$, $d = \gamma_2\delta_2$. Similarly, we can write $g(x)$ as we have done for the two previous cases.

2.3 Decompositions

Suppose we have the following rational function:

$$g(x) = \frac{h(x)}{1 - ax - bx^2 - cx^3 - dx^4}, \tag{4}$$

where $h(x)$ is a third degree polynomial and a, b, c, d are constants and $d \neq 0$. Then $g(x)$ can be decomposed into one of the following, depending on the multiplicity of each root of the denominator.

$$g(x) = \frac{A}{1 - \epsilon_1 x} + \frac{B}{1 - \epsilon_2 x} + \frac{C}{1 - \epsilon_3 x} + \frac{D}{1 - \epsilon_4 x}, \tag{5}$$

$$g(x) = \frac{A}{1 - \epsilon_1 x} + \frac{B}{(1 - \epsilon_1 x)^2} + \frac{C}{1 - \epsilon_2 x} + \frac{D}{(1 - \epsilon_2 x)^2}, \tag{6}$$

$$g(x) = \frac{A}{1 - \epsilon_1 x} + \frac{B}{(1 - \epsilon_1 x)^2} + \frac{C}{1 - \epsilon_2 x} + \frac{D}{1 - \epsilon_3 x}, \tag{7}$$

$$g(x) = \frac{A}{1 - \epsilon_1 x} + \frac{B}{(1 - \epsilon_1 x)^2} + \frac{C}{(1 - \epsilon_1 x)^3} + \frac{D}{1 - \epsilon_2 x}, \tag{8}$$

$$g(x) = \frac{A}{1 - \epsilon_1 x} + \frac{B}{(1 - \epsilon_1 x)^2} + \frac{C}{(1 - \epsilon_1 x)^3} + \frac{D}{(1 - \epsilon_1 x)^4}. \tag{9}$$

We can see that (5), (6), (7), (8), (9) are the decompositions for when $1-ax-bx^2-cx^3-dx^4$ has exactly four distinct roots, roots with multiplicity 2, roots with multiplicity 2 and 1, a root with multiplicity 3, or a root with multiplicity 4, respectively [23]. Note that each ϵ_i can be complex and that $\epsilon_i \neq 0$ for any i since $d \neq 0$.

2.3.1 Distinct roots

Here we will calculate the coefficients for the case of the distinct roots (5) of the partial fraction decomposition of (4). We can see from (4) and (5) that

$$\begin{aligned} h(x) = A(1-\epsilon_2 x)(1-\epsilon_3 x)(1-\epsilon_4 x) + B(1-\epsilon_1 x)(1-\epsilon_3 x)(1-\epsilon_4 x) \\ + C(1-\epsilon_1 x)(1-\epsilon_2 x)(1-\epsilon_4 x) + D(1-\epsilon_1 x)(1-\epsilon_2 x)(1-\epsilon_3 x). \end{aligned} \tag{10}$$

By setting $x = \frac{1}{\epsilon_i}$ for $i = 1, 2, 3, 4$, we obtain the following.

$$A = \frac{\epsilon_1^3 h(1/\epsilon_1)}{(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3)(\epsilon_1-\epsilon_4)}, B = \frac{\epsilon_2^3 h(1/\epsilon_2)}{(\epsilon_2-\epsilon_1)(\epsilon_2-\epsilon_3)(\epsilon_2-\epsilon_4)},$$
$$C = \frac{\epsilon_3^3 h(1/\epsilon_3)}{(\epsilon_3-\epsilon_1)(\epsilon_3-\epsilon_2)(\epsilon_3-\epsilon_4)}, D = \frac{\epsilon_4^3 h(1/\epsilon_4)}{(\epsilon_4-\epsilon_1)(\epsilon_4-\epsilon_2)(\epsilon_4-\epsilon_3)}$$

Once we calculate the coefficients, we can represent each term of (5) as a geometric series.

$$g(x) = A\sum_{n=0} \epsilon_1^n x^n + B\sum_{n=0} \epsilon_2^n x^n + C\sum_{n=0} \epsilon_3^n x^n + D\sum_{n=0} \epsilon_4^n x^n. \tag{11}$$

Thus from (1), we have

$$c_n = A\epsilon_1^n + B\epsilon_2^n + C\epsilon_3^n + D\epsilon_4^n. \tag{12}$$

Note that in Case 1 and Case 3 we have $\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4 = a = 0$, and in Case 1 and Case 2 we have $\epsilon_1\epsilon_2\epsilon_3 + \epsilon_1\epsilon_2\epsilon_4 + \epsilon_1\epsilon_3\epsilon_4 + \epsilon_2\epsilon_3\epsilon_4 = c = 0$, so restrictions on ϵ_i will arise from these constraints.

2.3.2 Multiplicity 2

Here we will explore the scenario in which the denominator of $g(x)$ has two roots each with multiplicity 2 and the necessary and sufficient conditions to obtain this scenario in each case. Consider the factorization of $1-ax-bx^2-cx^3-dx^4$ into

$$(1-\epsilon_1 x)^2(1-\epsilon_2 x)^2, \tag{13}$$

where ϵ_1, ϵ_2 are distinct. If we expand (13), then we have

$$1-2(\epsilon_1+\epsilon_2)x + (\epsilon_1^2+\epsilon_2^2+4\epsilon_1\epsilon_2)x^2 - 2(\epsilon_1\epsilon_2^2+\epsilon_1^2\epsilon_2)x^3 + \epsilon_1^2\epsilon_2^2 x^4. \tag{14}$$

We can see from each of the cases of the 2-Fibonacci sequences that $a = 0$ or $c = 0$. This implies that $\epsilon_1+\epsilon_2 = 0$ or $\epsilon_1\epsilon_2^2+\epsilon_1^2\epsilon_2 = 0$, both of which imply $\epsilon_2 = -\epsilon_1$. Since $\epsilon_2 = -\epsilon_1$, we have from (4) and (6) that

$$h(x) = A(1-\epsilon_1 x)(1+\epsilon_1 x)^2 + B(1+\epsilon_1 x)^2 + C(1+\epsilon_1 x)(1-\epsilon_1 x)^2 + D(1-\epsilon_1 x)^2. \tag{15}$$

Thus if we let $x = \frac{1}{\epsilon_1}$, then

$$B = \frac{h(1/\epsilon_1)}{4}. \tag{16}$$

If we let $x = -\frac{1}{\epsilon_1}$, then

$$D = \frac{h(-1/\epsilon_1)}{4}. \tag{17}$$

If we let $x = 0$, then

$$h(0) - B - D = A + C. \tag{18}$$

And if we let $x = \frac{2}{\epsilon_1}$, then

$$h(2/\epsilon_1) - 9B - D = -9A + 3C. \tag{19}$$

Then we can solve for A and C from (18) and (19) to obtain

$$A = \frac{3h(0) - h(2/\epsilon_1) + \frac{3}{2}h(1/\epsilon_1) - \frac{1}{2}h(-1/\epsilon_1)}{12}, \tag{20}$$

$$C = \frac{9h(0) + h(2/\epsilon_1) - \frac{9}{2}h(1/\epsilon_1) - \frac{5}{2}h(-1/\epsilon_1)}{12}. \tag{21}$$

As before, we can rewrite each term of (6) as a power series:

$$g(x) = A\sum_{n=0}\epsilon_1^n x^n + B\sum_{n=0}(n+1)\epsilon_1^n x^n + C\sum_{n=0}(-\epsilon_1)^n x^n + D\sum_{n=0}(n+1)(-\epsilon_1)^n x^n. \tag{22}$$

Thus from (1), we have

$$c_n = A\epsilon_1^n + B(n+1)\epsilon_1^n + C(-\epsilon_1)^n + D(n+1)(-\epsilon_1)^n. \tag{23}$$

Case 1 In this case, it is necessary from (14) that $\delta_1\gamma_1+\gamma_2+\delta_2 = b = 2\epsilon_1^2$ and $-\gamma_2\delta_2 = d = -\epsilon_1^4$. An example of the choices of the parameters for this case is outlined in Example 5 with a table of values for (23). Furthermore, this is a sufficient condition for obtaining two distinct roots, because if there exists some number ϵ_1 such that $b = 2\epsilon_1^2$ and $d = -\epsilon_1^4$, then the denominator of $g(x)$ becomes $1 - 2\epsilon_1^2x^2 + \epsilon_1^4x^4 = (1-\epsilon_1)^2(1+\epsilon_1)^2$.

Case 2 In this case, it is necessary from (14) that $\delta_1 + \gamma_1 = a = 0, -\delta_1\gamma_1 = b = 2\epsilon_1^2$, and $\gamma_2\delta_2 = d = -\epsilon_1^4$. An example of the choices of the parameters for this case is outlined in Example 6 with a table of values for (23). As in Case 1, we can see that this is also a sufficient condition.

Case 3 In this case, it is necessary from (14) that $\delta_1\gamma_2 + \delta_2\gamma_1 = c = 0, \delta_1\gamma_1 = b = 2\epsilon_1^2$, and $\gamma_2\delta_2 = d = -\epsilon_1^4$. An example of the choices for the parameters for this case is outlined in Example 7 with a table of values for (23). As in Case 1, we can see that this is also a sufficient condition.

2.3.3 Multiplicity 2 and 1

Here we will explore the scenario in which the denominator of $g(x)$ has three distinct roots and the necessary and sufficient conditions to obtain this scenario in each case. Consider the factorization of $1 - ax - bx^2 - cx^3 - dx^4$ into

$$(1-\epsilon_1 x)^2(1-\epsilon_2 x)(1-\epsilon_3 x), \tag{24}$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ are distinct. If we expand (24), then we have

$$1-(\epsilon_2+\epsilon_3+2\epsilon_1)x+(\epsilon_2\epsilon_3+2\epsilon_1\epsilon_2+2\epsilon_1\epsilon_3+\epsilon_1^2)x^2-(2\epsilon_1\epsilon_2\epsilon_3+\epsilon_1^2\epsilon_2+\epsilon_1^2\epsilon_3)x^3+\epsilon_1^2\epsilon_2\epsilon_3 x^4. \tag{25}$$

Next, we calculate the coefficients in (7). We can see from (4) and (7) that

$$\begin{aligned} h(x) &= A(1-\epsilon_1 x)(1-\epsilon_2 x)(1-\epsilon_3 x)+B(1-\epsilon_2 x)(1-\epsilon_3 x) \\ &\quad + C(1-\epsilon_1 x)^2(1-\epsilon_3 x)+D(1-\epsilon_1 x)^2(1-\epsilon_2 x). \end{aligned} \tag{26}$$

If we let $x=\frac{1}{\epsilon_1}$, then

$$B=\frac{\epsilon_1^2 h(1/\epsilon_1)}{(\epsilon_1-\epsilon_2)(\epsilon_1-\epsilon_3)}. \tag{27}$$

If we let $x=\frac{1}{\epsilon_2}$, then

$$C=\frac{\epsilon_2^3 h(1/\epsilon_2)}{(\epsilon_2-\epsilon_1)^2(\epsilon_2-\epsilon_3)}. \tag{28}$$

If we let $x=\frac{1}{\epsilon_3}$, then

$$D=\frac{\epsilon_3^3 h(1/\epsilon_3)}{(\epsilon_3-\epsilon_1)^2(\epsilon_3-\epsilon_2)}. \tag{29}$$

And if we let $x=0$, then

$$A=h(0)-B-C-D. \tag{30}$$

As with the previous cases, once we calculate the coefficients, we can represent each term in (7) as a power series:

$$g(x)=A\sum_{n=0}\epsilon_1^n x^n+B\sum_{n=0}(n+1)\epsilon_1^n x^n+C\sum_{n=0}\epsilon_2^n x^n+D\sum_{n=0}\epsilon_3^n x^n. \tag{31}$$

Thus from (1), we have

$$c_n=A\epsilon_1^n+B(n+1)\epsilon_1^n+C\epsilon_2^n+D\epsilon_3^n. \tag{32}$$

Case 1 It is necessary from (25) that $\epsilon_2+\epsilon_3+2\epsilon_1=a=0$ and $2\epsilon_1\epsilon_2\epsilon_3+\epsilon_1^2\epsilon_2+\epsilon_1^2\epsilon_3=c=0$, which means that $\epsilon_1=-\frac{\epsilon_2+\epsilon_3}{2}$ and $\epsilon_1=-\frac{2\epsilon_2\epsilon_3}{\epsilon_2+\epsilon_3}$, implying $\epsilon_2=\epsilon_3$, which contradicts our assumption that ϵ_2 and ϵ_3 are distinct. Thus the scenario of Multiplicity 2 and 1 is impossible for Case 1.

Case 2 It is necessary from (25) that $2\epsilon_1\epsilon_2\epsilon_3+\epsilon_1^2\epsilon_2+\epsilon_1^2\epsilon_3=c=0$, which means that $\epsilon_1=-\frac{2\epsilon_2\epsilon_3}{\epsilon_2+\epsilon_3}$. Then we must choose coefficients $\delta_1, \delta_2, \gamma_1, \gamma_2$ such that $\gamma_1+\delta_1=a=\epsilon_2+\epsilon_3-4\frac{\epsilon_2\epsilon_3}{\epsilon_2+\epsilon_3}$, $-\delta_1\gamma_1=b=3\epsilon_2\epsilon_3-4\frac{\epsilon_2^2\epsilon_3^2}{\epsilon_2+\epsilon_3}$, and $\gamma_2\delta_2=d=-4\frac{\epsilon_2^3\epsilon_3^3}{(\epsilon_2+\epsilon_3)^2}$ for some distinct numbers ϵ_2, ϵ_3. This also implies that $\epsilon_2\neq-\epsilon_3$. Additionally, this means ϵ_1 is distinct from ϵ_2 and ϵ_3, since otherwise we would have a root of multiplicity 3, and as we will show in the subsequent section, no real coefficients $\gamma_1, \gamma_2, \delta_1, \delta_2$ exist to yield this case. An example of the choices of the parameters for this case is outlined in Example 8 with a table of values for (32). As in Case 1 of the Multiplicity 2 scenario, we can show that this is also a sufficient condition.

Case 3 It is necessary from (25) that $\epsilon_2+\epsilon_3+2\epsilon_1 = a = 0$, which means that $\epsilon_1 = -\frac{\epsilon_2+\epsilon_3}{2}$. Then we must choose coefficients $\delta_1, \delta_2, \gamma_1, \gamma_2$ such that $\delta_1\gamma_1 = b = \epsilon_2\epsilon_3 + \frac{3(\epsilon_2+\epsilon_3)^2}{4}$, $\gamma_1\delta_2+\delta_1\gamma_2 = c = -(\epsilon_2+\epsilon_3)\epsilon_2\epsilon_3 + \frac{(\epsilon_2+\epsilon_3)^3}{4}$, and $\gamma_2\delta_2 = d = -\frac{(\epsilon_2+\epsilon_3)^2}{4}\epsilon_2\epsilon_3$ for some distinct numbers ϵ_2, ϵ_3. This also implies $\epsilon_2 \neq -\epsilon_3$ since otherwise $\delta_2\gamma_2 = d = 0$, and $\epsilon_1 = -\frac{\epsilon_2+\epsilon_3}{2}$ is distinct from ϵ_2, since otherwise if $\epsilon_1 = \epsilon_2$ we would have $\epsilon_3 = -3\epsilon_2$, resulting in $\gamma_1\delta_1 = b = 0$. By the same argument we also have $\epsilon_1 \neq \epsilon_3$. An example of the choices for δ_1, δ_2, γ_1, γ_2 for this case is outlined in Example 9 with a table of values for (32). As in Case 1 of the Multiplicity 2 scenario, we can show that this is also a sufficient condition.

2.3.4 Multiplicity 3

We will prove that the scenario outlined in (8) is impossible in each of our cases.

Lemma 2.2. *If $(\{a_n\}, \{b_n\})$ is a 2-Fibonacci sequence and g is a generating function for $\{a_n\}$ or $\{b_n\}$, and $\frac{1}{\epsilon_i}$ is a root of the denominator of $g(x)$, then $\frac{1}{\epsilon_i}$ does not have multiplicity 3.*

Proof. Consider the factorization of $1 - ax - bx^2 - cx^3 - dx^4$ into

$$(1-\epsilon_1 x)^3(1-\epsilon_2 x), \tag{33}$$

where ϵ_1, ϵ_2 are distinct. Additionally, we note that ϵ_1 and ϵ_2 are real numbers; otherwise, they would have to be complex conjugates since a, b, c, d are real numbers. Since the roots have different multiplicity, they must not be complex conjugates. If we expand (33), then we have

$$1 + (-\epsilon_2 - 3\epsilon_1)x + (3\epsilon_1\epsilon_2 + 3\epsilon_1^2)x^2 + (-3\epsilon_1^2\epsilon_2 - \epsilon_1^3)x^3 + \epsilon_1^3\epsilon_2 x^4. \tag{34}$$

Case 1 It is necessary from (34) that $\epsilon_2 + 3\epsilon_1 = a = 0$, which means that $\epsilon_2 = -3\epsilon_1$. Also, we can see that $3\epsilon_1^2\epsilon_2 + \epsilon_1^3 = c = 0$, as well, so we have that $\epsilon_2 = -\frac{\epsilon_1}{3}$. Then we have $-3\epsilon_1 = -\frac{\epsilon_1}{3}$, which implies that $\epsilon_1 = 0$, a contradiction. Thus the scenario of Multiplicity 3 is impossible for Case 1.

Case 2 It is necessary from (34) that $3\epsilon_1^2\epsilon_2 + \epsilon_1^3 = c = 0$, which means that $\epsilon_1 = -3\epsilon_2$. Thus we have that $\gamma_1 + \delta_1 = a = -8\epsilon_2$ and $-\delta_1\gamma_1 = b = -18\epsilon_2^2$, which means that $\gamma_1 = \frac{18\epsilon_2^2}{\delta_1}$. Then

$$\begin{aligned}
\frac{18\epsilon_2^2}{\delta_1} + \delta_1 &= -8\epsilon_2 \\
18\epsilon_2^2 + \delta_1^2 &= -8\epsilon_2\delta_1 \\
\delta_1^2 + 8\epsilon_2\delta_1 + 18\epsilon_2^2 &= 0 \\
\delta_1 &= \frac{-8\epsilon_2 \pm \sqrt{64\epsilon_2^2 - 72\epsilon_2^2}}{2} \\
\delta_1 &= \frac{-8\epsilon_2 \pm \sqrt{-8\epsilon_2^2}}{2},
\end{aligned}$$

which implies that δ_1 is complex or $\epsilon_2 = 0$, both of which are contradictions. Thus the scenario of Multiplicity 3 is impossible for Case 2, as well.

Case 3 It is necessary from (34) that $\epsilon_2 + 3\epsilon_1 = a = 0$, which means that $\epsilon_2 = -3\epsilon_1$. Thus we have that $\gamma_1\delta_1 = b = 6\epsilon_1^2$, so $\delta_1 = \frac{6\epsilon_1^2}{\gamma_1}$, $\delta_1\gamma_2 + \delta_2\gamma_1 = c = -8\epsilon_1^3$, and $\delta_2\gamma_2 = d = 3\epsilon_1^4$. Then

$$\begin{aligned}
\frac{6\epsilon_1^2\gamma_2}{\gamma_1} + \delta_2\gamma_1 &= -8\epsilon_1^3 \\
6\epsilon_1^2\gamma_2 + \delta_2\gamma_1^2 &= -8\epsilon_1^3\gamma_1 \\
\delta_2\gamma_1^2 + 8\epsilon_1^3\gamma_1 + 6\epsilon_1^2\gamma_2 &= 0 \\
\gamma_1 &= \frac{-8\epsilon_1^3 \pm \sqrt{64\epsilon_1^6 - 24\delta_2\gamma_2\epsilon_1^2}}{2\delta_2} \\
\gamma_1 &= \frac{-8\epsilon_1^3 \pm \sqrt{64\epsilon_1^6 - 72\epsilon_1^6}}{2\delta_2} = \frac{-8\epsilon_1^3 \pm \sqrt{-8\epsilon_1^6}}{2\delta_2},
\end{aligned}$$

which implies that γ_1 is complex or $\epsilon_1 = 0$, both of which are contradictions. Thus the scenario of Multiplicity 3 is impossible for Case 3, as well. □

2.3.5 Multiplicity 4

We will prove that the scenario outlined in (9) is impossible in each of our cases.

Lemma 2.3. *If $(\{a_n\}, \{b_n\})$ is a 2-Fibonacci sequence and g is a generating function for $\{a_n\}$ or $\{b_n\}$, and $\frac{1}{\epsilon_i}$ is a root of the denominator of $g(x)$, then $\frac{1}{\epsilon_i}$ does not have multiplicity 4.*

Proof. Consider the factorization of $1 - ax - bx^2 - cx^3 - dx^4$ into

$$(1 - \epsilon_1 x)^4, \tag{35}$$

where ϵ_1 is not 0. If we expand (35), then we have

$$1 + (-4\epsilon_1)x + (6\epsilon_1^2)x^2 + (-4\epsilon_1^3)x^3 + \epsilon_1^4 x^4. \tag{36}$$

Since for each of our cases we have $a = 0$ or $c = 0$, we can see that $\epsilon_1 = 0$, a contradiction. □

3 Periodic coefficient sequences

In [26], the authors showed that any sequence satisfying an order-k linear recurrence relation with period-p coefficients can be subdivided into p separate subsequences such that there is a single order-k linear recurrence relation with constant coefficients that each subsequence satisfies. In the case of 2-Fibonacci sequences, a similar result arises. We will define a 2-Fibonacci recurrence relation with period-p coefficients in the same way as a 2-Fibonacci recurrence relation in Section 2, but replacing the constant coefficients γ_i, δ_i with coefficient sequences $\{\gamma_{i,n}\}, \{\delta_{i,n}\}$ of period p. Likewise, we will similarly define 2-Fibonacci p-sequences.

Lemma 3.1. *If $(\{a_n\}, \{b_n\})$ is a 2-Fibonacci p-sequence, then there exist two order-4 linear recurrence relations with period-p coefficients such that $\{a_n\}$ and $\{b_n\}$ are each solutions to one of the relations.*

Proof. We will prove the case that is analogous to the case used in the proof of Theorem 2.1. The other proofs are similar and will be omitted. Suppose we have sequences $\{a_n\}, \{b_n\}$ such that for all n, the following holds:

$$a_{n+2} = \gamma_{1,n+1} a_{n+1} + \gamma_{2,n} b_n \tag{37}$$

$$b_{n+2} = \delta_{1,n+1} b_{n+1} + \delta_{2,n} a_n, \tag{38}$$

where $\{\gamma_{1,n}\}, \{\gamma_{2,n}\}, \{\delta_{1,n}\}, \{\delta_{2,n}\}$ are given sequences such that $\gamma_{i,n+p} = \gamma_{i,n}$ and $\delta_{i,n+p} = \delta_{i,n}$ for all i and all n. Using the same technique as in the proof of Theorem 2.1, we can show that

$$a_{n+4} = \left(\gamma_{1,n+3} + \frac{\gamma_{2,n+2}\delta_{1,n+1}}{\gamma_{2,n+1}}\right) a_{n+3} - \frac{\gamma_{2,n+2}\delta_{1,n+1}\gamma_{1,n+2}}{\gamma_{2,n+1}} a_{n+2} + \gamma_{2,n+2}\delta_{2,n} a_n,$$

$$b_{n+4} = \left(\delta_{1,n+3} + \frac{\delta_{2,n+2}\gamma_{1,n+1}}{\delta_{2,n+1}}\right) b_{n+3} - \frac{\delta_{2,n+2}\gamma_{1,n+1}\delta_{1,n+2}}{\delta_{2,n+1}} b_{n+2} + \delta_{2,n+2}\gamma_{2,n} b_n.$$

We can easily verify that the new coefficients in the obtained relations for $\{a_n\}$ and $\{b_n\}$ are period-p sequences. This means $\{a_n\}$ and $\{b_n\}$ are each solutions to order-4 recurrence relations with period-p coefficient sequences, assuming the original coefficient sequences have no zero elements. □

Note that unlike in the case with constant coefficients, the new coefficients for a_{n+4} and b_{n+4} are not necessarily equal. By the results in our previous paper, any order-4 linear recurrence relation with period-p coefficient sequences can be split into p separate order-4 linear recurrence relations with constant coefficients [25]. We will demonstrate this in the following example, the case where $p = 2$. For simplicity of notation, we will rename the coefficients.

Example 1 (Subsequences of a 2-Fibonacci sequence with period-2 coefficients)**.**

$$a_{n+4} = \begin{cases} Aa_{n+3} + Ba_{n+2} + Ca_{n+1} + Da_n, & \textit{if } n \textit{ is even} \\ Ea_{n+3} + Fa_{n+2} + Ga_{n+1} + Ha_n, & \textit{if } n \textit{ is odd} \end{cases} \tag{39}$$

We first construct matrices similar to the Fibonacci matrix.

$$A(1) = \begin{bmatrix} A & B & C & D \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ A(2) = \begin{bmatrix} E & F & G & H \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$Q(2) = A(2)A(1) = \begin{bmatrix} AE+F & BE+G & CE+H & DE \\ A & B & C & D \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

For $Q(2)$, the sum of the first principal minors is $AE + F + B$, the sum of the second principal minors is $BF - AG - CE - H - D$, the sum of the third principal minors is $CG - DF - BH$, and the sum of the fourth principal minors is DH. Then by the results of our previous paper, we have

$$a_{n+8} = (AE+B+F)a_{n+6} - (BF-AG-CE-H-D)a_{n+4} + (CG-DF-BH)a_{n+2} - (DH)a_n.$$

This can be applied to any of the cases of 2*-Fibonacci sequences with period-*2 *coefficient sequences and can be easily generalized to any period. A similar result can be obtained for* $\{b_n\}$.

Last, we will show that although the period-p coefficient sequences in the relations from Lemma 3.1 are not necessarily the same for $\{a_n\}$ and $\{b_n\}$, the derived constant coefficients for the subsequences are indeed the same.

Theorem 3.2. *If* $(\{a_n\}, \{b_n\})$ *is a* 2*-Fibonacci* p*-sequence, then there exists a single order-*4 *linear recurrence relation with constant coefficients such that both* $\{a_{k+np}\}$ *and* $\{b_{k+np}\}$ *are solutions for all* k.

Proof. We will prove the result in Case 1. The other results are similar. By Lemma 3.1, there exists an order-4 linear recurrence relation with period-p coefficients such that $\{a_n\}$ is a solution. Then by [26] there exist four constant coefficients α_i such that for each n we have $a_{n+4p} = \alpha_1 a_{n+3p} + \alpha_2 a_{n+2p} + \alpha_3 a_{n+p} + \alpha_4 a_n$. Then

$$\begin{aligned}
\gamma_{2,n+4p} b_{n+4p} &= a_{n+4p+2} - \gamma_{1,n+4p+1} a_{n+4p+1} \\
&= \sum_{i=1}^{4} \alpha_i a_{n+(4-i)p+2} - \gamma_{1,n+4p+1} \sum_{i=1}^{4} \alpha_i a_{n+(4-i)p+1} \\
&= \sum_{i=1}^{4} \alpha_i \big(a_{n+(4-i)p+2} - \gamma_{1,n+(4-i)p+1} a_{n+(4-i)p+1}\big) \\
&= \sum_{i=1}^{4} \alpha_i \gamma_{2,n+(4-i)p} b_{n+(4-i)p} = \gamma_{2,n+4p} \sum_{i=1}^{4} \alpha_i b_{n+(4-i)p}.
\end{aligned}$$

Since we assume $\gamma_{i,j} \neq 0$ for all i, j, this means that for each n we can divide both sides by $\gamma_{2,n+4p}$ to conclude $b_{n+4p} = \alpha_1 b_{n+3p} + \alpha_2 b_{n+2p} + \alpha_3 b_{n+p} + \alpha_4 b_n$. □

4 Examples

The following are examples of the 2-Fibonacci sequences to illustrate each possible Binet formula calculated in their respective sections. In each example,we will use initial conditions $a_0 = b_0 = 0$, $a_1 = b_1 = 1$. A list of the first few a_n are displayed in their respective columns of Table 1.

Example 2 (Four Distinct Roots, Case 1)**.** *Let* $\gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 1$. *Then we have* $1-3x^2+x^4$ *as our denominator for* $g(x)$*, and our distinct roots* $\frac{1}{\epsilon_i}$ *will be* $\pm\frac{1\pm\sqrt{5}}{2}$.

Example 3 (Four Distinct Roots, Case 2)**.** *Let* $\gamma_1 = \delta_1 = \gamma_2 = \delta_2 = 2$. *Then we have* $1 - 4x + 4x^2 - 4x^4$ *as our denominator for* $g(x)$*, and our distinct roots* $\frac{1}{\epsilon_i}$ *will be* $\frac{1}{2} \pm \frac{i}{2}, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$.

Example 4 (Four Distinct Roots, Case 3)**.** *Let* $\gamma_1 = \delta_1 = 1$*,* $\gamma_2 = \delta_2 = 2$. *Then we have* $1 - x^2 - 4x^3 - 4x^4$ *as our denominator for* $g(x)$*, and our distinct roots will be* $-1, \frac{1}{2}, -\frac{1}{4} \pm \frac{i\sqrt{7}}{4}$.

Example 5 (Two Distinct Roots, Case 1)**.** *Let* $\gamma_1 = 1$*,* $\delta_1 = -1$*,* $\gamma_2 = 4$ *and* $\delta_2 = 1$. *Then we have* $1 - 4x^2 + 4x^4$ *as our denominator for* $g(x)$*, and our distinct roots will be* $\pm\frac{\sqrt{2}}{2}$.

Example 6 (Two Distinct Roots, Case 2)**.** *Let* $\gamma_1 = 2$*,* $\delta_1 = -2$*,* $\gamma_2 = -2$ *and* $\delta_2 = 2$*. Then we have* $1 - 4x^2 + 4x^4$ *as our denominator for* $g(x)$*, and our distinct roots will be* $\pm\frac{\sqrt{2}}{2}$*.*

Example 7 (Two Distinct Roots, Case 3)**.** *Let* $\gamma_1 = 2$*,* $\delta_1 = 2$*,* $\gamma_2 = -2$ *and* $\delta_2 = 2$*. Then we have* $1 - 4x^2 + 4x^4$ *as our denominator for* $g(x)$*, and our distinct roots will be* $\pm\frac{\sqrt{2}}{2}$*.*

Example 8 (Three Distinct Roots, Case 2)**.** *Let* $\gamma_1 = 2$*,* $\delta_1 = 2$*,* $\gamma_2 = 1$ *and* $\delta_2 = 1$*. Then we have* $1 - 4x + 4x^2 - x^4$ *as our denominator for* $g(x)$*, our root of multiplicity* 2 *will be* 1*, and our remaining roots will be* $-1 \pm \sqrt{2}$*.*

Example 9 (Three Distinct Roots, Case 3)**.** *Let* $\gamma_1 = 2$*,* $\delta_1 = 2$*,* $\delta_2 = 1$ *and* $\gamma_2 = 1$*. Then we have* $1 - 4x^2 - 4x^3 - x^4$ *as our denominator for* $g(x)$*, our root of multiplicity* 2 *will be* 1*, and our remaining roots will be* $-1 \pm \sqrt{2}$*.*

Table 1. a_n and b_n values for 2-Fibonacci sequences

Roots	**Case**	**Coefficients**	ϵ_i	a_0 / b_0	a_1 / b_1	a_2 / b_2	a_3 / b_3	a_4 / b_4	a_5 / b_5	a_6 / b_6	… / …
4	1	$\delta_1 = \delta_2 = 1$	$\epsilon_1 = \frac{1}{2} + \frac{\sqrt{5}}{2} = -\epsilon_2$	0	1	1	2	3	5	8	…
		$\gamma_1 = \gamma_2 = 1$	$\epsilon_3 = \frac{1}{2} - \frac{\sqrt{5}}{2} = -\epsilon_4$	0	1	1	2	3	5	8	…
4	2	$\delta_1 = \delta_2 = 2$	$\epsilon_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} = -\epsilon_2,$	0	1	2	3	16	44	120	…
		$\gamma_1 = \gamma_2 = 2$	$\epsilon_3 = \frac{1}{2} - \frac{\sqrt{3}}{2} = -\epsilon_4$	0	1	2	3	16	44	120	…
4	3	$\delta_1 = \delta_2 = 2$	$\epsilon_1 = -1, \epsilon_2 = 2,$	0	1	1	3	5	11	21	…
		$\gamma_1 = \gamma_2 = 2$	$\epsilon_3, \epsilon_4 = -\frac{1}{2} \pm \frac{i\sqrt{7}}{2}$	0	1	1	3	5	11	21	…
2	1	$\delta_1 = -1, \gamma_1 = 1$	$\epsilon_1, \epsilon_2 = \pm\sqrt{2}$	0	1	1	3	4	8	12	…
		$\delta_2 = 1, \gamma_2 = 4$		0	1	-1	0	-4	-4	-12	…
2	2	$\delta_1 = \delta_2 = -2,$	$\epsilon_1, \epsilon_2 = \pm\sqrt{2}$	0	1	2	6	8	20	24	…
		$\gamma_1 = \gamma_2 = 2$		0	1	-2	2	-8	4	-24	…
2	3	$\delta_1 = \delta_2 = 2,$	$\epsilon_1, \epsilon_2 = \pm\sqrt{2}$	0	1	2	2	8	4	24	…
		$\gamma_1 = 2, \gamma_2 = -2$		0	1	2	6	8	20	24	…
3	2	$\delta_1 = \gamma_1 = 2,$	$\epsilon_1 = 1,$	0	1	2	5	12	29	70	…
		$\delta_2 = \gamma_2 = 1$	$\epsilon_2, \epsilon_3 = 1 \pm \sqrt{2}$	0	1	2	5	12	29	70	…
3	3	$\delta_1 = \gamma_1 = 2,$	$\epsilon_1 = -1,$	0	1	2	5	12	29	70	…
		$\delta_2 = \gamma_2 = 1$	$\epsilon_2, \epsilon_3 = 1 \pm \sqrt{2}$	0	1	2	5	12	29	70	…

5 Conclusion

We have shown that we can express a 2-Fibonacci sequence with arbitrary coefficients and initial conditions using a single order-4 linear recurrence relation. We have also constructed a generating function to obtain a closed form or Binet formula for each of these sequences. We have found necessary and sufficient conditions on the coefficients corresponding to each possible form of the Binet formula, and also which forms are not possible, along with examples for each possible scenario. We extend these concepts to the case where the coefficients are periodic sequences of period p.

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