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Some results on geometric circulant matrices involving the Leonardo numbers

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Abstract: In this study, by the motivation of the papers in the literature, we construct a special geometric circulant matrix Le_{r^*} whose entries are the Leonardo numbers. Then, we investigate some linear algebraic properties of these matrices. More specifically, we present some bounds for the spectral norm, as well as Euclidean norm of this matrix form. For this purpose, we benefit from the spectacular properties of the Leonardo numbers. Furthermore, we throw light on the obtained results with examples. In addition to all these, we give two Matlab code in order to calculate the results related norms more easily and more accurately in a short time in the computer environment.

Keywords: Circulant matrices, Geometric circulant matrices, Leonardo numbers, Matrix norm, Spectral norm.
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1 Introduction

1.1 A brief review on the development geometric circulant matrix

The circulant and r-circulant matrices have important applications in many fields of mathematics such as numerical analysis, probability, coding theory. An n-square matrix C_r is called an r-circulant matrix if it is defined as follows:

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}. \tag{1}$$

This matrix form was first proposed by Davis in [8], then one found it has many spectacular properties, and it is one of the most important research subject in the field of the computation and pure mathematics [1,20]. Afterwards, Kızılateş and Tuglu constructed a new geometric circulant matrix [15]. This matrix representation is characterized as follows:

$$C_{r^*} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}. \tag{2}$$

In addition, they analyzed the bounds for the spectral norms of geometric circulant matrices associated with the generalized Fibonacci number and Lucas numbers. It is well-known form (1) and (2) that C_r and C_{r^*} are determined by the coefficient r and the first row elements of the matrix. When the coefficient satisfies $r = 1$, we emphasize that this geometric circulant matrix rotates classically known as circulant matrix.

The r-circulant matrix and geometric circulant matrix with some commonly chosen values have been hot topic for many researchers in recent years. For example, many scholars have investigated the spectral norms of circulant matrices and r-circulant matrices involving famous sequences such as Fibonacci and Lucas sequences. In this sense, we can compile many studies such as these and others in the literature as follows. In [18], Shen and Cen have given upper and lower bounds for the spectral norms of r-circulant matrices whose entries are the Fibonacci and Lucas numbers. In [12], Jiang and Zhou studied spectral norms of even order r-circulant matrices. In [3], Bahsi computed the spectral norms of circulant and r-circulant matrices involving the hyperharmonic numbers. In addition to all these, in [4], Bahsi and Solak investigated norms of circulant and r-circulant matrices associated with the hyper-Fibonacci and hyper-Lucas numbers. In [9], He et al. present the upper bound estimation of the spectral norm for r-circulant matrices with Fibonacci and Lucas numbers. Recently, In [15], Kızılateş and Tuğlu have defined geometric circulant matrices and studied the bounds for the spectral norms of geometric circulant matrices involving the generalized Fibonacci number and Lucas numbers. Some great contributions for the spectral norms of r-circulant matrix and geometric circulant matrix can be found in references [5,9,13,14,16].

1.2 A brief review on the development Leonardo numbers

Leonardo numbers are introduced and given some properties by Catarino and Borges in [6]. When the studies in the existing literature, it is observed that there has been a great interest in the study of sequences of integers and their applications in various scientific fields. In this sense, there are many interesting sequences of integers in literature. But, one of the most widely investigated number sequence is the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ defined by the following recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \ \ n \geq 2,$$

together with the initial conditions $F_0 = 0$ and $F_1 = 1$.

In this paper, we will reckon with another sequence that has similar properties to the Fibonacci sequence. It is called Leonardo sequence and denoted with Le_n that is nth Leonardo numbers. In [19], Shannon has constructed generalized Leonardo numbers which are considered Asveld's extension and Horadam's generalized sequence. In [21], Vieira et. al. studied the two-dimensional recurrences relations of Leonardo numbers from its one-dimensional model. In [2], Alp and Koçer have given matrix representation of Leonardo numbers and obtained new identities of Leonardo numbers. In [7], the authors have introduced incomplete Leonardo numbers and given some properties of incomplete Leonardo numbers.

1.3 The motivation for our research

When the studies in the literature are examined, it is observed some well-known number sequences are generally used in the studies on geometric circulant matrix. From a similar mathematical point of view, we investigate a geometric circulant matrix such that its elements are a number sequence different from existing number sequences. More specifically, we take the elements of the geometric circulant matrix to be the Leonardo number sequence. In fact, considering the value of the number sequences, it sounds interesting to imagine geometric circulant matrices whose entries are the Leonardo numbers. Thus, many notable questions naturally arise. In this respect, we answer basic questions such as Euclidean norm and the bounds for spectral norm in this study. With the motivation provided by the studies in the literature, we fully believe that this theoretical contribution made for geometric circulant matrices will enrich the applications.

This paper has been organized as follows. The necessary definitions and results for some related sequences of numbers, circulant matrices and their norms are presented in Section 2. In Section 3, the main results obtained for some matrix norms are given. Finally, In Section 4, numerical examples with coding application are included to support the results we have obtained.

2 Preliminaries

In this section, we remind some preliminary details concerning matrix norms and Leonardo numbers for readers. In addition, we introduce some related basic notions and results Thus, we prepare a background for the theory that we will present in the rest of our paper.

2.1 Some notes on matrix norms

Definition 2.1. [10] *Let $M = (m_{ij})$ and $N = (n_{ij})$ be n-square real matrices. Then, the Hadamard product of these matrices is defined as*

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \circ \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} = \begin{bmatrix} m_{11}n_{11} & m_{12}n_{12} & m_{13}n_{13} \\ m_{21}n_{21} & m_{22}n_{22} & m_{23}n_{23} \\ m_{31}n_{31} & m_{32}n_{32} & m_{33}n_{33} \end{bmatrix}.$$

It is obvious that the norm of a matrix is a non-negative real number. There are several different methods of defining a matrix norm, but they all share the same definite characteristics. Now, let us remember some well-known matrix norm types.

Definition 2.2. [11] *Let $A = (a_{ij})$ be an n-square matrix. Then, the maximum column length norm, denoted by $c_1(.)$, and the maximum row length norm, denoted by $r_1(.)$, are respectively defined as follows:*

$$c_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2} \quad , \quad r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2}.$$

Definition 2.3. [11] *Let $A = (a_{ij})$ be an n-square matrix. Then, the well-known Frobenius (or Euclidean) norm of matrix A is defined by*

$$\|A\|_{\mathbb{E}} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.$$

Definition 2.4. [11] *Let $A = (a_{ij})$ be an n-square matrix. Then, the spectral norm of matrix A is defined by*

$$\|A\|_2 = \max\{\sqrt{\lambda} \ : \ \lambda \textit{ is an eigenvalue of } A^*A\},$$

where A^* is a conjugate transpose of matrix A.

Lemma 2.1. [11] *The following inequalities hold for the Euclidean norm and the spectral norm:*

$$\frac{1}{\sqrt{n}}\|A\|_{\mathbb{E}} \leq \|A\|_2 \leq \|A\|_{\mathbb{E}}, \tag{3}$$

and

$$\|A\|_2 \leq \|A\|_{\mathbb{E}} \leq \sqrt{n}\|A\|_2. \tag{4}$$

Lemma 2.2. [17,22] *Let A, B and C be $m \times n$ matrices. In this case, the following inequalities hold*

(i) If $A = B \circ C$, then

$$\|A\|_2 \leq r_1(B)c_1(C). \tag{5}$$

(ii) If $\|.\|$ is arbitrary norm on $n \times m$ matrices, then

$$\|A \circ B\| \leq \|A\|\|B\|.$$

2.2 Some notes on Leonardo numbers

Definition 2.5. [6] *Leonardo sequence is defined by the following recurrence relation for* $n \geq 2$

$$Le_n = Le_{n-1} + Le_{n-2} + 1$$

together with the initial contiditons $Le_0 = Le_1 = 1$.

In the following Table 1, we give some values of the Leonardo numbers.

Table 1. Some of the Leonardo numbers.

The value of n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\boldsymbol{Le_n}$	1	1	3	5	9	15	25	41	67	109	177	287	465

In this respect, it is well-known that there is an equation between the Leonardo numbers which is occasionally more appropriate as follows:

Lemma 2.3. *[6] For* $n \geq 2$*, the following equality is hold*

$$Le_{n+1} = 2Le_n - Le_{n-2} \tag{6}$$

where Le_n *is* n*th Leonardo number.*

Considering the above lemma, it can be easily seen that the characteristic equation of recurrence (6) is $\lambda^3 - 2\lambda^2 + 1 = 0$. Keeping this in mind, the Binet's formula of leonardo numbers is characterized as in the following theorem.

Theorem 2.1. [6] *The Binet's formula of the Leonardo number* Le_n *is characterized by*

$$Le_n = \frac{2\alpha^{n+1} - 2\beta^{n+1} - \alpha + \beta}{\alpha - \beta},$$

where α *and* β *are roots of characteristic equation of recurrence* (6).

After considering the motivation presented in Chapter 1.2, a question naturally arises: Is there a relationship between the fibonacci numbers and the leonardo numbers? In the following lemma, we give an affirmative answer to this question.

Lemma 2.4. [6] *The following equality holds*

$$Le_n = 2F_{n+1} - 1, \tag{7}$$

where F_n *is the* n*-th Fibonacci number.*

Proposition 2.1. [6] *For* $n \geq 0$*, the following identity holds*

$$\sum_{j=0}^{n} Le_k^2 = 4(F_{n+1} - 1)(F_{n+2} - 1) + (n + 1),$$

where Le_j *is the* j*-th Leonardo number and* F_j *is the* j*-th Fibonacci number.*

When the existing literature is examined, it is seen that Leonardo numbers have many properties, as well as it is well-known that there are many relationships between other famous number sequences other than Fibonacci numbers. In this sense, we refer the reader to the papers [5, 6, 14, 19] that provide a nice overview on this topic.

3 Main results

In this section, we consider the n-square geometric circulant matrix Le_{r^*} associated with the Leonardo numbers visualized in (8), at first. Afterwards, we present attractive results for Euclidean norms, as well as some bounds for the spectral norms of the matrix Le_{r^*}.

$$Le_{r^*} = \begin{pmatrix} Le_0 & Le_1 & Le_2 & \dots & Le_{n-2} & Le_{n-1} \\ rLe_{n-1} & Le_0 & Le_1 & \dots & Le_{n-3} & Le_{n-2} \\ r^2Le_{n-2} & rLe_{n-1} & Le_0 & \dots & Le_{n-4} & Le_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-1}Le_1 & r^{n-2}Le_2 & r^{n-3}Le_e & \dots & rLe_{n-1} & Le_0 \end{pmatrix}. \tag{8}$$

Theorem 3.1. *Let $Le_{r^*} = Circ_{r^*}(Le_0, Le_1, Le_2, \dots, Le_{n-1})$ be an $n \times n$ circulant matrix.*

(i) If $|r| > 1$, then

$$\sqrt{4(F_n-1)(F_{n+1}-1)+n} \leq \|Le_{r^*}\|_2 \leq \sqrt{\frac{(1-|r|^{2n})[4(F_n-1)(F_{n+1}-1)+n]}{1-|r|^2}}.$$

(ii) If $|r| < 1$, then

$$\frac{|r|}{\sqrt{5}}\sqrt{\frac{-4\alpha(|r|^{2n}-\alpha^n)}{|r|^2-\alpha}+\frac{-4\beta(|r|^{2n}-\beta^n)}{|r|^2-\beta}+\frac{-3|r|^{2n}+3}{|r|^2-1}+\frac{(6-2\sqrt{5})(|r|^{2n}-\beta^{2n})}{|r|^2-\beta^2}+\frac{(6+2\sqrt{5})(|r|^{2n}-\alpha^{2n})}{|r|^2-\alpha^2}}$$
$$\leq \|Le_{r^*}\|_2$$

and

$$\|Le_{r^*}\|_2 \leq \sqrt{4n(F_n-1)(F_{n+1}-1)+(n+1)}.$$

Proof. We have the matrix

$$Le_{r^*} = \begin{pmatrix} Le_0 & Le_1 & Le_2 & \dots & Le_{n-2} & Le_{n-1} \\ rLe_{n-1} & Le_0 & Le_1 & \dots & Le_{n-3} & Le_{n-2} \\ r^2Le_{n-2} & rLe_{n-1} & Le_0 & \dots & Le_{n-4} & Le_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-1}Le_1 & r^{n-2}Le_2 & r^{n-3}Le_e & \dots & rLe_{n-1} & Le_0 \end{pmatrix}.$$

(i) From $|r| > 1$ and the definition of Euclidean norm, we have

$$\begin{aligned} \|Le_{r^*}\|_E^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{i,j}|^2 \\ &= \sum_{k=0}^{n-1}(n-k)Le_k^2 + \sum_{k=1}^{n-1}k|r^{n-k}|^2Le_k^2 \\ &\geq \sum_{k=0}^{n-1}(n-k)Le_k^2 + \sum_{k=1}^{n-1}kLe_k^2 \\ &= n\sum_{k=0}^{n-1}Le_k^2 \\ &= n[4(F_n-1)(F_{n+1}-1)+n]. \end{aligned}$$

That is,
$$\frac{1}{\sqrt{n}}\|Le_{r^*}\|_E \geq \sqrt{4(F_n-1)(F_{n+1}-1)+n}$$
from (3), we have
$$\sqrt{4(F_n-1)(F_{n+1}-1)+n} \leq \|Le_{r^*}\|_2.$$
On the other hand, let the matrices $\mathbb{A}$ and $\mathbb{B}$ be defined by
$$\mathbb{A} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ r & 1 & 1 & \dots & 1 & 1 \\ r^2 & r & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & r & 1 \end{pmatrix}.$$
and
$$\mathbb{B} = \begin{pmatrix} Le_0 & Le_1 & Le_2 & \dots & Le_{n-2} & Le_{n-1} \\ Le_{n-1} & Le_0 & Le_1 & \dots & Le_{n-3} & Le_{n-2} \\ Le_{n-2} & Le_{n-1} & Le_0 & \dots & Le_{n-4} & Le_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Le_1 & Le_2 & Le_e & \dots & Le_{n-1} & Le_0 \end{pmatrix}.$$
That is, $Le_{r^*} = \mathbb{A} \circ \mathbb{B}$. Then, we obtain
$$\begin{aligned} r_1(\mathbb{A}) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} &= \sqrt{|r^{n-1}|^2 + \ldots + |r|^2 + 1} \\ &= \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}} \end{aligned}$$
and
$$\begin{aligned} c_1(\mathbb{B}) = \max_{1\leq j\leq n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} &= \sqrt{\sum_{k=0}^{n-1} Le_k^2} \\ &= \sqrt{4(F_n-1)(F_{n+1}-1)+n}. \end{aligned}$$
Hence, from Lemma 2.2, (5), we have
$$\|Le_{r^*}\|_2 \leq \sqrt{\frac{(1-|r|^{2n})[4(F_n-1)(F_{n+1}-1)+n]}{1-|r|^2}}.$$
Thus, we have
$$\sqrt{4(F_n-1)(F_{n+1}-1)+n} \leq \|Le_{r^*}\|_2 \leq \sqrt{\frac{(1-|r|^{2n})[4(F_n-1)(F_{n+1}-1)+n]}{1-|r|^2}}.$$
This completes the proof of (i).

(ii) From $|r|<1$, we have

$$\begin{aligned}
\|Le_{r^*}\|_E^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}|a_{i,j}|^2 \\
&= \sum_{k=0}^{n-1}(n-k)Le_k^2 + \sum_{k=1}^{n-1}k|r^{n-k}|^2 Le_k^2 \\
&\geq \sum_{k=0}^{n-1}(n-k)|r^{n-k}|^2 Le_k^2 + \sum_{k=1}^{n-1}k|r^{n-k}|Le_k^2 \\
&= n|r|^{2n}\sum_{k=0}^{n-1}\left(\frac{Le_k}{|r|^k}\right)^2 && \textit{(Using related Binet's formula)} \\
&= n|r|^{2n}\sum_{k=0}^{n-1}\left(\frac{\alpha(2\alpha^k-1)-\beta(2\beta^k-1)}{(\alpha-\beta)|r|^k}\right)^2 \\
&= \frac{n|r|^{2n}}{(\alpha-\beta)^2}\sum_{k=0}^{n-1}\left(\frac{\alpha(2\alpha^k-1)-\beta(2\beta^k-1)}{|r|^{2k}}\right)^2.
\end{aligned}$$

With a simple calculation, we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1}\left(\frac{\alpha(2\alpha^k-1)-\beta(2\beta^k-1)}{|r|^{2k}}\right)^2 =& -4\alpha\sum_{k=0}^{n-1}\left(\frac{\alpha}{|r|^2}\right)^k - 4\beta\sum_{k=0}^{n-1}\left(\frac{\beta}{|r|^2}\right)^k - 3\sum_{k=0}^{n-1}\left(\frac{1}{|r|^2}\right)^k \\
&+ (6+2\sqrt{5})\sum_{k=0}^{n-1}\left(\frac{\alpha^2}{|r|^2}\right)^k + (6-2\sqrt{5})\sum_{k=0}^{n-1}\left(\frac{\beta^2}{|r|^2}\right)^k.
\end{aligned}$$

Keeping this in mind, we reach the following equality.

$$\begin{aligned}
\sum_{k=0}^{n-1}\left(\frac{\alpha(2\alpha^k-1)-\beta(2\beta^k-1)}{|r|^{2k}}\right)^2 =& \frac{n|r|^2}{5}\left[\frac{-4\alpha(|r|^{2n}-\alpha^n)}{|r|^2-\alpha} + \frac{-4\beta(|r|^{2n}-\beta^n)}{|r|^2-\beta} + \frac{-3|r|^{2n}+3}{|r|^2-1}\right] \\
&+ \left[\frac{(6-2\sqrt{5})(|r|^{2n}-\beta^{2n})}{|r|^2-\beta^2} + \frac{(6+2\sqrt{5})(|r|^{2n}-\alpha^{2n})}{|r|^2-\alpha^2}\right].
\end{aligned}$$

That is,

$$\frac{1}{\sqrt{n}}\|Le_{r^*}\|_E \geq \frac{|r|}{\sqrt{5}}\sqrt{\left(\frac{-4\alpha(|r|^{2n}-\alpha^n)}{|r|^2-\alpha} + \frac{-4\beta(|r|^{2n}-\beta^n)}{|r|^2-\beta} + \frac{-3|r|^{2n}+3}{|r|^2-1} + \frac{(6-2\sqrt{5})(|r|^{2n}-\beta^{2n})}{|r|^2-\beta^2} + \frac{(6+2\sqrt{5})(|r|^{2n}-\alpha^{2n})}{|r|^2-\alpha^2}\right)}.$$

From (3), we have

$$\frac{|r|}{\sqrt{5}}\sqrt{\left(\frac{-4\alpha(|r|^{2n}-\alpha^n)}{|r|^2-\alpha} + \frac{-4\beta(|r|^{2n}-\beta^n)}{|r|^2-\beta} + \frac{-3|r|^{2n}+3}{|r|^2-1} + \frac{(6-2\sqrt{5})(|r|^{2n}-\beta^{2n})}{|r|^2-\beta^2} + \frac{(6+2\sqrt{5})(|r|^{2n}-\alpha^{2n})}{|r|^2-\alpha^2}\right)} \leq \|Le_{r^*}\|_2.$$

On the other hand, let the matrices $\mathbb{C}$ and $\mathbb{D}$ be defined by

$$\mathbb{C} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ r & 1 & 1 & \dots & 1 & 1 \\ r^2 & r & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \dots & r & 1 \end{pmatrix}.$$

and

$$\mathbb{D} = \begin{pmatrix} Le_0 & Le_1 & Le_2 & \dots & Le_{n-2} & Le_{n-1} \\ Le_{n-1} & Le_0 & Le_1 & \dots & Le_{n-3} & Le_{n-2} \\ Le_{n-2} & Le_{n-1} & Le_0 & \dots & Le_{n-4} & Le_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Le_1 & Le_2 & Le_3 & \dots & Le_{n-1} & Le_0 \end{pmatrix}.$$

That is, $Le_{r^*} = \mathbb{C} \circ \mathbb{D}$. Then, we obtain

$$r_1(\mathbb{C}) = \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{n}$$

and

$$c_1(\mathbb{D}) = \max_{1 \le j \le n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} Le_k^2} = \sqrt{4(F_n - 1)(F_{n+1} - 1) + n}.$$

Hence, from Lemma 2.2, (5), we have

$$\|Le_{r^*}\| \le \sqrt{n[4(F_n - 1)(F_{n+1} - 1) + n]}.$$

Thus,

$$\frac{|r|}{\sqrt{5}} \sqrt{\frac{-4\alpha(|r|^{2n} - \alpha^n)}{|r|^2 - \alpha} + \frac{-4\beta(|r|^{2n} - \beta^n)}{|r|^2 - \beta} + \frac{-3|r|^{2n} + 3}{|r|^2 - 1} + \frac{(6 - 2\sqrt{5})(|r|^{2n} - \beta^{2n})}{|r|^2 - \beta^2} + \frac{(6 + 2\sqrt{5})(|r|^{2n} - \alpha^{2n})}{|r|^2 - \alpha^2}}$$
$$\le \|Le_{r^*}\|_2$$

and

$$\|Le_{r^*}\|_2 \le \sqrt{n[4(F_n - 1)(F_{n+1} - 1) + n]}.$$

These also complete the proof of (ii). □

4 Numerical examples with a coding application

In this section, we aim to be our paper more comprehensible for the readers, at first. For this purpose, we add some illustrative numerical examples for the bounds of the spectral norm, as well as Euclidean norm of geometric circulant matrix whose entries are the Leonardo numbers.

Table 2. Some bounds for the norms of Le_{r^*} in case of $|r| > 1$.

$n \ge 2$	The lower bound for $\|Le_{r^*}\|_E$	The upper bound for $\|Le_{r^*}\|_2$	The lower bound for $\|Le_{r^*}\|_2$
2	2	$\sqrt{2(r^2+1)}$	$\sqrt{2}$
3	$\sqrt{33}$	$\sqrt{11(r^4+r^2+1)}$	$\sqrt{11}$
4	12	$\sqrt{36(r^6+r^4+r^2+1)}$	6

Table 3. Some bounds for the norms of Le_{r^*} in case of $|r| < 1$.

$n \geq 2$	The lower bound for $\|Le_{r^*}\|_E$	The upper bound for $\|Le_{r^*}\|_2$	The lower bound for $\|Le_{r^*}\|_2$
2	$\sqrt{2r^4 + 2r^2}$	2	$\sqrt{r^4 + r^2}$
3	$\sqrt{3r^6 + 3r^4 + 27r^2}$	$\sqrt{33}$	$\sqrt{r^6 + r^4 + 9r^2}$
4	$2\sqrt{r^8 + r^6 + 9r^4 + 25r^2}$	12	$\sqrt{r^8 + r^6 + 9r^4 + 25r^2}$

Table 4. Code 1: MATLAB-R2023a code for the matrix Le_{r^*} in case of $|r| > 1$.

```
clc;
clear all;
n=input("Enter the value of n=");
syms x r;
Le(1) = x;
Le(2) = 1;
for i = 3:n
      Le(i)=subs(Le(i-1) + Le(i-2)+1,Le(1),1);
      Le(i);
end
for i=1:n
      for j=1:n
            if i==j
              a(i,j)=subs(Le(1),Le(1),1);
            elseif i<j
              a(i,j)=subs(Le(mod(j-i,n)+1),Le(1),1);
            elseif i>j
              a(i,j)=subs(r^(mod(i-j,n))*Le(mod(j-i,n)+1),Le(1),1);
            end
      end
end
display(a,"Geometric circulant matrix with the Leonardo numbers for n")
disp("For |r|>1;")
b = subs(simplify(Le(1:n)),Le(1),1);
c = simplify(b.^2);
sum_Leonardo = cumsum(c);
sum_Leonardo_1 = simplify((sum_Leonardo(n))^(1/2));
row_norm_1 = 0;
for k = 0:n-1
         row_norm_1 = row_norm_1 + (r^2)^k;
end
row_norm = sqrt(row_norm_1)
column_norm = sum_Leonardo(n)
Euclidean_norm_greather_than = simplify(sqrt(n)*sum_Leonardo_1)
Spectral_norm_less_than = simplify(row_norm*column_norm)
Spectral_norm_greater_than=simplify(1/(sqrt(n))*Euclidean_norm_greather_than)
```

Table 5. Code 2: MATLAB-R2023a code for the matrix Le_{r^*} in case of $|r| < 1$.

```
clc;
clear all;
n=input("Enter the value of n=");
syms x r;
Le(1) = x;
Le(2) = 1;
for i = 3:n
      Le(i)=subs(Le(i-1) + Le(i-2)+1,Le(1),1);
      Le(i);
end
for i=1:n
      for j=1:n
            if i==j
              a(i,j)=subs(Le(1),Le(1),1);
            elseif i<j
              a(i,j)=subs(Le(mod(j-i,n)+1),Le(1),1);
            elseif i>j
              a(i,j)=subs(r^(mod(i-j,n))*Le(mod(j-i,n)+1),Le(1),1);
            end
      end
end
display(a,"Geometric circulant matrix with the Leonardo numbers for n")
disp("For |r|<1;")
d = subs(simplify(Le(1:n)),Le(1),1);
e = simplify(d.^2);
sum = cumsum(e);
for i=1:n
      e(i)=e(i)/r^(2*(i-1));
      e(i);
end
sum_Leonardo = cumsum(e);
sum_Leonardo_2 = simplify((n*r^(2*n)*sum_Leonardo(n))^(1/2));
row_norm = sqrt(n)
column_norm = simplify(sum(n)^(1/2))
Euclidean_norm_greather_than = simplify(sum_Leonardo_2)
Spectral_norm_less_than = simplify(row_norm*column_norm)
Spectral_norm_greater_than=simplify(1/(sqrt(n))*Euclidean_norm_greather_than)
```

5 Concluding remarks

In this paper, we take into account Leonardo numbers and we construct a special geometric circulant matrix Le_{r^*} whose elements are the Leonardo numbers. Afterwards, we investigate some linear algebraic properties of these matrices. More specifically, we give some inequalities for bounds of their spectral and Euclidean norms. Moreover, we give two MATLAB-R2023a codes for the matrices Le_{r^*} which is given the norm calculations of this matrix (see, Table 4 and Table 5). Here, we write a new MATLAB-R2023a code which is not contained in the existing Matlab libraries. For the value n entered into the code given,

- **Code;**

  1. writes the matrix Le_{r^*}.

  2. Code 1: For the matrix Le_{r^*} in case of $|r| > 1$,

    i. computes $r_1(\mathbb{A})$ and $c_1(\mathbb{B})$.

    ii. gives an upper and a lower bounds for $\|Le_{r^*}\|_2$.

    iii. gives a lower bound for $\|Le_{r^*}\|_E$.

  3. Code 2: For the matrix Le_{r^*} in case of $|r| < 1$,

    i. computes $r_1(\mathbb{D})$ and $c_1(\mathbb{C})$.

    ii. gives an upper and a lower bounds for $\|Le_{r^*}\|_2$.

    iii. gives a lower bound for $\|Le_{r^*}\|_E$.

We believe that all these can throw light on the researches that can be done about these topics in the future. In this respect, we expect applications of our results in several branches of mathematics.

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