Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2024, Volume 30, Number 1, 20–33
DOI: 10.7546/nntdm.2024.30.1.20-33

Proving the existence of Euclidean knight's tours on $n \times n \times \cdots \times n$ chessboards for $n < 4$

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Received: 5 September 2023 **Revised:** 10 January 2024
Accepted: 24 February 2024 **Online First:** 26 February 2024

Abstract: The Knight's Tour problem consists of finding a Hamiltonian path for the knight on a given set of points so that the knight can visit exactly once every vertex of the mentioned set. In the present, we provide a 5-dimensional alternative to the well-known statement that it is not ever possible for a knight to visit once every vertex of $C(3,k) := \{\underbrace{\{0,1,2\} \times \{0,1,2\} \times \cdots \times \{0,1,2\}}_{k\text{-times}}\}$ by performing a sequence of $3^k - 1$ jumps of standard length, since the most accurate answer to the original question actually depends on which mathematical assumptions we are making at the beginning of the game when we decide to extend a planar chess piece to the third dimension and above. Our counterintuitive outcome follows from the observation that we can alternatively define a 2D knight as a piece that moves from one square to another on the chessboard by covering a fixed Euclidean distance of $\sqrt{5}$ so that also the statement of Theorem 3 in [Erde, J., Golénia, B., & Golénia, S. (2012), The closed knight tour problem in higher dimensions, The Electronic Journal of Combinatorics, 19(4), #P9] does not hold anymore for such a Euclidean knight, as long as a $2 \times 2 \times \cdots \times 2$ chessboard with at least 2^7 cells is given. Moreover, we construct a classical closed knight's tour on $C(3,4) - \{(1,1,1,1)\}$ whose arrival is at a distance of 2 from $(1,1,1,1)$, and then we show a closed Euclidean knight's tour on $\{\{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}\} \subseteq \mathbb{Z}^7$.

Keywords: Knight's tour, Euclidean distance, Knight metric, Hamiltonian path.
2020 Mathematics Subject Classification: 05C12, 05C38, 05C57.

1 Introduction

Given a set $C \subseteq \mathbb{R}^k$ consisting of $m \in \mathbb{Z}^+$ points, it is commonly agreed that every knight's tour is a sequence of $m-1$ knight jumps, of Euclidean length $\sqrt{5}$ chessboard units each (where one chessboard unit is the distance between the centers of adjacent squares of the chessboard), that let the knight visit exactly once all the m vertices of C. In particular, we say that the knight's tour is closed if and only if the m-th visited vertex of C (including the starting vertex) is at a unit knight-distance from the beginning point, otherwise we have an open knight's tour on C.

The origins of the Knight's Tour problem are lost in the centuries, this being a thousands of years old puzzle [2] that lists among its contributors some very big names in mathematics, such as Abraham de Moivre, Alexandre-Théophile Vandermonde, Adrien-Marie Legendre, and Leonard Euler himself, who found one solution for the planar 8×8 configuration in 1759 [7, 18]. Euler's solution is an open knight's tour since the center of the last square visited by the knight is not at a distance of $\sqrt{2^2+1^2}$ chessboard units from the center of its starting square (in the most common sense, for arbitrary k, being the beginning vertex at a Euclidean distance of $\sqrt{5}$ from the arrival would represent a necessary but not sufficient condition for having a closed knight's tour).

Now, if we agree that the Euclidean $\sqrt{5}$-rule (see [8], Article 3.6, that uses the superlative of *near* as a criterion for the official knight move rule) defines also the knight metric for any k-dimensional $n \times n \times \cdots \times n$ chessboard (while a customizable definition of the discrete knight pattern is at the bottom of the generalized knight's tour problem in two and three dimensions, as described in [3] and [1], respectively), we trivially have that a k-knight is a mathematical object whose move rule consists of performing only jumps having Euclidean length equal to $\sqrt{5}$ chessboard units [9], from a cell of the given chessboard to one of the remaining $n^k - 1$ cells. Then, we can move our favorite chess piece from the vertex V_1, identified by the k-tuple of Cartesian coordinates $(x_1, x_2, \dots, x_k)$: $x_1, x_2, \dots, x_k \in \{0, 1, \dots, n-1\}$, to another one, $\mathrm{V}_2 \equiv (y_1, y_2, \dots, y_k)$ also belonging to $\{\underbrace{\{0,1,\dots,n-1\} \times \{0,1,\dots,n-1\} \times \cdots \times \{0,1,\dots,n-1\}}_{k\text{-times}}\}$, if and only if

$$\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+\cdots+(y_k-x_k)^2} = \sqrt{5}. \tag{1}$$

Hence, (1) can be compactly rewritten as

$$\sum_{j=1}^{k}(x_j-y_j)^2 = 5. \tag{2}$$

It is easily possible to show that our *Euclidean knight* produces a metric for every pair (n, k) that allows the usual knight to do so (i.e., $n \geq 4 \wedge k \geq 2$ represents a sufficient condition) [15], but it also induces a metric space on any given $3 \times 3 \times \cdots \times 3$ chessboard with at least 3^5 cells (Section 2) and even on every $2 \times 2 \times \cdots \times 2$ chessboard consisting of at least 2^7 cells (see Section 4, Theorem 4.1).

Furthermore, we construct an open Euclidean knight's tour on $\{\{0,1,2\} \times \{0,1,2\} \times \{0,1,2\} \times \{0,1,2\} \times \{0,1,2\}\} \subseteq \mathbb{Z}^5$ and provide a closed Euclidean knight's tour on $\{\{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}\} \subseteq \mathbb{Z}^7$ (which can be smoothly generalized, for any given $k > 6$, to $\{\{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}\} \subseteq \mathbb{Z}^k$).

Accordingly, Section 2 is devoted to proving the existence of Euclidean knight's tours when $n = 3$ is given, and then a pair of corollaries will follow, while Section 3 proposes a variation of the main problem [16] by removing the central vertex from any grid $\{\{0,1,2\} \times \{0,1,2\} \times \cdots \times \{0,1,2\}\} \subseteq \mathbb{Z}^k$ [12], under the additional constraint of ending the path in a vertex that is at a Euclidean distance of $\sqrt{k}$ from the missing point $(1,1,\ldots,1)$ (for the related problem of determining the existence of closed, conventional, knight's tours on boxes, see Theorem 1 of [13]). Finally, Section 4 entirely covers the $n = 2$ case.

2 Euclidean knight's tours on a $3 \times 3 \times 3 \times 3 \times 3$ chessboard

Here we consider the problem of finding (possibly open) Euclidean knight's tours on $3 \times 3 \times \cdots \times 3$ chessboards. Then, in order to constructively show that a Euclidean knight's tour exists only if $k > 4$, it is sufficient to point out that the (Euclidean) distance between the vertex $(1,1,\ldots,1)$ and one of the farthest 2^k vertices (*corners*) of the k-cube $\{[0,2] \times [0,2] \times \cdots \times [0,2]\} \subseteq \mathbb{R}^k$ (i.e., the distance between $(1,1,\ldots,1)$ and any element of $\{\{0,2\} \times \{0,2\} \times \cdots \times \{0,2\}\} \subseteq \mathbb{Z}^k$) is equal to $\sqrt{k}$, for any positive integer k.

Hence, starting at $(1,1,\ldots,1)$, by Equation (2), 5 is the minimum value of k such that our Euclidean knight can make one single move on the $3 \times 3 \times \cdots \times 3$ grid.

In this regard, let us observe how Qing and Watkins indicated a different way to extend in 3D the planar knight's move pattern by proposing, in [13], pages 45-46, that every knight jump has to mandatorily change all the $k = 3$ Cartesian coordinates of its starting vertex (by 2^0, 2^1, and 2^2). Although this personal interpretation of Article 3.6 of [8] is obviously not compatible with the existence of any knight's tour on $3 \times 3 \times \cdots \times 3$ grids (since the Euclidean distance between $(1,1,\ldots,1)$ and $(0,0,\ldots,0)$ is equal to $\sqrt{k}$, which is clearly smaller than $\sqrt{(2^0)^2 + (2^1)^2 + (2^2)^2 + \cdots + (2^{k-1})^2} = \sqrt{\sum_{j=0}^{k-1} 4^j} = \frac{\sqrt{4^k-1}}{\sqrt{3}}$ for any $k > 1$), the underlying idea of a k-knight that can change (or has to mandatorily change) the values of all its k Cartesian coordinates by performing a single move is fascinating and useful [11].

Now we are ready to prove that an open Euclidean knight's tour actually exists if $n = 3$ and k is set at 5, so the trivial consideration that the knight graph is not connected for any $3 \times 3 \times \cdots \times 3$ board does not apply anymore, as the k-knight is a Euclidean k-knight.

Theorem 2.1. *Let $h \in \{0,1,2,\ldots,3^k - 1\}$ and assume that the knight move rule from the vertex $\mathrm{V}_h \equiv (x_1, x_2, \ldots, x_k)$ to the next vertex, $\mathrm{V}_{h+1} \equiv (y_1, y_2, \ldots, y_k)$, of $C(3,k) := \{\{0,1,2\} \times \{0,1,2\} \times \cdots \times \{0,1,2\}\}$ is given by $d(\mathrm{V}_h, \mathrm{V}_{h+1}) := \sqrt{\sum_{j=1}^{k}(x_j - y_j)^2} = \sqrt{5}$. Then, the minimum value of k that produces a knight's tour on $C(3,k)$ is 5.*

Proof. As we have already observed, $d((0,0,\ldots,0),(1,1,\ldots,1)) = \sqrt{k}$ (for any $k \in \mathbb{Z}^+$) and this implies that $d((0,0,\ldots,0),(1,1,\ldots,1)) < \sqrt{5}$ if and only if $k < 5$. Thus, k cannot be less than 5.

Accordingly, let us constructively prove Theorem 2.1 by simply providing the sequence of the 243 Cartesian coordinates that describe an open knight's tour on the set $C(3,5)$ (see Figure 1 for a graphical proof), since every vertex is visited exactly once and the Euclidean length of each knight jump is equal to $\sqrt{5}$.

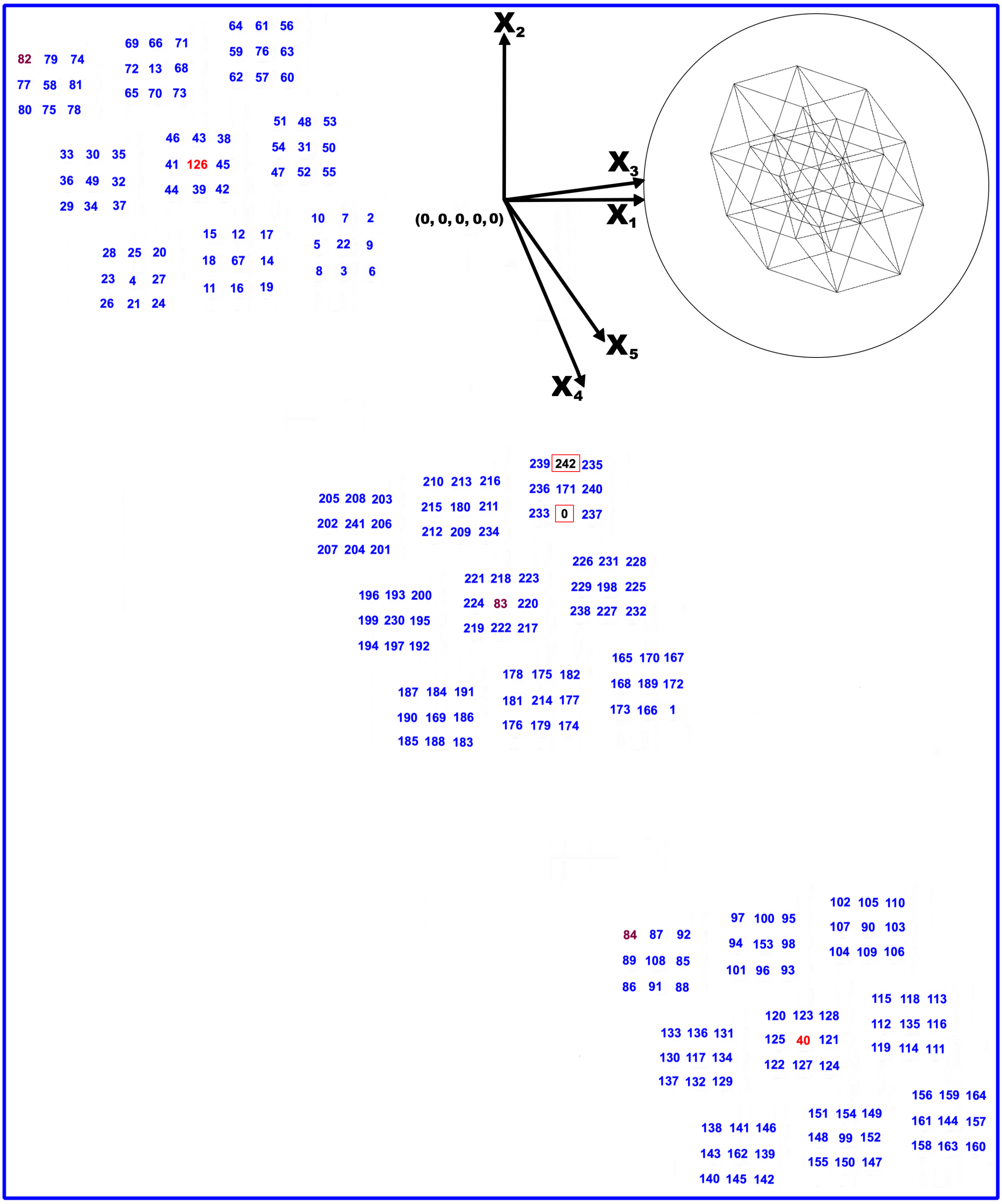

Figure 1. A graphical representation of an open Euclidean knight's tour on $C(3, k) := \{\{0, 1, 2\} \times \{0, 1, 2\} \times \cdots \times \{0, 1, 2\}\}$.

Then, the polygonal chain $P_o(3, 5) := (1, 0, 2, 0, 1) \to (2, 0, 2, 2, 1) \to (2, 2, 2, 2, 0) \to (1, 0, 2, 2, 0) \to (1, 1, 0, 2, 0) \to (0, 1, 2, 2, 0) \to (2, 0, 2, 2, 0) \to (1, 2, 2, 2, 0) \to (0, 0, 2, 2, 0) \to (2, 1, 2, 2, 0) \to (0, 2, 2, 2, 0) \to (0, 0, 1, 2, 0) \to (1, 2, 1, 2, 0) \to (1, 1, 1, 0, 0) \to (2, 1, 1, 2, 0) \to (0, 2, 1, 2, 0) \to (1, 0, 1, 2, 0) \to (2, 2, 1, 2, 0) \to (0, 1, 1, 2, 0) \to (2, 0, 1, 2, 0) \to (2, 2, 0, 2, 0) \to (1, 0, 0, 2, 0) \to (1, 1, 2, 2, 0) \to (0, 1, 0, 2, 0) \to (2, 0, 0, 2, 0) \to (1, 2, 0, 2, 0) \to (0, 0, 0, 2, 0) \to (2, 1, 0, 2, 0) \to (0, 2, 0, 2, 0) \to (0, 0, 0, 1, 0) \to (1, 2, 0, 1, 0) \to (1, 1, 2, 1, 0) \to (2, 1, 0, 1, 0) \to$

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has link length 242 and represents a knight's tour for the given set (let us highlight that also $d(\mathrm{V}_{82}, \mathrm{V}_{83}) = d(\mathrm{V}_{83}, \mathrm{V}_{84}) = \sqrt{5}$, even if this time the taxicab length [10, 17] of our knight jump is equal to 5, instead of 3 as for any other knight move), while the Euclidean distance between its starting point and ending point is 2, so the knight tour is open.

Therefore, a knight tour exists for the set $\{\{0,1,2\} \times \{0,1,2\} \times \{0,1,2\} \times \{0,1,2\} \times \{0,1,2\}\} \subseteq \mathbb{Z}^5$, while it is impossible to achieve it on $C(3,k)$ if $k < 5$, and this concludes the proof of the theorem. $\square$

Corollary 2.1. *For any given $k \in \mathbb{N} - \{0, 1\}$, it does not exist any closed Euclidean knight's tour on $C(3, k)$.*

Proof. In order to prove Corollary 2.1, let us introduce the well-known parity argument [13]. Then, we have that the generic vertex $(x_1, x_2, \ldots, x_k)$ of $\{\{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, n-1\} \times \cdots \times \{0, 1, \ldots, n-1\}\} \subseteq \mathbb{Z}^k$ is a dark vertex if and only if $\sum_{j=1}^{k} x_j \equiv 0 \pmod 2$, whereas any light vertex is such that $\sum_{j=1}^{k} x_j \equiv 1 \pmod 2$ (i.e., if we add together the k Cartesian coordinates of a dark vertex of C the result is always an even number, otherwise the given vertex is a light vertex). Figure 2 shows how to consistently represent the set $C(3, 5)$ as a 5D chessboard.

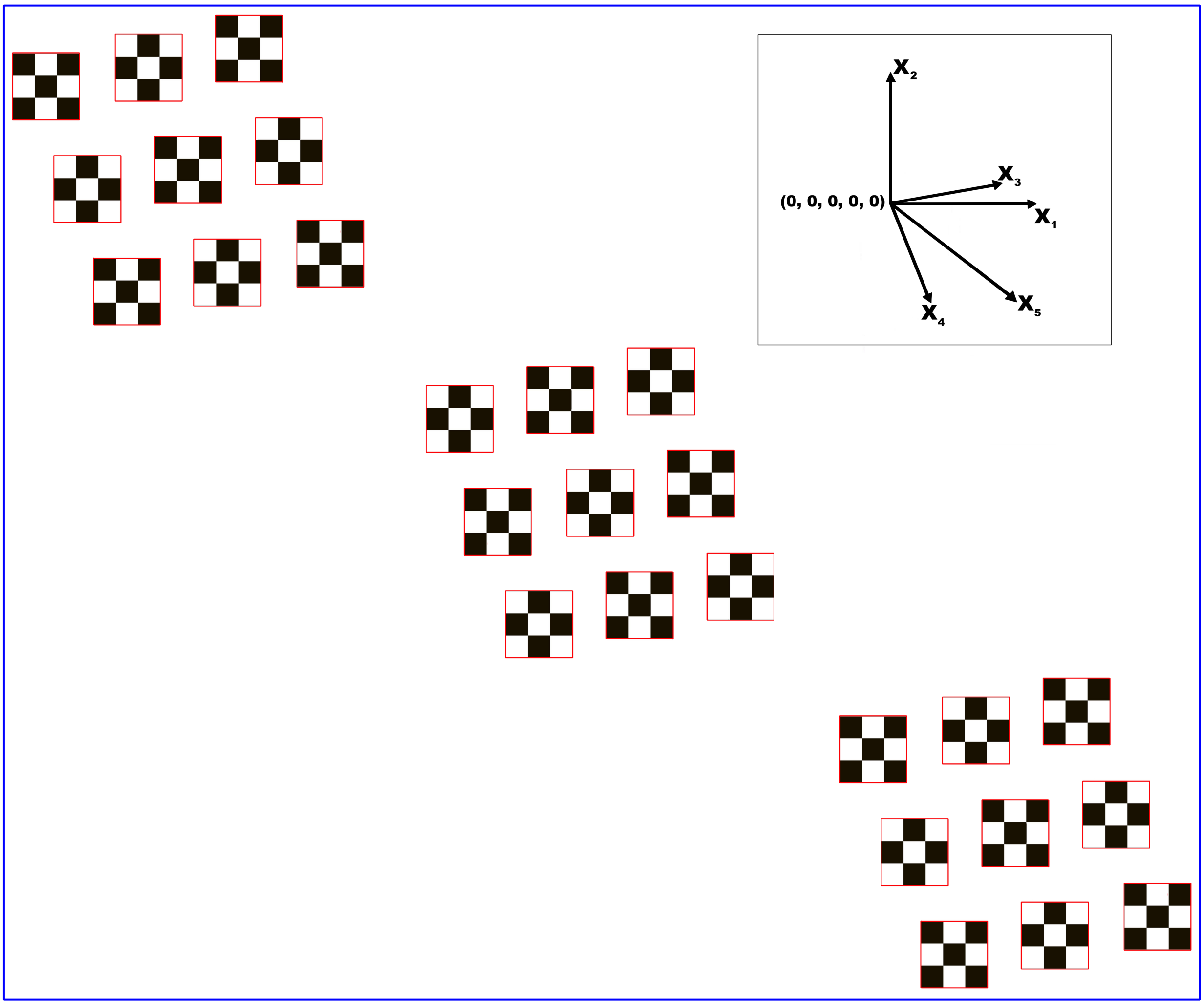

Figure 2. Coloring the $3 \times 3 \times 3 \times 3 \times 3$ chessboard in a proper way (thanks to the parity argument).

Thus, we invoke the parity argument by observing that the knight (including the Euclidean knight which is subject to the same constraint by construction, since $|1|+|1|+|1|+|1|+|1|$ is odd as $|1| + |2|$) can only move from a dark square to a light square and vice versa (see [4], Figure 2). For this purpose, let us observe that the taxicab length [17] of any Euclidean knight jump has to necessarily be 3 or 5, since all the Diophantine equations of the form $5 = t_1^2 + t_2^2 + \cdots + t_k^2$ (see

Equation (2)) admit only two types of solutions, one having only 2 non-zero terms (i.e., a ± 1 term and a ± 2 term), while the second set of solutions is characterized by 5 non-zero terms that are all elements of $\{-1, 1\}$.

Hence, a necessary (but not sufficient) condition for having a closed (possibly Euclidean) knight's tour on $C(n, k)$ is that the number of light vertices is equal to the number of dark vertices [13], and this is obviously impossible if $n = 3$ since 3^k is odd for any k.

Therefore, every $3 \times 3 \times \cdots \times 3$ chessboard does not admit any closed Euclidean knight's tour. $\square$

Corollary 2.2. *A Euclidean knight can visit every vertex of* $C(3, 5)$ *and then return to the vertex on which it began by performing no more than* $3^5 + 1$ *jumps.*

Proof. It is sufficient to observe that we can close the polygonal chain $P_o(3, 5)$ (see proof of Theorem 2.1 and Figure 1) by visiting the vertex $(1, 1, 0, 0, 1)$ on move 243, finally reaching the starting vertex, $(1, 0, 2, 0, 1)$, on move 244 (i.e., we visit twice the vertex $(1, 1, 0, 0, 1)$). Then, the statement of Corollary 2.2 trivially follows. $\square$

3 Closed knight's tours on $C(3, k) - \{(1, 1, \ldots, 1)\}$

In this section, we introduce the problem of finding closed (possibly Euclidean) knight's tours on given k-dimensional grids of the form $\{\{0, 1, 2\} \times \{0, 1, 2\} \times \cdots \times \{0, 1, 2\}\} - \{(1, 1, \ldots, 1)\}$ (for planar knight's tours on rectangular chessboards with holes, see [5] and [12]).

Then, we constructively show that a closed regular knight's tour certainly exists on $C(3, k) - \{(1, 1, \ldots, 1)\}$ if $k \in \{2, 4\}$ (i.e., here we consider the usual kD grids of rank 3 without their central vertex, together with the classical knight that can only make L-shaped moves of taxicab length 3), where the existence of a closed (regular) knight's tour represents a sufficient condition for the existence of an open knight's tour on the same set of vertices, even under the constraint of having the arrival of the polygonal chain at a Euclidean distance of $\sqrt{k}$ from $(1, 1, \ldots, 1)$.

Since any closed/open Euclidean knight's tour is also a closed/open regular knight's tour as long as $k < 5$, for the sake of simplicity, we take into account here only the sets $C(3, 2) - \{(1, 1)\}$, $C(3, 3) - \{(1, 1, 1)\}$, and $C(3, 4) - \{(1, 1, 1, 1)\}$, showing that a closed knight's tour exists for the aforementioned 2D and 4D cases.

Thus, if $k = 2$, then a valid solution to the stated problem is given by the trivial polygonal chain $P_c(3, 2; \{(1, 1)\}) := (2, 1) \to (0, 2) \to (1, 0) \to (2, 2) \to (0, 1) \to (2, 0) \to (1, 2) \to (0, 0)$ (see Figure 3, and also Figure 1 of Reference [12] for the related cycle $P_c(3, 2; \{(1, 1)\}) \cup \{(0, 0) \to (2, 1)\}$).

If $k = 3$, the best achievable sequence of knight jumps has length 24 (see [14], page 66, Figures 4&5). In addition, Figure 3 shows another polygonal chain, $\overline{P}(3, 3; \{(1, 1, 1), (2, 0, 0)\}) := (0, 0, 0) \to (1, 2, 0) \to (1, 1, 2) \to (0, 1, 0) \to (2, 2, 0) \to (1, 0, 0) \to (0, 2, 0) \to (2, 1, 0) \to (0, 1, 1) \to (2, 2, 1) \to (1, 0, 1) \to (0, 2, 1) \to (2, 1, 1) \to (0, 0, 1) \to (1, 2, 1) \to (2, 0, 0) \to (2, 2, 2) \to (1, 0, 2) \to (0, 2, 2) \to (2, 1, 2) \to (0, 0, 2) \to (1, 2, 2) \to (1, 1, 0) \to (0, 1, 2) \to (2, 0, 2)$, consisting of 25 nodes and 24 links, too. Now, by looking at the arrival of $\overline{P}(3, 3; \{(1, 1, 1), (2, 0, 0)\})$, it is immediate to note that we would have got an open Euclidean knight's tour on $C(3,3)$ if the distance between the corners of $\{\{0,1,2\} \times \{0,1,2\} \times \{0,1,2\}\}$ and the center of the same grid would have been equal to $\sqrt{5}$, instead of $\sqrt{3}$.

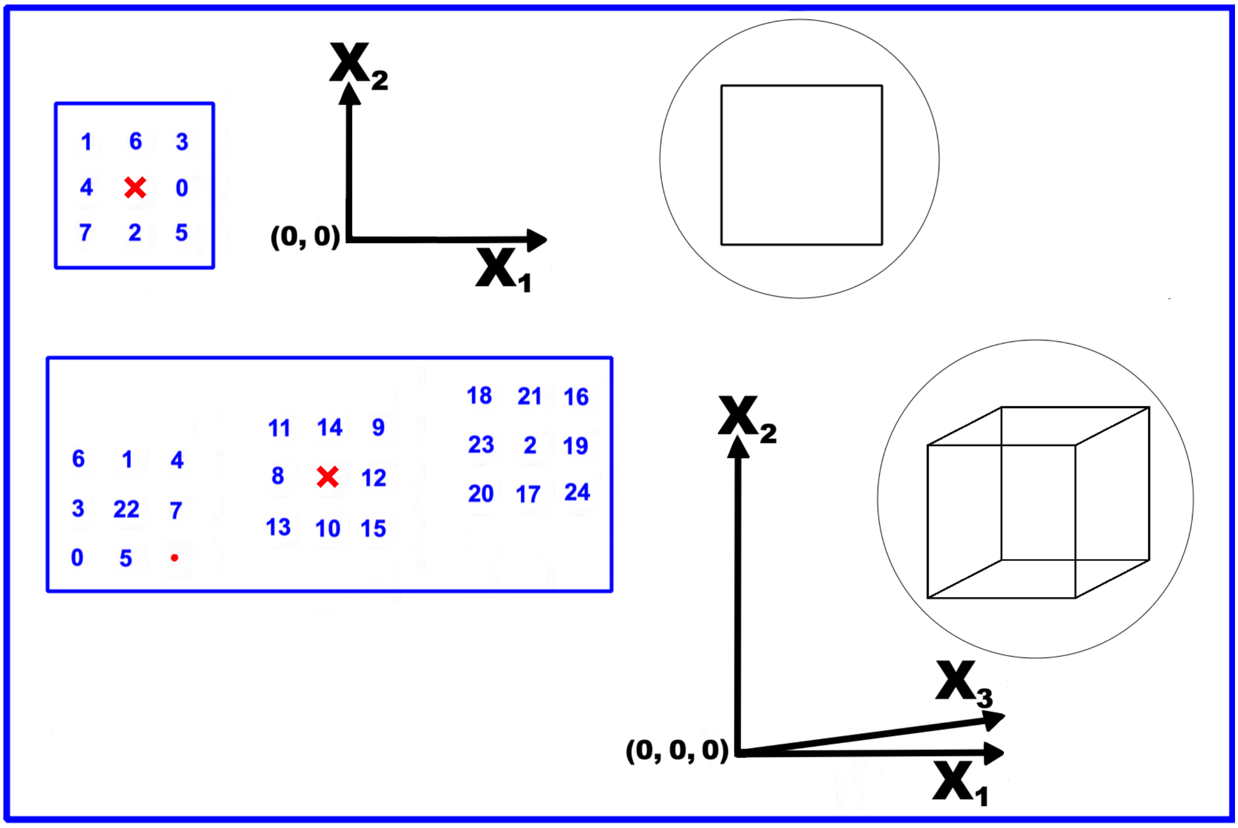

Figure 3. The polygonal chains $P_c(3,2;\{(1,1)\})$ and $\overline{P}(3,3;\{(1,1,1),(2,0,0)\})$ visit all the vertices of $C(3,2)-\{(1,1)\}$ and $C(3,3)-\{(1,1,1),(2,0,0)\}$, respectively. Both of them follow the regular (and thus also the Euclidean) knight move rule and end at the corners of C.

Now, let $k = 4$. Then, it is certainly possible to achieve closed knight's tours on $C(3,4)-\{(1,1,1,1)\}$, as shown by the Hamiltonian cycle $P_c(3,4;\{(1,1,1,1)\}) \cup \{(0,2,0,1) \rightarrow (0,0,0,2)\}$ (see Figure 4), where $P_c(3,4;\{(1,1,1,1)\}) := (0,0,0,2) \rightarrow (2,1,0,2) \rightarrow (0,2,0,2) \rightarrow (1,0,0,2) \rightarrow (2,2,0,2) \rightarrow (0,1,0,2) \rightarrow (1,1,2,2) \rightarrow (1,2,0,2) \rightarrow (2,0,0,2) \rightarrow (0,0,1,2) \rightarrow (1,2,1,2) \rightarrow (2,0,1,2) \rightarrow (0,1,1,2) \rightarrow (2,2,1,2) \rightarrow (1,0,1,2) \rightarrow (1,1,1,0) \rightarrow (2,1,1,2) \rightarrow (0,2,1,2) \rightarrow (0,0,2,2) \rightarrow (1,2,2,2) \rightarrow (2,0,2,2) \rightarrow (0,1,2,2) \rightarrow (2,2,2,2) \rightarrow (1,0,2,2) \rightarrow (1,1,2,0) \rightarrow (2,1,2,2) \rightarrow (0,2,2,2) \rightarrow (0,0,2,1) \rightarrow (1,2,2,1) \rightarrow (2,0,2,1) \rightarrow (0,1,2,1) \rightarrow (2,2,2,1) \rightarrow (1,0,2,1) \rightarrow (1,1,0,1) \rightarrow (2,1,2,1) \rightarrow (0,2,2,1) \rightarrow (0,0,2,0) \rightarrow (1,2,2,0) \rightarrow (2,0,2,0) \rightarrow (0,1,2,0) \rightarrow (2,2,2,0) \rightarrow (1,0,2,0) \rightarrow (1,1,0,0) \rightarrow (2,1,2,0) \rightarrow (0,2,2,0) \rightarrow (0,0,1,0) \rightarrow (2,1,1,0) \rightarrow (0,2,1,0) \rightarrow (1,0,1,0) \rightarrow (2,2,1,0) \rightarrow (0,1,1,0) \rightarrow (1,1,1,2) \rightarrow (1,2,1,0) \rightarrow (2,0,1,0) \rightarrow (0,0,1,1) \rightarrow (1,2,1,1) \rightarrow (2,0,1,1) \rightarrow (0,1,1,1) \rightarrow (2,2,1,1) \rightarrow (1,0,1,1) \rightarrow (0,2,1,1) \rightarrow (2,1,1,1) \rightarrow (0,1,0,1) \rightarrow (2,0,0,1) \rightarrow (1,2,0,1) \rightarrow (0,0,0,1) \rightarrow (2,1,0,1) \rightarrow (1,1,2,1) \rightarrow (1,0,0,1) \rightarrow (2,2,0,1) \rightarrow (2,0,0,0) \rightarrow (1,2,0,0) \rightarrow (0,0,0,0) \rightarrow (2,1,0,0) \rightarrow (0,2,0,0) \rightarrow (1,0,0,0) \rightarrow (1,1,0,2) \rightarrow (0,1,0,0) \rightarrow (2,2,0,0) \rightarrow (0,2,0,1)$.

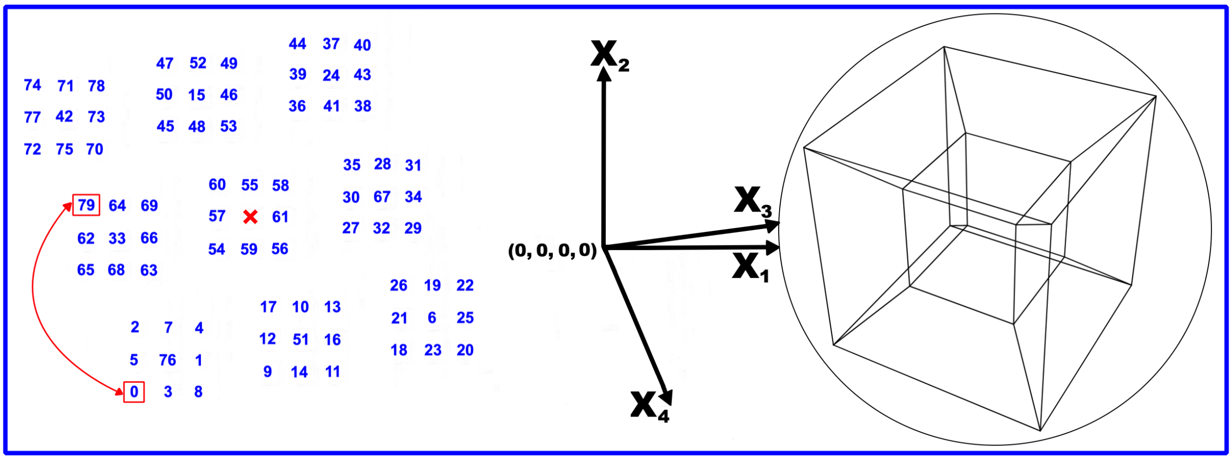

Figure 4. A graphical representation of the closed (regular) knight's tour $P_c(3, 4; \{(1, 1, 1, 1)\})$ on $C(3, 4) - \{(1, 1, 1, 1)\}$.

Thus, we have constructively proven the existence of a closed knight's tour also for $C(3, 4) - \{(1, 1, 1, 1)\}$, and we can conjecture the existence of closed Euclidean knight's tours on $C(3, k) - \{(1, 1, \ldots, 1)\}$ for every $k \in \mathbb{Z}^+ : k \equiv 0 \pmod 2$.

4 Closed Euclidean knight's tours on any $C(2, k) : k \geq 7$

This section is devoted to providing an alternative result to the statement of Theorem 3 of [6] by assuming that the knight is a Euclidean knight, as defined in Section 1. Furthermore, we prove a more general theorem on the existence of closed Euclidean knight's tours for any k-dimensional metric space $C(2, k) \subseteq \mathbb{Z}^k$, as long as k is above 6.

In detail, the aforementioned Theorem 3 of Reference [6] assumes $k \geq 3$ and states that a k-dimensional rectangular grid of the form $n_1 \times n_2 \times \cdots \times n_k$, such that $2 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, admits a closed knight's tour if and only if the following three conditions hold:

1. $\prod_{j=1}^{k} n_j \equiv 0 \pmod 2$,
2. $n_{k-1} \geq 3$,
3. $n_k \geq 4$.

Thus, if $n_1 = n_2 = \cdots = n_k = 2$, Theorem 3 of [6] would imply that $C(2, k)$ does not admit any closed knight tour, but this is no longer true if closed Euclidean knight's tours are included.

Theorem 4.1. *Let $k \in \mathbb{N} - \{0, 1, 2, 3, 4, 5, 6\}$. Then, closed Euclidean knight's tours exist on any $\underbrace{2 \times 2 \times \cdots \times 2}_{k\text{-times}}$ chessboard.*

Proof. Let us preliminary point out that a Euclidean knight can move also on a $2 \times 2 \times \cdots \times 2$ chessboard, but this is possible only if $k \geq 5$, since here is mandatory that a Euclidean knight changes by one 5 Cartesian coordinates at any move, given the fact that $n < 3$ does not let the knight make its well-known L-shaped move.

We prove Theorem 4.1 by providing the Hamiltonian cycle $P_c(2,7) \cup \{(0,1,1,1,1,1,0) \rightarrow (0,0,0,0,0,0,0)\}$ (see below) for $C(2,7) := \{\{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}\}$ to show the existence of a closed Euclidean knight's tour on the mentioned set of 128 vertices. Then, it will be very easy to see that closed Euclidean knight's tours exist also for any other $C(2,k)$ such that $k > 7$.

In detail, the polygonal chain $P_c(2,7) := (0,0,0,0,0,0,0) \rightarrow (0,0,1,1,1,1,1) \rightarrow (0,1,1,0,0,0,0) \rightarrow (0,1,0,1,1,1,1) \rightarrow (1,1,0,0,0,0,0) \rightarrow (1,1,1,1,1,1,1) \rightarrow (1,0,1,0,0,0,0) \rightarrow (1,0,0,1,1,1,1) \rightarrow (1,1,1,1,0,0,0) \rightarrow (1,1,0,0,1,1,1) \rightarrow (1,0,0,1,0,0,0) \rightarrow (1,0,1,0,1,1,1) \rightarrow (0,0,1,1,0,0,0) \rightarrow (0,0,0,0,1,1,1) \rightarrow (0,1,0,1,0,0,0) \rightarrow (0,1,1,0,1,1,1) \rightarrow (0,0,0,1,1,0,0) \rightarrow (0,0,1,0,0,1,1) \rightarrow (0,1,1,1,1,0,0) \rightarrow (0,1,0,0,0,1,1) \rightarrow (1,1,0,1,1,0,0) \rightarrow (1,1,1,0,0,1,1) \rightarrow (1,0,1,1,1,0,0) \rightarrow (1,0,0,0,0,1,1) \rightarrow (1,1,1,0,1,0,0) \rightarrow (1,1,0,1,0,1,1) \rightarrow (1,0,0,0,1,0,0) \rightarrow (1,0,1,1,0,1,1) \rightarrow (0,0,1,0,1,0,0) \rightarrow (0,0,0,1,0,1,1) \rightarrow (0,1,0,0,1,0,0) \rightarrow (0,1,1,1,0,1,1) \rightarrow (0,0,0,0,1,1,0) \rightarrow (0,0,1,1,0,0,1) \rightarrow (0,1,1,0,1,1,0) \rightarrow (0,1,0,1,0,0,1) \rightarrow (1,1,0,0,1,1,0) \rightarrow (1,1,1,1,0,0,1) \rightarrow (1,0,1,0,1,1,0) \rightarrow (1,0,0,1,0,0,1) \rightarrow (1,1,1,1,1,1,0) \rightarrow (1,1,0,0,0,0,1) \rightarrow (1,0,0,1,1,1,0) \rightarrow (1,0,1,0,0,0,1) \rightarrow (0,0,1,1,1,1,0) \rightarrow (0,0,0,0,0,0,1) \rightarrow (0,1,0,1,1,1,0) \rightarrow (0,1,1,0,0,0,1) \rightarrow (0,0,0,1,0,1,0) \rightarrow (0,0,1,0,1,0,1) \rightarrow (0,1,1,1,0,1,0) \rightarrow (0,1,0,0,1,0,1) \rightarrow (1,1,0,1,0,1,0) \rightarrow (1,1,1,0,1,0,1) \rightarrow (1,0,1,1,0,1,0) \rightarrow (1,0,0,0,1,0,1) \rightarrow (1,1,1,0,0,1,0) \rightarrow (1,1,0,1,1,0,1) \rightarrow (1,0,0,0,0,1,0) \rightarrow (1,0,1,1,1,0,1) \rightarrow (0,0,1,0,0,1,0) \rightarrow (0,0,0,1,1,0,1) \rightarrow (0,1,0,0,0,1,0) \rightarrow (0,1,1,1,1,0,1) \rightarrow (0,0,0,0,0,1,1) \rightarrow (0,0,1,1,1,0,0) \rightarrow (0,1,1,0,0,1,1) \rightarrow (0,1,0,1,1,0,0) \rightarrow (1,1,0,0,0,1,1) \rightarrow (1,1,1,1,1,0,0) \rightarrow (1,0,1,0,0,1,1) \rightarrow (1,0,0,1,1,0,0) \rightarrow (1,1,1,1,0,1,1) \rightarrow (1,1,0,0,1,0,0) \rightarrow (1,0,0,1,0,1,1) \rightarrow (1,0,1,0,1,0,0) \rightarrow (0,0,1,1,0,1,1) \rightarrow (0,0,0,0,1,0,0) \rightarrow (0,1,0,1,0,1,1) \rightarrow (0,1,1,0,1,0,0) \rightarrow (0,0,0,1,1,1,1) \rightarrow (0,0,1,0,0,0,0) \rightarrow (0,1,1,1,1,1,1) \rightarrow (0,1,0,0,0,0,0) \rightarrow (1,1,0,1,1,1,1) \rightarrow (1,1,1,0,0,0,0) \rightarrow (1,0,1,1,1,1,1) \rightarrow (1,0,0,0,0,0,0) \rightarrow (1,1,1,0,1,1,1) \rightarrow (1,1,0,1,0,0,0) \rightarrow (1,0,0,0,1,1,1) \rightarrow (1,0,1,1,0,0,0) \rightarrow (0,0,1,0,1,1,1) \rightarrow (0,0,0,1,0,0,0) \rightarrow (0,1,0,0,1,1,1) \rightarrow (0,1,1,1,0,0,0) \rightarrow (0,0,0,0,1,0,1) \rightarrow (0,0,1,1,0,1,0) \rightarrow (0,1,1,0,1,0,1) \rightarrow (0,1,0,1,0,1,0) \rightarrow (1,1,0,0,1,0,1) \rightarrow (1,1,1,1,0,1,0) \rightarrow (1,0,1,0,1,0,1) \rightarrow (1,0,0,1,0,1,0) \rightarrow (1,1,1,1,1,0,1) \rightarrow (1,1,0,0,0,1,0) \rightarrow (1,0,0,1,1,0,1) \rightarrow (1,0,1,0,0,1,0) \rightarrow (0,0,1,1,1,0,1) \rightarrow (0,0,0,0,0,1,0) \rightarrow (0,1,0,1,1,0,1) \rightarrow (0,1,1,0,0,1,0) \rightarrow (0,0,0,1,0,0,1) \rightarrow (0,0,1,0,1,1,0) \rightarrow (0,1,1,1,0,0,1) \rightarrow (0,1,0,0,1,1,0) \rightarrow (1,1,0,1,0,0,1) \rightarrow (1,1,1,0,1,1,0) \rightarrow (1,0,1,1,0,0,1) \rightarrow (1,0,0,0,1,1,0) \rightarrow (1,1,1,0,0,0,1) \rightarrow (1,1,0,1,1,1,0) \rightarrow (1,0,0,0,0,0,1) \rightarrow (1,0,1,1,1,1,0) \rightarrow (0,0,1,0,0,0,1) \rightarrow (0,0,0,1,1,1,0) \rightarrow (0,1,0,0,0,0,1) \rightarrow (0,1,1,1,1,1,0)$ represents a closed Euclidean knight's tour on $C(2,7)$, since the distance between its arrival and the starting vertex, $\mathrm{V}_0 \equiv (0,0,0,0,0,0,0)$, is $d(\mathrm{V}_{127}, \mathrm{V}_0) = d((0,1,1,1,1,1,0),(0,0,0,0,0,0,0)) = \sqrt{0^2+1^2+1^2+1^2+1^2+1^2+0^2} = \sqrt{5}$ (see Equation 1).

Consequently, we have constructed a closed Euclidean knight's tour on $C(2,7)$, the set of the 2^7 corners of a 7-cube.

Now, we note that any vertex of a 7-face belonging to an 8-cube is connected to some other vertices of the opposite 7-face of the same 8-cube by as many minor diagonals, including those that are characterized by a Euclidean length equal to the length of the major diagonal of a 5-cube (since $5 < 8$, trivially).

Thus, we can simply take the solution for the $k = 7$ case and reproduce it also on the opposite

7-face of the mentioned 8-cube, and then we are free to mirror/rotate it so that the endpoints of both the covering paths of the two 7-faces are connected by a pair of minor diagonals of (Euclidean) length $\sqrt{5}$.

As an example, we can extend the $k = 7$ solution $P_c(2,7) = (0,0,0,0,0,0,0) \rightarrow (0,0,1,1,1,1,1) \rightarrow \cdots \rightarrow (0,1,1,1,1,1,0)$ to $k = 8$ as follows.

1. First of all, we move $P_c(2,7)$ from $\mathbb{R}^7$ to $\mathbb{R}^8$ and duplicate it as $S_1(2,8) = \{(0,0,0,0,0,0,0,0) \rightarrow (0,0,1,1,1,1,1,0) \rightarrow \cdots \rightarrow (0,1,1,1,1,1,0,0)\}$ and $S_2(2,8) = \{(0,0,0,0,0,0,0,1) \rightarrow (0,0,1,1,1,1,1,1) \rightarrow \cdots \rightarrow (0,1,1,1,1,1,0,1)\}$.

2. We apply $S_1(2,8)$ to the first 7-face of $C(2,8)$ as it is (i.e., $\{(0,0,0,0,0,0,0,0) \rightarrow (0,0,1,1,1,1,1,0) \rightarrow \cdots \rightarrow (0,1,1,1,1,1,0,0)\}$), while we switch between $(0 \leftrightarrow 1)$ exactly $(5-1)$ more times out of $(8-1)$ Cartesian coordinates left (i.e., we modify 4 other coordinates at our choice and we keep doing the switch $(0 \leftrightarrow 1)$ on the same, selected, coordinates of every node of $S_2(2,8)$, for the entire transformation of the aforementioned path) in order to place in a legit spot the arrival of the final path, since we can do this by simply reflect/rotate and then reverse the whole polygonal chain $S_2(2,8)$ (e.g., $\{(0,0,1,1,1,1,0,1) \rightarrow (0,0,0,0,0,0,1,1) \rightarrow \cdots \rightarrow (0,1,0,0,0,0,0,1)\}$ indicates a valid geometric transformation of $S_2(2,8)$ that we can later reverse and finally apply on the proper 7-face).

3. As a result, we get the Hamiltonian path for the opposite 7-face of $C(2,8)$, $\hat{S}_2(2,8) := \{(0,0,1,1,1,1,0,1) \leftarrow (0,0,0,0,0,0,1,1) \leftarrow \cdots \leftarrow (0,1,0,0,0,0,0,1)\}$, which is a polygonal chain whose starting point is at a distance of $\sqrt{(0-0)^2+(1-1)^2+(0-1)^2+(0-1)^2+(0-1)^2+(0-1)^2+(0-0)^2+(1-0)^2}$ from the ending point of $S_1(2,8)$ and whose arrival is, again, at a distance of $\sqrt{(0-0)^2+(0-0)^2+(1-0)^2+(1-0)^2+(1-0)^2+(1-0)^2+(0-0)^2+(1-0)^2}$ from the beginning of $S_1(2,8)$, letting our Euclidean knight jump onto $\hat{S}_2(2,8)$ at the end of $S_1(2,8)$ and vice versa, over and over.

4. Accordingly, we join the ending point of $S_1(2,8)$ and the starting point of $\hat{S}_2(2,8)$ to get the closed Euclidean knight's tour described by $S_1(2,8) \cup \{(0,1,0,0,0,0,0,1) \rightarrow (0,1,1,1,1,1,0,0)\} \cup \hat{S}_2(2,8)$, a Hamiltonian path $\{(0,0,0,0,0,0,0,0) \rightarrow \cdots \rightarrow (0,0,1,1,1,1,0,1)\}$ for $C(2,8)$ that, without loss of generality, we have constructed from a closed Euclidean knight's tour on $C(2,7)$ beginning from $(0,0,0,0,0,0,0) \in \mathbb{R}^7$.

Finally, we can repeat the same process to extend the 8-cube solution to the 9-cube, and so forth. We are allowed to do so since any vertex of a k-cube is also a corner, and thus the symmetry is preserved.

Therefore, there exist closed Euclidean knight's tours on any $2 \times 2 \times \cdots \times 2$ chessboard with at least 2^7 cells and the proof of Theorem 4.1 is complete. □

5 Conclusion

The move pattern of the 2D knight, the piece that also appears in the FIDE logo, is described by Article 3.6 of Reference [8] as follows: "*The knight may move to one of the squares nearest to that on which it stands but not on the same rank, file or diagonal*". Thus, we have defined the knight by assuming the standard Euclidean metric and, consequently, the resulting distance covered by any jump is $\sqrt{5}$.

Then, it seems legitimate to assume that a multidimensional knight can also be defined in a more inclusive way than the usual description of a piece that merely moves, from a vertex of $C(n,k) := \{\{0,1,\ldots,n-1\} \times \{0,1,\ldots,n-1\} \times \cdots \times \{0,1,\ldots,n-1\}\}$ to another one of the same set, by adding or subtracting 2 to one of the k Cartesian coordinates of the starting vertex and, simultaneously, adding or subtracting 1 to another of the remaining $k-1$ elements of the mentioned k-tuple. Accordingly, we have provided the alternative definition of the knight as a piece that is allowed to move from $\mathrm{V}_m \in C(n,k)$ to $\mathrm{V}_{m+1} \in C(n,k)$ if and only if the condition $d(\mathrm{V}_m, \mathrm{V}_{m+1}) = d(\mathrm{V}_{m+1}, \mathrm{V}_m) = \sqrt{5}$ is satisfied (where $d(\mathrm{A}, \mathrm{B})$ indicates the Euclidean distance between the point A and the point B, as usual).

Consequently, the present paper has shown that such a Euclidean knight can produce a knight's tour also on k-dimensional grids of the form $\{\{0,1,2\} \times \{0,1,2\} \times \cdots \times \{0,1,2\}\}$, for some $k \geq 5$. In particular, by Corollary 2.2, every Euclidean knight's tour for any $3 \times 3 \times \cdots \times 3$ chessboard has to necessarily be an open tour.

Now, if this is not enough to fully accept the $\sqrt{5}$ knight metric, since the knight may not be allowed to move along one of the major diagonals of a 5-cube by unimaginatively extending beyond the $k = 2$ case the semantic meaning of Article 3.6 of [8], then Theorem 4.1 shows that it is possible to construct closed Euclidean knight's tours on any (k-dimensional) $2 \times 2 \times \cdots \times 2$ chessboard, as long as $k > 6$, avoiding by construction to move along any major diagonal of the given k-cube. Furthermore, we could invoke the same argument to suggest that, for any $k > 4$, the central vertex of a $3 \times 3 \times \cdots \times 3$ k-cube is (Euclidean) knight-connected to any other element of the set $C(3,k)$.

As a result, Theorem 3 of [6] can no longer be invoked on any metric space $C(2,k) : k \geq 7$, where the distance between two vertices $\mathrm{A} \in C(n,k)$ and $\mathrm{B} \in C(n,k)$ is given by the minimum number of jumps of length $\sqrt{5}$ that the described Euclidean k-knight requires in order to go from A to B (and vice versa).

Lastly, a related open problem is to prove the existence of a closed (possibly Euclidean) knight's tour also for every $C(3,k) - \{(1,1,\ldots,1)\}$ such that k is even (in Section 3, we have verified the correctness of this conjecture for the cases $k = 2$ and $k = 4$, but making such inferences in low dimensions can be notoriously misleading).

Acknowledgements

The author sincerely thanks Aldo Roberto Pessolano for his invaluable effort, greatly helping him to prove Theorem 4.1 by performing a computational analysis that has returned, in just a few seconds, independent closed Euclidean knight's tours for $2 \times 2 \times \cdots \times 2$ chessboards of 2^7, 2^8,

2^9, 2^{10}, 2^{11}, and 2^{12} cells (with the only exception of $P_c(2,7)$, we have not included any of them in the present paper to save space).

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