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Discatenated and lacunary recurrences

Hakan Akkuş[1], Ömür Deveci[2], Engin Özkan[3] and Anthony G. Shannon[4]

[1] Department of Mathematics, Graduate School of Natural and Applied Sciences
Erzincan Binali Yıldırım University, Yalnızbağ Campus, 24100, Erzincan, Türkiye
e-mail: hakan.akkus@ogr.ebyu.edu.tr

[2] Department of Mathematics, Faculty of Sciences Arts, Kafkas University
36100 Kars, Türkiye
e-mail: odeveci36@hotmail.com

[3] Department of Mathematics, Faculty of Sciences Arts, Erzincan Binali Yıldırım University,
Yalnızbağ Campus, 24100, Erzincan, Türkiye
e-mail: eozkan@erzincan.edu.tr

[4] Warrane College, The University of New South Wales
Kensington, NSW 2033, Australia
e-mail: t.shannon@warrane.unsw.edu.au

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Abstract: Recursive sequences with gaps have been studied previously. This paper considers some elementary properties of such sequences where the gaps have been created on a regular basis from sequence to sequence – ‘discatenated’ (systematic gaps) and ‘lacunary’ (general gaps). In particular, their generating functions are developed in order to open up their general terms and relations with other properties.
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1 Introduction

The Fibonacci and Lucas sequences are famous sequences of numbers. These sequences have intrigued scientists for a long time. Fibonacci and Lucas sequences have been applied in various fields such as algebraic coding theory, phylotaxis, biomathematics, computer science, and so on. For $n \geq 0$, Fibonacci numbers F_n and Lucas numbers L_n are defined by the recurrence relations, respectively,

$$F_{n+2} = F_{n+1} + F_n \text{, with } F_0 = 0 \text{ and } F_1 = 1,$$

$$L_{n+2} = L_{n+1} + L_n \text{, with } L_0 = 2 \text{ and } L_1 = 1.$$

For F_n and L_n the Binet formulas are given by the following relations, respectively,

$$F_n = \frac{\varphi^n - \omega^n}{\varphi - \omega} \text{ and } L_n = \varphi^n + \omega^n$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\omega = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $r^2 - r - 1 = 0$. Here the number φ is the known golden ratio. In [10, 11, 16, 18], are many studies on Fibonacci and Lucas sequences related to this present paper.

'Discatenation', the opposite of concatenation, deals with gaps in Fibonacci and Lucas sequences and their generalizations. It belongs to the category of lacunary recurrence relations [5, 6, 13, 19], though we shall distinguish them arbitrarily. We begin with a ratio of Fibonacci numbers:

$$f_{k,n} = \frac{F_{kn+k}}{F_k} = \frac{\alpha^{nk+k} - \beta^{nk+k}}{\alpha^k - \beta^k}, \; k \geq 1, n \geq 0, \tag{1.1}$$

in which F_n is an ordinary Fibonacci number, and where α, β are the roots, assumed distinct, of $x^2 - px + q = 0$, in which p, q are arbitrary integers. We start to see the discatenated sequences $\{F_k f_{k,n}\} \subseteq \{F_{kn+k}\}$ in Table 1.

Table 1. Examples of discatenated Fibonacci sequences

$F_k f_{k,n}$	Gaps	$n = 1$	2	3	4	5	6	Sloane [10]
$F_1 f_{1,n}$	0	1	2	3	5	8	13	A000045
$F_2 f_{2,n}$	1	3	8	21	55	144	377	A001906
$F_3 f_{3,n}$	2	8	34	144	610	2584	10946	A014445
$F_4 f_{4,n}$	3	21	144	987	6765	46368	317811	A033888
Sloane A……		001906	014445	033888	102312	134492	134498	

We notice the initial terms $1x1$, $3x1$, $4x2$, $7x3$, and so on, and that the horizontal and vertical sequences have the same number of gaps. These are artifacts of their construction, and they are related to sequences found in Sloane for different reasons. The initial idea for expressing these sequences came from Section 3 of Craveiro et al [4]. Although some parts of the notation in the latter are somewhat similar, the focus here is quite different. For reasons which will become obvious we also define a companion sequence for { $f_{k,n}$ }, namely,

$$g_{k,n} = \frac{F_{n+k}}{F_k} = \frac{\alpha^{n+k} - \beta^{n+k}}{\alpha^k - \beta^k}, \ k \geq 1, n \geq 0, \tag{1.2}$$

so that the $g_{1,n}$ are also ordinary Fibonacci numbers, which are open to extensions to Lucas analogues, such as [7], as we shall see in one case in Section 4.

In [1], Aistleitner et al, the analysis of lacunary sequences and their applications in probability and number theory were studied. In addition, Sharma identified a new lacunary sequences and found many features of this sequence [15]. In [9], studies were carried out on lacunary type polynomials, and interesting properties of these polynomials were obtained.

2 Lacunary numbers

Lucas [8] studied the second order primordial sequence $\left\{U_{0,n}^{(2)}\right\}$ and one of the basic fundamental sequences $\left\{U_{2,n}^{(2)}\right\}$ which satisfy the linear recurrence with arbitrary integers *p, q*

$$U_{s,n}^{(2)} = pU_{s,n-1}^{(2)} - qU_{s,n-2}^{(2)}, \ s \in \{0,2\}, \tag{2.1}$$

with initial values $U_{2,0}^{(2)} = 0, U_{2,1}^{(2)} = 1, U_{0,0}^{(2)} = 2, U_{0,1}^{(2)} = \alpha + \beta$, where α, β are the roots, assumed distinct, of the auxiliary equation associated with (2.1). For ease of subsequent notation, we shall represent $\{U_{2,n+2}^{(2)}\}$ by $\left\{u_n^{(2)}\right\}$ and $\{U_{2,0}^{(2)}\}$ by $\{v_n^{(2)}\}$ which are more akin to Lucas' original notation. The superscripts are there to encourage interested readers to extend the results to higher orders. In terms of their relations with the Lucas sequences, we have

$$f_{k+1,n+1} = v_k^{(2)} f_{k+1,n} - q^k f_{k+1,n-1} \tag{2.2}$$

$$g_{k+1,n+1} = pg_{k+1,n} - qg_{k+1,n-1}. \tag{2.3}$$

We call these numbers, $\left\{f_{k,n}\right\}$, 'lacunary' to distinguish them from the corresponding 'discatenated' numbers, $\left\{F_k f_{k,n}\right\}$.

Proof of (2.2):

$$\begin{aligned} v_k^{(2)} f_{k+1,n} - q^k f_{k+1,n-1} &= \frac{\left[\left(\alpha^k + \beta^k\right)\left(\alpha^{nk+k} - \beta^{nk+k}\right) - (\alpha\beta)^k\left(\alpha^{nk} - \beta^{nk}\right)\right]}{\left(\alpha^k - \beta^k\right)} \\ &= \frac{\left[\left(\alpha^{nk+2k} - \beta^{nk+2k} + (\alpha\beta)^k\left(\alpha^{nk} - \beta^{nk}\right) - (\alpha\beta)^k\left(\alpha^{nk} - \beta^{nk}\right)\right)\right]}{\left(\alpha^k - \beta^k\right)} \\ &= \frac{\left(\alpha^{nk+2k} - \beta^{nk+2k}\right)}{\left(\alpha^k - \beta^k\right)} \\ &= f_{k+1,n+1}. \end{aligned}$$

We can now see the lacunary sequences $\left\{f_{k,n}\right\} \subseteq \left\{F_k f_{k,n}\right\} \subseteq \left\{F_{kn+k}\right\}$ in Table 2.

Table 2. Lacunary numbers with their linear recurrence relations

$f_{k,n}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	Recurrence relations
$f_{1,n}$	1	2	3	5	8	13	$f_{1,n} = 1f_{1,n-1} + f_{1,n-2}$
$f_{2,n}$	3	8	21	55	144	377	$f_{2,n} = 3f_{2,n-1} - f_{2,n-2}$.
$f_{3,n}$	4	17	72	305	1292	5473	$f_{3,n} = 4f_{3,4-2} + f_{3,n-2}$
$f_{4,n}$	7	48	329	2255	15456	105937	$f_{4,n} = 7f_{4,n-1} - f_{4,n-2}$
A……	000032	261876	083564	103326	---	028412	↔ Sloane [10]

From the inductive argument on k by considering definition of the sequence $\{f_{k,n}\}$, we also write the following general recurrence relations:

$$f_{k,n} = \begin{cases} a_{\frac{k+1}{2}} f_{k,n-1} + f_{k,n-2} & \text{if } k \text{ is odd,} \\ b_{\frac{k}{2}} f_{k,n-1} - f_{k,n-2} & \text{if } k \text{ is even,} \end{cases}$$

where the sequences $\{a_k\}$ and $\{b_k\}$ are as follows, respectively:

$$a_1 = 1,\ a_2 = 4;\ a_k = 3a_{k-1} - a_{k-2},\ k \geq 3$$

and

$$b_1 = 3,\ b_2 = 7;\ b_k = 3b_{k-1} - b_{k-2},\ k \geq 3 .$$

Theorem 2.1. (Binet Formulas) *Let* $n \in N$ *. Then, the Binet formulas of the* $U_{0,n}^{(2)}$*, and* $U_{2,n}^{(2)}$ *sequences are as follows:*

(i). $U_{0,n}^{(2)} = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$,

(ii). $U_{2,n}^{(2)} = \alpha^n + \beta^n$.

Proof. (i). The Binet form of a sequence is as follows

$$U_{0,n}^{(2)} = x\alpha^n + y\beta^n .$$

Here, the scalars x and y can be obtained by substituting the initial conditions and solving the given system of equations. For $n = 0$, $U_{0,0}^{(2)} = 0$ and $n = 1$, $U_{0,1}^{(2)} = 1$. So, $x = \frac{1}{\alpha-\beta}$ and $y = -\frac{1}{\alpha-\beta}$. Thus, we obtain

$$U_{0,n}^{(2)} = \frac{\alpha^n - \beta^n}{\alpha - \beta} .$$

The proof of the other may be found similarly. □

Theorem 2.2. *Let* $n \in N$ *. The following equations are true:*

(i). $U_{2,n}^{(2)} = \dfrac{1}{p} U_{0,n+1}^{(2)} + \dfrac{q}{p} U_{0,n-1}^{(2)}$, (ii). $U_{0,n}^{(2)}U_{2,n}^{(2)}=U_{0,2n}^{(2)}$,

(iii). $\left(p^2-4q\right)U_{0,n}^{(2)}=U_{2,n+1}^{(2)}-U_{2,n-1}^{(2)}$,

(iv). $\sqrt{p^2-4q}U_{0,n}^{(2)}+U_{2,n}^{(2)}=2\alpha^n$,

(v). $\sqrt{p^2-4q}U_{0,n}^{(2)}-U_{2,n}^{(2)}=-2\beta^n$.

Proof. (i). If the Binet formulas are used for proof, we obtain:

$$\begin{aligned}U_{0,n+1}^{(2)}-qU_{0,n-1}^{(2)}&=\frac{\alpha^{n+1}-\beta^{n+1}}{p(\alpha-\beta)}-q\frac{\alpha^{n-1}-\beta^{n-1}}{p(\alpha-\beta)}\\&=\frac{\alpha^n\left(\alpha-\frac{q}{\alpha}\right)-\beta^n\left(\beta-\frac{q}{\beta}\right)}{p(\alpha-\beta)}\\&=\alpha^n+\beta^n\\&=U_{2,n}^{(2)}.\end{aligned}$$

The proofs of the others may be found similarly. □

Theorem 2.3. *Let* $m,n\in N$ *and* $m>n$. *The following equations are satisfied:*

(i). $U_{0,m+n+1}^{(2)}=U_{0,m+1}^{(2)}U_{0,n+1}^{(2)}-qU_{0,m}^{(2)}U_{0,n}^{(2)}$,

(ii). $U_{2,m+n+1}^{(2)}=U_{2,n+1}^{(2)}U_{0,m+1}^{(2)}-qU_{2,n}^{(2)}U_{0,m}^{(2)}$.

Proof. If the Binet formulas are used for proof, we obtain:

(i). $$\begin{aligned}U_{0,m+1}^{(2)}U_{0,n+1}^{(2)}-qU_{0,m}^{(2)}U_{0,n}^{(2)}&=\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}-q\frac{\alpha^m-\beta^m}{\alpha-\beta}\frac{\alpha^n-\beta^n}{\alpha-\beta}\\&=\frac{\alpha^{m+n+2}-\alpha^{m+1}\beta^{n+1}-\alpha^{n+1}\beta^{m+1}+\beta^{m+n+2}-q\alpha^{m+n}+q\alpha^m\beta^n+q\alpha^n\beta^m-q\beta^{m+n}}{(\alpha-\beta)(\alpha-\beta)}\\&=\frac{\alpha^{m+n+1}\left(\alpha-\frac{q}{\alpha}\right)+\beta^{m+n+1}\left(\beta-\frac{q}{\beta}\right)}{(\alpha-\beta)(\alpha-\beta)}\\&=\frac{\alpha^{m+n+1}-\beta^{m+n+1}}{\alpha-\beta}\\&=U_{0,m+n+1}^{(2)}.\end{aligned}$$

(ii). $$\begin{aligned}U_{2,n+1}^{(2)}U_{0,m+1}^{(2)}-qU_{2,n}^{(2)}U_{0,m}^{(2)}&=\left(\alpha^{n+1}+\beta^{n+1}\right)\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}-q\left(\alpha^n+\beta^n\right)\frac{\alpha^m-\beta^m}{\alpha-\beta}\\&=\frac{\alpha^{m+n+2}-\alpha^{n+1}\beta^{m+1}+\alpha^{m+1}\beta^{n+1}-\beta^{m+n+2}-q\alpha^{m+n}+q\alpha^n\beta^m-q\alpha^m\beta^n+q\beta^{m+n}}{\alpha-\beta}\\&=\frac{\alpha^{m+n+1}\left(\alpha-\frac{q}{\alpha}\right)+\beta^{m+n+1}\left(\beta-\frac{q}{\beta}\right)}{\alpha-\beta}\\&=\alpha^{m+n+1}+\beta^{m+n+1}\\&=U_{2,m+n+1}^{(2)}.\end{aligned}$$ □

Theorem 2.4. *Let* $n \in N$ *. The following equations are true:*

(i). $U_{0,-n}^{(2)} = -\frac{1}{q^n} U_{0,n}^{(2)}$,

(ii). $U_{2,-n}^{(2)} = \frac{1}{q^n} U_{2,n}^{(2)}$.

Proof. (i). If the Binet formulas are used for proof, we obtain:

$$U_{0,-n}^{(2)} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = -\frac{1}{\alpha^n \beta^n} \frac{\alpha^n - \beta^n}{\alpha - \beta} = -\frac{1}{q^n} U_{0,n}^{(2)}.$$

(ii).
$$U_{2,-n}^{(2)} = \alpha^{-n} + \beta^{-n} = \frac{1}{\alpha^n} + \frac{1}{\beta^n} = \frac{1}{\alpha^n \beta^n}\left(\alpha^n + \beta^n\right) = \frac{1}{q^n} U_{2,n}^{(2)}.$$
□

Theorem 2.5. *Let* $m, n \in N$ *and* $m > n$*. We obtain*

(i). $2U_{2,m+n}^{(2)} = \left(p^2 - 4q\right) U_{0,n}^{(2)} U_{0,m}^{(2)} + U_{2,n}^{(2)} U_{2,m}^{(2)}$,

(ii). $2q^n U_{2,m-n}^{(2)} = U_{2,n}^{(2)} U_{2,m}^{(2)} - \left(p^2 - 4q\right) U_{0,n}^{(2)} U_{0,m}^{(2)}$,

(iii). $2q^n U_{0,m-n}^{(2)} = U_{2,n}^{(2)} U_{0,m}^{(2)} - U_{2,m}^{(2)} U_{0,n}^{(2)}$.

Proof. If the Binet formulas are used for proof, we obtain:

(i). $\left(p^2 - 4q\right) U_{0,n}^{(2)} U_{0,m}^{(2)} + U_{2,n}^{(2)} U_{2,m}^{(2)} = \left(p^2 - 4q\right) \frac{\alpha^n - \beta^n}{\alpha - \beta} \frac{\alpha^m - \beta^m}{\alpha - \beta} + \left(\alpha^n + \beta^n\right)\left(\alpha^m + \beta^m\right)$

$$= \alpha^{m+n} - \alpha^n \beta^m - \alpha^m \beta^n + \beta^{m+n} + \alpha^{m+n} + \alpha^n \beta^m + \alpha^m \beta^n + \beta^{m+n}$$
$$= 2\alpha^{m+n} + 2\beta^{m+n}$$
$$= 2U_{2,m+n}^{(2)}.$$

(ii). $U_{2,n}^{(2)} U_{2,m}^{(2)} - \left(p^2 - 4q\right) U_{0,n}^{(2)} U_{0,m}^{(2)} = \left(\alpha^n + \beta^n\right)\left(\alpha^m + \beta^m\right) - \left(p^2 - 4q\right) \frac{\alpha^n - \beta^n}{\alpha - \beta} \frac{\alpha^m - \beta^m}{\alpha - \beta}$

$$= \alpha^{m+n} + \alpha^n \beta^m + \alpha^m \beta^n + \beta^{m+n} - \alpha^{m+n} + \alpha^n \beta^m + \alpha^m \beta^n - \beta^{m+n}$$
$$= 2\alpha^n \beta^m + 2\alpha^m \beta^n$$
$$= 2\alpha^n \beta^n \left(\alpha^{m-n} + \beta^{m-n}\right)$$
$$= 2q^n U_{2,m-n}^{(2)}.$$

(iii). $U_{2,n}^{(2)} U_{0,m}^{(2)} - U_{2,m}^{(2)} U_{0,n}^{(2)} = \left(\alpha^n + \beta^n\right) \frac{\alpha^m - \beta^m}{\alpha - \beta} - \left(\alpha^m + \beta^m\right) \frac{\alpha^n - \beta^n}{\alpha - \beta}$

$$= \frac{\alpha^{m+n} - \alpha^n \beta^m + \alpha^m \beta^n - \beta^{m+n} - \alpha^{m+n} - \alpha^n \beta^m + \alpha^m \beta^n + \beta^{m+n}}{\alpha - \beta}$$
$$= \frac{2\alpha^n \beta^n \left(\alpha^{m-n} - \beta^{m-n}\right)}{\alpha - \beta}$$
$$= 2q^n U_{0,m-n}^{(2)}.$$
□

Theorem 2.6. (Cassini Identity) *Let* $n \in N$. *We obtain*

(i). $U_{0,n+1}^{(2)}U_{0,n-1}^{(2)} - U_{0,n}^{(2)}U_{0,n}^{(2)} = -q^{n-1}$,

(ii). $U_{2,n+1}^{(2)}U_{2,n-1}^{(2)} - U_{2,n}^{(2)}U_{2,n}^{(2)} = q^{n-1}\left(p^2 - 4q\right)$.

Proof. If the Binet formulas are used for proof, we obtain:

(i).
$$U_{0,n+1}^{(2)}U_{0,n-1}^{(2)} - U_{0,n}^{(2)}U_{0,n}^{(2)} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} - \frac{\alpha^{n} - \beta^{n}}{\alpha - \beta}\frac{\alpha^{n} - \beta^{n}}{\alpha - \beta}$$
$$= \frac{\alpha^n\beta^n\left(-\frac{\alpha}{\beta} - \frac{\beta}{\alpha} + 2\right)}{\left(\alpha - \beta\right)^2}$$
$$= -q^{n-1}.$$

(ii).
$$U_{2,n+1}^{(2)}U_{2,n-1}^{(2)} - U_{2,n}^{(2)}U_{2,n}^{(2)} = \left(\alpha^{n+1} + \beta^{n+1}\right)\left(\alpha^{n-1} - \beta^{n-1}\right) - \left(\alpha^{n} - \beta^{n}\right)\left(\alpha^{n} - \beta^{n}\right)$$
$$= \alpha^n\beta^n\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} - 2\right)$$
$$= q^{n-1}\left(p^2 - 4q\right).$$
□

Theorem 2.7. (Catalan Identity) *Let* $n, m \in N$. *We obtain*

(i). $U_{0,n+m}^{(2)}U_{0,n-m}^{(2)} - U_{0,n}^{(2)}U_{0,n}^{(2)} = -q^{n-m}U_{0,m}^{(2)}U_{0,m}^{(2)}$,

(ii). $U_{2,n+m}^{(2)}U_{2,n-m}^{(2)} - U_{2,n}^{(2)}U_{2,n}^{(2)} = q^{n-m}\left(p^2 - 4q\right)U_{0,m}^{(2)}U_{0,m}^{(2)}$.

Proof. If the Binet formulas are used for proof, we obtain:

(i).
$$U_{0,n+m}^{(2)}U_{0,n-m}^{(2)} - U_{0,n}^{(2)}U_{0,n}^{(2)} = \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta}\frac{\alpha^{n-m} - \beta^{n-m}}{\alpha - \beta} - \frac{\alpha^{n} - \beta^{n}}{\alpha - \beta}\frac{\alpha^{n} - \beta^{n}}{\alpha - \beta}$$
$$= \frac{\alpha^n\beta^n}{\left(\alpha - \beta\right)\left(\alpha - \beta\right)}\left(-\frac{\alpha^m}{\beta^m} - \frac{\beta^m}{\alpha^m} + 2\right)$$
$$= -\alpha^{n-m}\beta^{n-m}\frac{\alpha^m - \beta^m}{\alpha - \beta}\frac{\alpha^m - \beta^m}{\alpha - \beta}$$
$$= -q^{n-m}U_{0,m}^{(2)}U_{0,m}^{(2)}.$$

(ii).
$$U_{2,n+m}^{(2)}U_{2,n-m}^{(2)} - U_{2,n}^{(2)}U_{2,n}^{(2)} = \left(\alpha^{n+m} + \beta^{n+m}\right)\left(\alpha^{n-m} + \beta^{n-m}\right) - \left(\alpha^{n} + \beta^{n}\right)\left(\alpha^{n} + \beta^{n}\right)$$
$$= \alpha^n\beta^n\left(\frac{\alpha^m}{\beta^m} + \frac{\beta^m}{\alpha^m} - 2\right)$$
$$= -\alpha^{n-m}\beta^{n-m}\frac{\alpha^m - \beta^m}{\alpha - \beta}\frac{\alpha^m - \beta^m}{\alpha - \beta}\left(p^2 - 4q\right)$$
$$= q^{n-m}\left(p^2 - 4q\right)U_{0,m}^{(2)}U_{0,m}^{(2)}.$$
□

Theorem 2.8. (D'Ocagne Identity) *Let* $n,m\in N$ *and* $n>m$. *We obtain*

(i). $U_{0,n+1}^{(2)}U_{0,m}^{(2)}-U_{0,n}^{(2)}U_{0,m+1}^{(2)}=-q^{m}U_{0,n-m}^{(2)}$,

(ii). $U_{2,n+1}^{(2)}U_{2,m}^{(2)}-U_{2,n}^{(2)}U_{2,m+1}^{(2)}=q^{m}\left(p^{2}-4q\right)U_{0,n-m}^{(2)}$.

Proof. If the Binet formulas are used for proof, we obtain:

(i).
$$\begin{aligned}U_{0,n+1}^{(2)}U_{0,m}^{(2)}-U_{0,n}^{(2)}U_{0,m+1}^{(2)}&=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\frac{\alpha^{m}-\beta^{m}}{\alpha-\beta}-\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta}\\&=\frac{-\alpha^{n+1}\beta^{m}-\alpha^{m}\beta^{n+1}+\alpha^{n}\beta^{m+1}+\alpha^{m+1}\beta^{n}}{(\alpha-\beta)(\alpha-\beta)}\\&=\frac{\alpha^{n}\beta^{m}(\beta-\alpha)+\alpha^{m}\beta^{n}(\alpha-\beta)}{(\alpha-\beta)(\alpha-\beta)}\\&=\frac{-(\alpha-\beta)\alpha^{m}\beta^{m}\left(\alpha^{n-m}-\beta^{n-m}\right)}{(\alpha-\beta)(\alpha-\beta)}\\&=-q^{m}U_{0,n-m}^{(2)}.\end{aligned}$$

(ii).
$$\begin{aligned}U_{2,n+1}^{(2)}U_{2,m}^{(2)}-U_{2,n}^{(2)}U_{2,m+1}^{(2)}&=\left(\alpha^{n+1}+\beta^{n+1}\right)\left(\alpha^{m}+\beta^{m}\right)-\left(\alpha^{n}+\beta^{n}\right)\left(\alpha^{m+1}+\beta^{m+1}\right)\\&=\alpha^{n+1}\beta^{m}+\alpha^{m}\beta^{n+1}-\alpha^{n}\beta^{m+1}-\alpha^{m+1}\beta^{n}\\&=\alpha^{n}\beta^{m}(\alpha-\beta)+\alpha^{m}\beta^{n}(\beta-\alpha)\\&=\frac{(\alpha-\beta)(\alpha-\beta)\alpha^{m}\beta^{m}\left(\alpha^{n-m}-\beta^{n-m}\right)}{\alpha-\beta}\\&=q^{m}\left(p^{2}-4q\right)U_{0,n-m}^{(2)}.\end{aligned}$$
□

Theorem 2.9. (Vajda Identity) *Let* $n,m,p\in N$. *We obtain*

(i). $U_{0,n+m}^{(2)}U_{0,n+p}^{(2)}-U_{0,n}^{(2)}U_{0,n+m+p}^{(2)}=q^{n}U_{0,m}^{(2)}U_{0,p}^{(2)}$,

(ii). $U_{2,n+m}^{(2)}U_{2,n+p}^{(2)}-U_{2,n}^{(2)}U_{2,n+m+p}^{(2)}=-q^{n}\left(p^{2}-4q\right)U_{0,m}^{(2)}U_{0,p}^{(2)}$.

Proof. If the Binet formulas are used for proof, we obtain:

(i).
$$\begin{aligned}U_{0,n+m}^{(2)}U_{0,n+p}^{(2)}-U_{0,n}^{(2)}U_{0,n+m+p}^{(2)}&=\frac{\alpha^{n+m}-\beta^{n+m}}{\alpha-\beta}\frac{\alpha^{n+p}-\beta^{n+p}}{\alpha-\beta}\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}\frac{\alpha^{n+m+p}-\beta^{n+m+p}}{\alpha-\beta}\\&=\frac{\alpha^{n}\beta^{n}\left(\alpha^{p}-\beta^{p}\right)\left(\alpha^{m}-\beta^{m}\right)}{(\alpha-\beta)(\alpha-\beta)}\\&=q^{n}U_{0,m}^{(2)}U_{0,p}^{(2)}.\end{aligned}$$

(ii).
$$\begin{aligned}U_{2,n+m}^{(2)}U_{2,n+p}^{(2)}-U_{2,n}^{(2)}U_{2,n+m+p}^{(2)}&=\left(\alpha^{n+m}+\beta^{n+m}\right)\left(\alpha^{n+p}+\beta^{n+p}\right)-\left(\alpha^{n}-\beta^{n}\right)\left(\alpha^{n+m+p}+\beta^{n+m+p}\right)\\&=-\frac{\alpha^{n}\beta^{n}\left(\alpha^{p}-\beta^{p}\right)\left(\alpha^{m}-\beta^{m}\right)(\alpha-\beta)(\alpha-\beta)}{(\alpha-\beta)(\alpha-\beta)}\\&=-q^{n}\left(p^{2}-4q\right)U_{0,m}^{(2)}U_{0,p}^{(2)}.\end{aligned}$$
□

3 Key properties

The ordinary generating functions for these sequences are represented formally by

$$\sum_{n=0}^{\infty} f_{k+1,n} x^n = \frac{1}{1-v_k^{(2)}x+x^2}, \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} g_{k+1,n} x^n = \frac{\left(1+\left(g_{k+1,1}-p\right)x\right)}{1-px+qx^2}. \tag{3.2}$$

Proofs: Let $f(x)=\sum_{n=0}^{\infty} f_{k+1,n}x^n$ and $g(x)=\sum_{n=0}^{\infty} g_{k+1,n}x^n$, so that

$$\begin{aligned}(1-v_k^{(2)}x+x^2)f(x) &= f_{k+1,0}+\left(f_{k+1,1}-f_{k+1,0}v_k^{(2)}\right)x \\ &= 1+(1-1)x \\ &= 1, \qquad \text{as required for (3.1).}\end{aligned}$$

Similarly,

$$\begin{aligned}\left(1-px+qx^2\right)g(x) &= g_{k+1.0}+\left(g_{k+1.1}-pg_{k+1.0}\right)x \\ &= 1+\left(\frac{\alpha^{k+k}-\beta^{k+k}}{\alpha^k-\beta^k}-(\alpha+\beta)\right)x \\ &= 1+\left(g_{k+1,1}-p\right)x, \qquad \text{as required for (3.2).}\end{aligned}$$

As an analogue of Catalan's Identity, we have

$$g_{k+1,n}^2 - g_{k+1,n-k}g_{k+1,n+k} = q^n. \tag{3.3}$$

Proof. The numerator of the left-hand side reduces to

$$\begin{aligned}(\alpha\beta)^n\alpha^{2k}+(\alpha\beta)^n\beta^{2k}-2(\alpha\beta)^n(\alpha\beta)^k &= (\alpha\beta)^n\left(\alpha^k-\beta^k\right)^2 \\ &= q^n\left(\alpha^k-\beta^k\right)^2\end{aligned}$$

which is q times the denominator of the left-hand side.

When $p = -q = R$, say, we are able to relate the $f_{k+1,n}$ to ordinary Lucas fundamental numbers, $u_n^{(2)}$, by means of a generalization of Barakat [2], who proved that

$$u_n^{(2)} = \sum_{0\le 2m\le n}\binom{n-m}{m}p^{n-2m}(-q)^m = \sum_{0\le 2m\le n}\binom{n-m}{m}R^{n-m}.$$

For notational convenience, we further define formally

$$u_n^{(2)} = Rx_n = Ry_n,$$

so that from (3.3), we have that

$$x_k y_{k-1} - x_{k-1}y_k = (-R)^{k-2}. \tag{3.4}$$

We are now in a position to assert a property which relates these lacunary Fibonacci numbers to the ordinary Fibonacci numbers and which yields an iterative formula for the general terms [12].

4 General terms

With the notation from the previous section, we have

Lemma 4.1.
$$v_k^{(2)} = \frac{Ry_{k-1} + R^2 y_{k-3}}{\alpha^2 - \beta^2} . \tag{4.1}$$

Proof. The numerator of the right-hand side is

$$\begin{aligned}
&(\alpha+\beta)\left(\alpha^{k+1}-\beta^{k+1}\right)+(-\alpha\beta)(\alpha+\beta)\left(\alpha^{k-1}-\beta^{k-1}\right)\\
&\qquad =\alpha^{k+2}-\beta^{k+2}-\alpha\beta^{k+1}+\alpha^{k+1}\beta+\alpha^2\beta^k-\alpha^k\beta^2+\alpha\beta^{k+1}-\alpha^{k+1}\beta\\
&\qquad =\left(\alpha^k+\beta^k\right)\left(\alpha^2-\beta^2\right)\\
&\qquad =v_k^{(2)}\left(\alpha^2-\beta^2\right).
\end{aligned}$$

□

Lemma 4.2.
$$f_{k+1,n} z^n = \sum_{0\le m+s\le n} \binom{m}{s}\binom{n-m}{s}\left(u_{k-1}^{(2)}\right)^{2s}\left(u_k^{(2)}\right)^{n-m-s} R^m . \tag{4.2}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} f_{k+1,n} z^n &= \frac{1}{1-v_k^{(2)}x+x^2} && \textit{from (3.1)}\\
&=\left(1-\left(R^2x_{k-2}+Ry_{k-1}\right)z+(-R)^k z^2\right)^{-1} && \textit{from Lemma 4.1}\\
&= \left(1-\left(R^2x_{k-2}+Ry_{k-1}\right)z+R^3\left(x_{k-2}y_{k-1}-x_{k-1}y_{k-2}\right)z^2\right)^{-1} && \textit{from (3.4)}\\
&= \left(\left(1-R^2x_{k-2}z\right)\left(1-Ry_{k-1}z\right)-R^3x_{k-1}y_{k-2}z^2\right)^{-1}\\
&= \sum_{s=0}^{\infty}\left(\left(1-R^2x_{k-2}z\right)\left(1-Ry_{k-1}z\right)\right)^{-s-1}\left(x_{k-1}y_{k-2}R^3z^2\right)^s\\
&= \sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+s}{s}\left(1-Ry_{k-1}\right)^{-s-1} x_{k-1}^s x_{k-2}^m y_{k-2}^s R^{3s+2m} z^{2s+m}\\
&= \sum_{m=0}^{\infty}\sum_{s=0}^{m}\binom{m}{s}\left(1-Ry_{k-1}\right)^{-s-1} x_{k-1}^s x_{k-2}^{m-s} y_{k-2}^s R^{s+2m} z^{s+m}\\
&= \sum_{m=0}^{\infty}\sum_{s=0}^{m}\sum_{t=0}^{\infty}\binom{m}{s}\binom{t+s}{s} x_{k-1}^s x_{k-2}^{m-s} y_{k-1}^t y_{k-2}^s R^{s+2m+t} z^{s+m+t}\\
&= \sum_{n=0}^{\infty}\sum_{0\le m+s\le n}\binom{m}{s}\binom{n-m}{s} x_{k-1}^s x_{k-2}^{m-s} y_{k-1}^{n-m-s} y_{k-2}^s R^{n+m} z^n .
\end{aligned}$$

So, by equating coefficients of z^n, we find that

$$f_{k+1,n} = \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} \frac{\left(u_{k-1}^{(2)}\right)^s \left(u_{k-2}^{(2)}\right)^{m-s} \left(u_{k}^{(2)}\right)^{n-m-s} \left(u_{k-1}^{(2)}\right)^s R^{n+m}}{R^{s+m-s+n-m-s+s}}$$

$$= \sum_{0 \le m+s \le n} \binom{m}{s}\binom{n-m}{s} \left(u_{k-2}^{(2)}\right)^{m-s} \left(u_{k}^{(2)}\right)^{n-m-s} \left(u_{k-1}^{(2)}\right)^{2s} R^{m}. \qquad \square$$

as required. For example, when $k = 1$,

$$f_{2,n} = \sum_{0 \le 2m \le n} \binom{n-m}{m} R^m = \sum_{0 \le 2m \le n} \binom{n-m}{m} R^{n-m}$$

as in Barakat [2].

5 Conclusion

Other identities can be readily developed to relate these discatenated and lacunary Fibonacci sequnces to other recursive properties. We shall conclude here with one such example.

$$f_{k+1,n} z^n = f_{k+1,n} = \alpha^{kn} \left(\underline{\left(\frac{\alpha}{\beta}\right)^k} \right)_{n+1} \tag{5.1}$$

in which $\underline{x}_n$ represents the n-th reduced Fermatian of index x, defined formally by

$$\underline{x}_n = 1 + x + x^2 + \cdots + x^{n-1}, \tag{5.2}$$

as used by Whitney [17] and extended by Shannon [14].

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