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# On the pulsating Padovan sequence

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**Abstract:** A novel kind of Padovan sequence is introduced, and precise formulas for the form of its members are given and proven. Furthermore, the pulsating Padovan sequence in its most general form is introduced and the obtained identity is proved.
**Keywords:** Fibonacci sequence, Jacobsthal sequence, Padovan sequence.
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## 1 Introduction

The Fibonacci sequence is a well-known integer sequence with amazing features that have been extensively researched by authors. In the present work we focus on the pulsating Fibonacci sequence, which was first defined by K.T. Atanassov in [1]. After this work, there are some studies on the pulsating Fibonacci sequences [2–4, 7, 8, 11–13].

In the present work, we define the pulsating Padovan sequence and some of its properties. We also introduced $n$-pulsted Padovan sequences. We recall the definitions of the $k$-Fibonacci and Padovan sequences as follows:

The $k$-Fibonacci and Padovan sequences $\{F_{n,k}\}$ and $\{P_n\}$ are defined by two and three order recurrences for $n \geq 0$, respectively,

$$F_{n+2,k} = F_{n+1,k} + kF_{n,k},$$
$$P_{n+3} = P_{n+1} + P_n,$$

with the initial conditions being given as follows, respectively:

$$F_{0,k} = 0 \quad \text{and} \quad F_{1,k} = 1,$$
$$P_0 = 1, \quad P_1 = 1 \quad \text{and} \quad P_2 = 1.$$

The first few members of these sequences are given as follows, respectively.

| $\boldsymbol{n}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{F_{n,k}}$ | 0 | 1 | 1 | $k+1$ | $2k+1$ | $k^2+3k+1$ | $3k^2+4k+1$ | $k^3+6k^2+5k+1$ | ... |
| $\boldsymbol{P_n}$ | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | ... |

Table 1. The first few members of $F_n$ and $P_n$

More information about these sequences can be found in [5, 6, 9, 10].

## 2 The pulsating Padovan sequence

The Padovan sequence has undergone numerous extensions and modifications over the past decade. In this paper, a new type of Padovan-like sequence is introduced to continue this line of research on Padovan sequences.

Let $a$ and $b$ be two fixed real numbers. The two sequences we will build are as follows:

$$\begin{aligned} \Upsilon_1 = \Omega_1 &= a, \\ \Upsilon_2 = b, \Omega_2 &= c, \\ \Upsilon_{2k+1} = \Omega_{2k+1} &= \Upsilon_{2k} + \Omega_{2k}, \\ \Upsilon_{2k} &= \Omega_{2k-2} + \Upsilon_{2k-3}, \\ \Omega_{2k} &= \Upsilon_{2k-2} + \Omega_{2k-3}, \end{aligned}$$

for the positive natural number $k \geq 1$. A pulsating Padovan sequence is the name given to this pair of sequences. The first few values of the two sequences are given in Table 2 below.

| $\boldsymbol{n}$ | $\boldsymbol{\Upsilon_n}$ | $\boldsymbol{\Upsilon_n = \Omega_n}$ | $\boldsymbol{\Omega_n}$ |
|---|---|---|---|
| 1 | | $a$ | |
| 2 | $b$ | | $c$ |
| 3 | | $b+c$ | |
| 4 | $a+c$ | | $a+b$ |
| 5 | | $2a+b+c$ | |
| 6 | $a+2b+c$ | | $a+b+2c$ |
| 7 | | $2a+3b+3c$ | |
| 8 | $3a+2b+3c$ | | $3a+3b+2c$ |
| 9 | | $6a+5b+5c$ | |
| 10 | $5a+6b+5c$ | | $5a+5b+6c$ |
| 11 | | $10a+11b+11c$ | |
| 12 | $11a+10b+11c$ | | $11a+11b+10c$ |
| 13 | | $22a+21b+21c$ | |
| 14 | $21a+22b+21c$ | | $21a+21b+22c$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 2. The first few members of the two sequences

**Theorem 2.1.** *For $k \geq 1$,*

$$\begin{aligned} \Upsilon_{2k+1} = \Omega_{2k+1} &= J_k(b+c) + 2J_{k-1}a, \\ \Upsilon_{2k} &= J_{k-1}(a+c) + 2J_{k-2}b, \\ \Omega_{2k} &= J_{k-1}(a+b) + 2J_{k-2}c, \end{aligned}$$

*where $J_k$ is the $k$-th $2-$Fibonacci number which is called the Jacobsthal number.*

*Proof.* The assertion is obviously valid when $k = 0$. Assume that for some positive natural number $k \geq 1$ are correct. Now, we check for the positive natural number $k+1$. First,

$$\begin{aligned} \Upsilon_{2k+1} = \Omega_{2k+1} &= \Upsilon_{2k} + \Omega_{2k} \\ &= J_{k-1}(2a+b+c) + 2J_{k-2}(b+c) \\ &= (J_{k-1} + 2J_{k-2})\,(b+c) + 2J_{k-1}a \\ &= J_k(b+c) + 2J_{k-1}a. \end{aligned}$$

Second,

$$\begin{aligned} \Upsilon_{2k+2} &= \Omega_{2k} + \Upsilon_{2k-1} \\ &= J_{k-1}(a+b) + 2J_{k-2}c + J_{k-1}(b+c) + 2J_{k-2}a \\ &= (J_{k-1} + 2J_{k-2})\,(a+c) + 2J_{k-1}b \\ &= J_k(a+c) + 2J_{k-1}b. \end{aligned}$$

All other equalities are checked in the same way. $\square$

For instance, if $c=-b$, the pulsating Padovan sequence is as follows:

| $\boldsymbol{n}$ | $\boldsymbol{\Upsilon_n}$ | $\boldsymbol{\Upsilon_n=\Omega_n}$ | $\boldsymbol{\Omega_n}$ |
|---|---|---|---|
| 1 | | $a$ | |
| 2 | $b$ | | $-b$ |
| 3 | | $0$ | |
| 4 | $a-b$ | | $a+b$ |
| 5 | | $2a$ | |
| 6 | $a+b$ | | $a-b$ |
| 7 | | $2a$ | |
| 8 | $3a-b$ | | $3a+b$ |
| 9 | | $6a$ | |
| 10 | $5a+b$ | | $5a-b$ |
| 11 | | $10a$ | |
| 12 | $11a-b$ | | $11a+b$ |
| 13 | | $22a$ | |
| 14 | $21a+b$ | | $21a-b$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

If $c=b$, the pulsating Padovan sequence takes the following form:

| $\boldsymbol{n}$ | $\boldsymbol{\Upsilon_n}$ | $\boldsymbol{\Upsilon_n=\Omega_n}$ | $\boldsymbol{\Omega_n}$ |
|---|---|---|---|
| 1 | | $a$ | |
| 2 | $b$ | | $b$ |
| 3 | | $2b$ | |
| 4 | $a+b$ | | $a+b$ |
| 5 | | $2a+2b$ | |
| 6 | $a+3b$ | | $a+3b$ |
| 7 | | $2a+6b$ | |
| 8 | $3a+5b$ | | $3a+5b$ |
| 9 | | $6a+10b$ | |
| 10 | $5a+11b$ | | $5a+11b$ |
| 11 | | $10a+22b$ | |
| 12 | $11a+21b$ | | $11a+21b$ |
| 13 | | $22a+42b$ | |
| 14 | $21a+43b$ | | $21a+43b$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

## 3 The $k$-pulsating Padovan sequences

Let $a$ and $b_1, b_2, b_3, \ldots, b_k$ be $k+1$ fixed real numbers and $1 \leq i \leq k$. The $k$-pulsating Padovan sequences $\{\tau_{i,n}\}_{n\geq 1}$ are defined by the recurrences relations:

$$\begin{aligned}
\tau_{1,2m+1} = \tau_{2,2m+1} = \cdots = \tau_{k,2m+1} = \tau_{1,2m} + \tau_{2,2m} + \cdots + \tau_{k,2m},\\
\tau_{1,2m} = \tau_{k,2m-2} + \tau_{1,2m-3},\\
\tau_{2,2m} = \tau_{k-1,2m-2} + \tau_{2,2m-3},\\
\vdots\\
\tau_{k,2m} = \tau_{1,2m-2} + \tau_{k,2m-3},
\end{aligned}$$

with initial conditions

$$\begin{aligned}
\tau_{1,1} = \tau_{2,1} = \cdots = \tau_{k,1} = a,\\
\tau_{1,2} = b_1, \tau_{2,2} = b_2, \ldots, \tau_{k,2} = b_k.
\end{aligned}$$

Let

$$B = \sum_{i=1}^{k} b_i.$$

The first values of the new sequence are shown in Table 3.

| $m$ | $\tau_{1,m}$ | $\tau_{2,m}$ | $\ldots$ | $\tau_{k,m}$ |
|---|---|---|---|---|
| 1 | $a$ | $a$ | $\ldots$ | $a$ |
| 2 | $b_1$ | $b_2$ | $\ldots$ | $b_k$ |
| 3 | $B$ | $B$ | $\ldots$ | $B$ |
| 4 | $a+b_k$ | $a+b_{k-1}$ | $\ldots$ | $a+b_1$ |
| 5 | $ka+B$ | $ka+B$ | $\ldots$ | $ka+B$ |
| 6 | $a+b_1+B$ | $a+b_2+B$ | $\ldots$ | $a+b_k+B$ |
| 7 | $ka+(k+1)B$ | $ka+(k+1)B$ | $\ldots$ | $ka+(k+1)B$ |
| 8 | $(k+1)a+b_k+2B$ | $(k+1)a+b_{k-1}+2B$ | $\ldots$ | $(k+1)a+b_1+2B$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 3. The first few members of $\{\tau_{i,n}\}_{n\geq 1}$

**Theorem 3.1.** *For $m \geq 1$, we have*

$$\begin{aligned}
\tau_{i,4m} &= F_{2m-1,k}a + b_{k-i+1} + \left(\sum_{j=0}^{2m-2} F_{j,k}\right) B,\\
\tau_{i,4m-1} &= kF_{2m-2,k}a + F_{2m-1,k}B,\\
\tau_{i,4m-2} &= F_{2m-2,k}a + b_i + \left(\sum_{j=0}^{2m-3} F_{j,k}\right) B,\\
\tau_{i,4m-3} &= kF_{2m-3,k}a + F_{2m-2,k}B,
\end{aligned}$$

*where $F_{m,k}$ is the $k$-Fibonacci number.*

*Proof.* The assertion is obviously valid when $k = 0$. Assume that for some positive natural number $k \geq 1$, are correct. Now, we check for the positive natural number $k + 1$.

$$\begin{aligned}\tau_{1,4m-1} = \tau_{2,4m-1} = \cdots = \tau_{k,4m-1} &= \tau_{1,4m-2} + \tau_{2,4m-2} + \cdots + \tau_{k,4m-2} \\ &= kF_{2m-2,k}a + B + k\left(\sum_{j=0}^{2m-3} F_{j,k}\right)B \\ &= kF_{2m-2,k}a + F_{2m-1,k}B,\end{aligned}$$

and

$$\begin{aligned}\tau_{1,4m-3} = \tau_{2,4m-3} = \cdots = \tau_{k,4m-3} &= \tau_{1,4m-4} + \tau_{2,4m-4} + \cdots + \tau_{k,4m-4} \\ &= kF_{2m-3,k}a + B + k\left(\sum_{j=0}^{2m-4} F_{j,k}\right)B \\ &= kF_{2m-3,k}a + F_{2m-2,k}B.\end{aligned}$$

□

# 4 Conclusion

In the present work, we introduce the pulsating Padovan sequences. We establish some accurate formulas. Moreover, we define the $k$-pulsating Padovan sequences.


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