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On tertions and other algebraic objects

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To Tony for his 85th anniversary,
without whose support this idea would not be realized.

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Abstract: The concept of the object called "tertion" is discussed. Some operations over tertions are introduced and their properties are studied. The relationship between tertions, complex numbers are quaternions are discussed.
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1 Introduction

The present idea (of course, in a simpler form), was generated by me around 1971 – 1972, when I was at the Burgas High School of Mathematics. It took me more than 20 years to finally publish it in 1994 in 9 preprints [8–16].

Some years after this, on the basis of [10], Prof. Anthony (Tony) Shannon (Australia) and I published the paper [17], on basis of which in the next years a number of publications by other colleagues appeared.

Finally, I started work on a book in which I will collect, correct, extend and systematize my results. The present paper contains a part of the new, unpublished results.

The ultimate goal of my research is to construct models of objects, called **"tertions"** (from different types), having in some sense an intermediate place between the complex numbers and quaternions. The word "tertion" (initially, I wrote "tercion", but after discussion with Tony, I changed it to "tertion") was invented by me as a name of the new object by analogy with the word "quaternion".

In [17], Tony offered to refer to a part of the new objects as "matrix-tertion" and as "matrix-noitret" ("noitret being the mirror of "tertion"), but here I will keep the terminology from [8–16] with the following changes: "A-matrix" with "A-tertion" and "V-matrix" with "V-tertion".

2 Definition of the concept of an A-tertion

The objects "vector" and "matrix" are well known (see, e.g., [19, 28]). Their simplest, but not trivial forms are the 2-dimensional vector $\langle a, b\rangle$ and the (2×2)-dimensional matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now, following [8], we shall describe an object that is intermediate between both of them:

$$\Big/ \begin{matrix} & a & \\ b & & c \end{matrix} \Big\backslash ,$$

where a, b and c are elements of a fixed set. Let us call this object "an A-tertion", having in mind its form.

Let

$$A_2 = \left\{ \Big/ \begin{matrix} & \alpha & \\ \beta & & \gamma \end{matrix} \Big\backslash \mid \alpha, \beta, \gamma \in \mathcal{R} \right\},$$

where, here and below $\mathcal{R}$ is the set of the real numbers.

The subscript "2" of A_2 corresponds to the tertion dimension, that is related to the number of the A-tertion rows.

By analogy, we can define the sets $A_3, A_4, ...$

Let everywhere $a, b, c, d, e, f, g, h, i \in \mathcal{R}$ be arbitrary numbers.

Following [8], we define the first operation that can be defined over the elements of A_2 as follows:

$$\Big/ \begin{matrix} & a & \\ b & & c \end{matrix} \Big\backslash + \Big/ \begin{matrix} & d & \\ e & & f \end{matrix} \Big\backslash = \Big/ \begin{matrix} & a+d & \\ b+e & & c+f \end{matrix} \Big\backslash .$$

Obviously, the A-tertion $\Big/ \begin{matrix} & 0 & \\ 0 & & 0 \end{matrix} \Big\backslash$ is a (left and right) identity element.

The operation "multiplication" of a real number α with an A-tertion is defined as follows:

$$\alpha \Big/ \begin{matrix} & a & \\ b & & c \end{matrix} \Big\backslash = \Big/ \begin{matrix} & \alpha a & \\ \alpha b & & \alpha c \end{matrix} \Big\backslash .$$

Let us define the following constant A_2-tertion :

$$O = \left/ \begin{array}{ccc} & 0 & \\ 0 & & 0 \end{array} \right\backslash ,$$

$$E = \left/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \right\backslash ,$$

$$I = \left/ \begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array} \right\backslash ,$$

$$J = \left/ \begin{array}{ccc} & 0 & \\ 0 & & 1 \end{array} \right\backslash .$$

From the above equalities it follows that every A-tertion $\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash$ can be presented in the form:

$$\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash = a \left/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \right\backslash + b \left/ \begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array} \right\backslash + c \left/ \begin{array}{ccc} & 0 & \\ 0 & & 1 \end{array} \right\backslash .$$

It also follows that

$$\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash = aE + bI + cJ.$$

From here, it can also be seen that

$$a \left/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \right\backslash + b \left/ \begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array} \right\backslash + c \left/ \begin{array}{ccc} & 0 & \\ 0 & & 1 \end{array} \right\backslash = \left/ \begin{array}{ccc} & 0 & \\ 0 & & 0 \end{array} \right\backslash$$

if and only if $a = b = c = 0$.

In [9–13], by 1994, five operations "multiplication" are defined over A_2-tertions. In the next years, three new operations were introduced. Below, we will define a ninth operation and discuss its properties and applications.

3 Operation $\circ_9$ over A-tertions

Here, for a first time, we introduce the ninth operation "$\circ$" over A_2-tertions:

$$\left/ \begin{array}{ccc} & a & \\ b & & c \end{array} \right\backslash \circ_9 \left/ \begin{array}{ccc} & d & \\ e & & f \end{array} \right\backslash = \left/ \begin{array}{ccc} & ad - be & \\ ae + bd & & bf + ce \end{array} \right\backslash .$$

We will describe the properties of this operation in more detail, because we will use them below.

Obviously, A_2 is closed with respect to operation "$\circ_9$".

The operation is commutative and distributive with respect to operation "+", and in general, it is not associative, because:

$$
\begin{aligned}
&\left(\left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & f \end{array} \right\backslash \right) \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{ad - be} \\ ae + bd & bf + ce \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{adg - beg - aeh - bdh} \\ adh - beh + aeg + bdg & aei + bdi + bfh + ceh \end{array} \right\backslash \\
&\qquad \neq \left/ \begin{array}{cc} \multicolumn{2}{c}{adg - beg - aeh - bdh} \\ adh - beh + aeg + bdg & bei + cei + bfh + cfh \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{dg - eh} \\ dh + eg & ei + fh \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash \circ_9 \left(\left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & f \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash \right),
\end{aligned}
$$

$$
\begin{aligned}
&\left(\left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash + \left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & f \end{array} \right\backslash \right) \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash = \left/ \begin{array}{cc} \multicolumn{2}{c}{a + d} \\ b + e & c + f \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{ad + dg - bh - eh} \\ ah + dh + bg + eg & bi + ei + fh + ch \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{ad - bh} \\ ah + bg & bi + ch \end{array} \right\backslash + \left/ \begin{array}{cc} \multicolumn{2}{c}{dg - eh} \\ dh + eg & ei + fh \end{array} \right\backslash \\
&\qquad = \left(\left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & c \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash \right) + \left(\left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & f \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash \right).
\end{aligned}
$$

On the other hand, and this is important for the discussion below,

$$
\begin{aligned}
&\left(\left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & 0 \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & 0 \end{array} \right\backslash \right) \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & 0 \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{ad - be} \\ ae + bd & 0 \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & i \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{adg - beg - aeh - bdh} \\ adh - beh + aeg + bdg & 0 \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & 0 \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{dg - eh} \\ dh + eg & 0 \end{array} \right\backslash \\
&\qquad = \left/ \begin{array}{cc} \multicolumn{2}{c}{a} \\ b & 0 \end{array} \right\backslash \circ_9 \left(\left/ \begin{array}{cc} \multicolumn{2}{c}{d} \\ e & 0 \end{array} \right\backslash \circ_9 \left/ \begin{array}{cc} \multicolumn{2}{c}{g} \\ h & 0 \end{array} \right\backslash \right),
\end{aligned}
$$

i.e., in this case, when $c = f = i = 0$, the operation is associative.

For $b \neq 0$ and $a^2 + b^2 \neq 0$ (obviously, the second condition follows from the validity of the first one), i.e., for

$$\Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash \in A_{2,\circ_9} \equiv \left\{ \Big/ \begin{array}{ccc} & \alpha & \\ \beta & & \gamma \end{array} \Big\backslash \, |\alpha, \beta, \gamma \in \mathcal{R} \;\&\; \beta \neq 0 \right\},$$

the solutions of the equations

$$\Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & x & \\ y & & z \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & p & \\ q & & r \end{array} \Big\backslash$$

and

$$\Big/ \begin{array}{ccc} & x & \\ y & & z \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & p & \\ q & & r \end{array} \Big\backslash$$

are

$$\Big/ \begin{array}{ccc} & x & \\ y & & z \end{array} \Big\backslash = \Bigg/ \begin{array}{ccc} & \frac{ap+bq}{a^2+b^2} & \\ \frac{aq-bp}{a^2+b^2} & & \frac{a^2r-acq+bcp+b^2r}{b(a^2+b^2)} \end{array} \Bigg\backslash .$$

In particular,

$$\Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & \frac{a}{a^2+b^2} & \\ \frac{-b}{a^2+b^2} & & \frac{c}{a^2+b^2} \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \Big\backslash$$

$$= \Big/ \begin{array}{ccc} & \frac{a}{a^2+b^2} & \\ \frac{-b}{a^2+b^2} & & \frac{c}{a^2+b^2} \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash$$

$$\Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & \frac{b}{a^2+b^2} & \\ \frac{a}{a^2+b^2} & & \frac{-ac}{a^2+b^2} \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array} \Big\backslash$$

$$= \Big/ \begin{array}{ccc} & \frac{b}{a^2+b^2} & \\ \frac{a}{a^2+b^2} & & \frac{-ac}{a^2+b^2} \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash$$

$$\Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & 0 & \\ 0 & & \frac{1}{b} \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & 0 & \\ 0 & & 1 \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & 0 & \\ 0 & & \frac{1}{b} \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash$$

and

$$\Big/ \begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & 0 & \\ -1 & & 0 \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \Big\backslash .$$

The left and right identity element of A is $\Big/ \begin{array}{ccc} & 1 & \\ 0 & & \frac{c}{b} \end{array} \Big\backslash$, because

$$\Big/ \begin{array}{ccc} & 1 & \\ 0 & & \frac{c}{b} \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & a & \\ b & & c \end{array} \Big\backslash \circ_9 \Big/ \begin{array}{ccc} & 1 & \\ 0 & & \frac{c}{b} \end{array} \Big\backslash .$$

Obviously, the equation

$$\Big/ \begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array} \Big\backslash \circ_1 \Big/ \begin{array}{ccc} & x & \\ y & & z \end{array} \Big\backslash = \Big/ \begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array} \Big\backslash$$

for which, e.g., in [18] is mentioned that it does not have any solution*, here has the solution

$$\Big/\begin{array}{ccc} & x & \\ y & & z \end{array}\Big\backslash = \Big/\begin{array}{ccc} & 0 & \\ -1 & & 0 \end{array}\Big\backslash .$$

Also,

$$\Big/\begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array}\Big\backslash \circ_9 \Big/\begin{array}{ccc} & a & \\ b & & c \end{array}\Big\backslash = \Big/\begin{array}{ccc} & a & \\ b & & 0 \end{array}\Big\backslash = \Big/\begin{array}{ccc} & a & \\ b & & c \end{array}\Big\backslash \circ_9 \Big/\begin{array}{ccc} & 1 & \\ 0 & & 0 \end{array}\Big\backslash$$

$$\Big/\begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array}\Big\backslash \circ_9 \Big/\begin{array}{ccc} & a & \\ b & & c \end{array}\Big\backslash = \Big/\begin{array}{ccc} & -b & \\ a & & c \end{array}\Big\backslash = \Big/\begin{array}{ccc} & a & \\ b & & c \end{array}\Big\backslash \circ_9 \Big/\begin{array}{ccc} & 0 & \\ 1 & & 0 \end{array}\Big\backslash$$

$$\Big/\begin{array}{ccc} & 0 & \\ 0 & & 1 \end{array}\Big\backslash \circ_9 \Big/\begin{array}{ccc} & a & \\ b & & c \end{array}\Big\backslash = \Big/\begin{array}{ccc} & 0 & \\ 0 & & b \end{array}\Big\backslash = \Big/\begin{array}{ccc} & a & \\ b & & c \end{array}\Big\backslash \circ_9 \Big/\begin{array}{ccc} & 0 & \\ 0 & & 1 \end{array}\Big\backslash .$$

Proposition 1. For each natural number n and i taking values from 1 to 8:

$$\Big/\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}\Big\backslash^{(8n+i+1,\circ_9)} = \begin{cases} \Big/\begin{array}{ccc} & 0 & \\ 2^{4n+1} & & 2 \end{array}\Big\backslash , & \text{if } i=1 \\ \Big/\begin{array}{ccc} & -2^{4n+1} & \\ 2^{4n+1} & & 2^{4n+1}+2 \end{array}\Big\backslash , & \text{if } i=2 \\ \Big/\begin{array}{ccc} & -2^{4n+2} & \\ 0 & & 2^{4n+2}+2 \end{array}\Big\backslash , & \text{if } i=3 \\ \Big/\begin{array}{ccc} & -2^{4n+2} & \\ -2^{4n+2} & & 2^{4n+2}+2 \end{array}\Big\backslash , & \text{if } i=4 \\ \Big/\begin{array}{ccc} & 0 & \\ -2^{4n+3} & & 2 \end{array}\Big\backslash , & \text{if } i=5 \\ \Big/\begin{array}{ccc} & 2^{4n+3} & \\ -2^{4n+3} & & -2^{4n+3}+2 \end{array}\Big\backslash , & \text{if } i=6 \\ \Big/\begin{array}{ccc} & 2^{4n+4} & \\ 0 & & -2^{4n+4}+2 \end{array}\Big\backslash , & \text{if } i=7 \\ \Big/\begin{array}{ccc} & 2^{4n+4} & \\ 2^{4n+4} & & -2^{4n+4}+2 \end{array}\Big\backslash , & \text{if } i=8 \end{cases} .$$

Proof. When $n=0$, we check directly that

$$\Big/\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}\Big\backslash^{(2,\circ_9)} = \Big/\begin{array}{ccc} & 0 & \\ 2 & & 2 \end{array}\Big\backslash$$

$$\Big/\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}\Big\backslash^{(3,\circ_9)} = \Big/\begin{array}{ccc} & -2 & \\ 2 & & 4 \end{array}\Big\backslash$$

* In [18], the equation is given in the form $(0+1i+0j)x = 1+0i+0j$.

$$\left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(4,\circ_9)} = \left/\begin{matrix} & -4 & \\ 0 & & 6 \end{matrix}\right\backslash$$

$$\left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(5,\circ_9)} = \left/\begin{matrix} & -4 & \\ -4 & & 6 \end{matrix}\right\backslash$$

$$\left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(6,\circ_9)} = \left/\begin{matrix} & 0 & \\ -8 & & 2 \end{matrix}\right\backslash$$

$$\left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(7,\circ_9)} = \left/\begin{matrix} & 8 & \\ -8 & & -6 \end{matrix}\right\backslash$$

$$\left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(8,\circ_9)} = \left/\begin{matrix} & 16 & \\ 0 & & -14 \end{matrix}\right\backslash$$

$$\left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(9,\circ_9)} = \left/\begin{matrix} & 16 & \\ 16 & & -14 \end{matrix}\right\backslash$$

Let us assume that the proposition is valid for some natural number n. Then

$$\begin{aligned}
\left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(8(n+1)+2,\circ_9)} &= \left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(8(n+1)+1,\circ_9)} \circ_9 \left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash \\
&= \left/\begin{matrix} & 2^{4n+4} & \\ 2^{4n+4} & & -2^{4n+4}+2 \end{matrix}\right\backslash \circ_9 \left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash \\
&= \left/\begin{matrix} & 2^{4n+4}-2^{4n+4} & \\ 2^{4n+4}+2^{4n+4} & & 2^{4n+4}+(-2^{4n+4}+2) \end{matrix}\right\backslash \\
&= \left/\begin{matrix} & 0 & \\ 2^{4n+5} & & 2) \end{matrix}\right\backslash .
\end{aligned}$$

$$\begin{aligned}
\left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(8(n+1)+3,\circ_9)} &= \left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash^{(8(n+1)+2,\circ_9)} \circ_9 \left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash \\
&= \left/\begin{matrix} & 0 & \\ 2^{4n+5} & & 2 \end{matrix}\right\backslash \circ_9 \left/\begin{matrix} & 1 & \\ 1 & & 1 \end{matrix}\right\backslash = \left/\begin{matrix} & -2^{4n+5} & \\ 2^{4n+5} & & 2^{4n+5}+2 \end{matrix}\right\backslash ,
\end{aligned}$$

etc., for the rest of the cases. □

It is interesting to mention that for every $a, b, c \in \mathcal{R}$

$$\left/\begin{matrix} & a & \\ 0 & & b \end{matrix}\right\backslash \circ_9 \left/\begin{matrix} & 0 & \\ 0 & & c \end{matrix}\right\backslash = \left/\begin{matrix} & 0 & \\ 0 & & 0 \end{matrix}\right\backslash .$$

Finally, we note that the following table is valid for operation "$\circ_9$":

$\circ_9$	E	I	J
E	E	I	O
I	I	$-E$	J
J	O	J	O

.

4 Definition of the concept of a V-tertion

Below, following [14], we will introduce a new object, having the form

$$\diagdown \begin{array}{ccc} b & & c \\ & a & \end{array} \diagup .$$

Let us call this object a "V-tertion". For it, we can re-define all operations over the A-tertions, but now, over V-tertions.

We will mention only the definitions of operations "$+$" and "$\circ_9$", but now, over V-tertions, elements of set

$$V_2 = \left\{ \diagdown \begin{array}{ccc} \beta & & \gamma \\ & \alpha & \end{array} \diagup \, | \alpha, \beta, \gamma, \in \mathcal{R} \right\}.$$

They are:

$$\diagdown \begin{array}{ccc} b & & c \\ & a & \end{array} \diagup + \diagdown \begin{array}{ccc} e & & f \\ & d & \end{array} \diagup \quad = \diagdown \begin{array}{ccc} b+e & & c+f \\ & a+d & \end{array} \diagup ,$$

$$\diagdown \begin{array}{ccc} b & & c \\ & a & \end{array} \diagup \circ_9 \diagdown \begin{array}{ccc} e & & f \\ & d & \end{array} \diagup \quad = \diagdown \begin{array}{ccc} ae+bd & & bf+ce \\ & ad-be & \end{array} \diagup .$$

5 On the representations of the complex numbers by A- and V-tertions

Let us juxtapose the A-tertion $\diagup \begin{array}{ccc} & a & \\ b & & 0 \end{array} \diagdown$ to the complex number $a + b\,\mathbf{i}$. Then, we can see immediately that the well-known complex number equalities

$$(a + b\,\mathbf{i}) + (c + \,\mathbf{i}) = (a + c) + (b + d)\,\mathbf{i},$$

$$(a + b\,\mathbf{i})(c + d\,\mathbf{i}) = (ac - bd) + (ad + bc)\,\mathbf{i},$$

and

$$\alpha(a + b\,\mathbf{i}) = \alpha a + \alpha b\,\mathbf{i}$$

have the following respective A-tertion forms:

$$\diagup \begin{array}{ccc} & a & \\ b & & 0 \end{array} \diagdown + \diagup \begin{array}{ccc} & c & \\ d & & 0 \end{array} \diagdown \quad = \diagup \begin{array}{ccc} & a+c & \\ b+d & & 0 \end{array} \diagdown ,$$

$$\diagup \begin{array}{ccc} & a & \\ b & & 0 \end{array} \diagdown \circ_9 \diagup \begin{array}{ccc} & c & \\ d & & 0 \end{array} \diagdown \quad = \diagup \begin{array}{ccc} & ac-bd & \\ ad+bc & & 0 \end{array} \diagdown = \diagup \begin{array}{ccc} & c & \\ d & & 0 \end{array} \diagdown \circ_9 \diagup \begin{array}{ccc} & a & \\ b & & 0 \end{array} \diagdown ,$$

$$\alpha \diagup \begin{array}{ccc} & a & \\ b & & 0 \end{array} \diagdown \quad = \diagup \begin{array}{ccc} & \alpha a & \\ \alpha b & & 0 \end{array} \diagdown .$$

For the representation of the fourth basic complex number equality

$$(a+b\,\mathbf{i})^{-1} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}\,\mathbf{i},$$

using the equality

$$\Big/\begin{matrix} & a & \\ b & & c \end{matrix}\Big\backslash \circ_9 \Big/\begin{matrix} & \frac{a}{a^2+b^2} & \\ \frac{-b}{a^2+b^2} & & \frac{-c}{a^2+b^2} \end{matrix}\Big\backslash = \Big/\begin{matrix} & 1 & \\ 0 & & 0 \end{matrix}\Big\backslash ,$$

we can define

$$\Big/\begin{matrix} & a & \\ b & & c \end{matrix}\Big\backslash^{-1} = \Big/\begin{matrix} & \frac{a}{a^2+b^2} & \\ \frac{-b}{a^2+b^2} & & \frac{-c}{a^2+b^2} \end{matrix}\Big\backslash .$$

Therefore, its representation is

$$\Big/\begin{matrix} & a & \\ b & & 0 \end{matrix}\Big\backslash^{-1} = \Big/\begin{matrix} & \frac{a}{a^2+b^2} & \\ \frac{-b}{a^2+b^2} & & 0 \end{matrix}\Big\backslash .$$

Proposition 2. For each natural number $n \geq 1$ and for every two real numbers a, b:

$$\Big/\begin{matrix} & a & \\ b & & 0 \end{matrix}\Big\backslash^{(n,\circ_9)} = \Big/\begin{matrix} & \alpha_n & \\ \beta_n & & 0 \end{matrix}\Big\backslash ,$$

where

$$\begin{aligned} \alpha_1 &= a, \\ \beta_1 &= b, \\ \alpha_n &= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k} a^{n-2k} b^{2k}, \\ \beta_n &= \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k C_n^{2k+1} a^{n-2k-1} b^{2k+1}, \end{aligned}$$

where

$$C_p^q = \frac{q!}{(q-p)!p!}$$

for p, q – natural numbers and $q \geq p$.

Proof. For $n = 1$, it is obvious that

$$\Big/\begin{matrix} & a & \\ b & & 0 \end{matrix}\Big\backslash^{(1,\circ_9)} = \Big/\begin{matrix} & \alpha_1 & \\ \beta_1 & & 0 \end{matrix}\Big\backslash .$$

Let us assume that the assertion is valid for some natural number n. Then

$$\begin{aligned} \Big/\begin{matrix} & a & \\ b & & 0 \end{matrix}\Big\backslash^{(n+1,\circ_9)} &= \Big/\begin{matrix} & a & \\ b & & 0 \end{matrix}\Big\backslash^{(n,\circ_9)} \circ_9 \Big/\begin{matrix} & a & \\ b & & 0 \end{matrix}\Big\backslash \\ &= \Big/\begin{matrix} & \alpha_n & \\ \beta_n & & 0 \end{matrix}\Big\backslash \circ_9 \Big/\begin{matrix} & a & \\ b & & 0 \end{matrix}\Big\backslash \\ &= \Big/\begin{matrix} & \alpha_n a - \beta_n b & \\ \alpha_n b + \beta_n a & & 0 \end{matrix}\Big\backslash . \end{aligned}$$

Now, we see that

$$\begin{aligned}
\alpha_n a - \beta_n b &= a \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k} a^{n-2k} b^{2k} - b \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k C_n^{2k+1} a^{n-2k-1} b^{2k+1} \\
&= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k} a^{n-2k+1} b^{2k} - \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k C_n^{2k+1} a^{n-2k-1} b^{2k+2} \\
&= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k} a^{n-2k+1} b^{2k} - \sum_{k=1}^{[\frac{n+1}{2}]} (-1)^{k+1} C_n^{2k-1} a^{n-2k+1} b^{2k} \\
&= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k} a^{n-2k+1} b^{2k} + \sum_{k=1}^{[\frac{n+1}{2}]} (-1)^k C_n^{2k-1} a^{n-2k+1} b^{2k} \\
&= \sum_{k=0}^{[\frac{n+1}{2}]} (-1)^k C_{n+1}^{2k} a^{n-2k+1} b^{2k} = \alpha_{n+1}
\end{aligned}$$

and in the same way

$$\begin{aligned}
\alpha_n b + \beta_n a &= b \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k} a^{n-2k} b^{2k} + a \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k C_n^{2k+1} a^{n-2k-1} b^{2k+1} \\
&= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k} a^{n-2k} b^{2k+1} + \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k C_n^{2k+1} a^{n-2k} b^{2k+1} \\
&= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_{n+1}^{2k+1} a^{n-2k-1} b^{2k+1} = \beta_{n+1}.
\end{aligned}$$

□

On the other hand, in a similar way we can check that

$$(a + b\,\mathbf{i})^n = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k C_n^{2k} a^{n-2k} b^{2k} + \left(\sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k C_n^{2k+1} a^{n-2k-1} b^{2k+1} \right) \mathbf{i}.$$

Second, we can represent the complex number $a + b\,\mathbf{i}$ with the A-tertion $\diagup\!\begin{smallmatrix} & a & \\ 0 & & b \end{smallmatrix}\!\diagdown$.

Now, the above formulas obtain, respectively, the forms

$$\diagup\!\begin{matrix} & a & \\ 0 & & b \end{matrix}\!\diagdown + \diagup\!\begin{matrix} & c & \\ 0 & & d \end{matrix}\!\diagdown = \diagup\!\begin{matrix} & a+c & \\ 0 & & b+d \end{matrix}\!\diagdown ,$$

$$\diagup\!\begin{matrix} & a & \\ 0 & & b \end{matrix}\!\diagdown \circ_9 \diagup\!\begin{matrix} & c & \\ 0 & & d \end{matrix}\!\diagdown = \diagup\!\begin{matrix} & ac-bd & \\ 0 & & ad+bc \end{matrix}\!\diagdown ,$$

$$\alpha \diagup\!\begin{matrix} & a & \\ 0 & & b \end{matrix}\!\diagdown = \diagup\!\begin{matrix} & \alpha a & \\ 0 & & \alpha b \end{matrix}\!\diagdown ,$$

$$\left/\begin{matrix} & a & \\ 0 & & b\end{matrix}\right\backslash^{-1} = \left/\begin{matrix} & \frac{a}{a^2+b^2} & \\ 0 & & \frac{-b}{a^2+b^2}\end{matrix}\right\backslash ,$$

$$\left/\begin{matrix} & a & \\ 0 & & b\end{matrix}\right\backslash^{(n,\circ_9)} = \left/\begin{matrix} & \alpha_n & \\ 0 & & \beta_n\end{matrix}\right\backslash .$$

Therefore, each complex number can be represented by an A-tertion. The opposite is not valid, since the A- (and V-) tertions are composed of 3 components while the complex numbers are composed of just two.

With regard to V_2-tertions, we will mention that a V-tertion $\left\backslash\begin{matrix} b & & 0\\ & a & \end{matrix}\right/$ can be juxtaposed to each complex number $a + b\,\mathbf{i}$. Therefore, the above expressions for A-tertions will obtain the forms

$$\left\backslash\begin{matrix} b & & 0\\ & a & \end{matrix}\right/ + \left\backslash\begin{matrix} d & & 0\\ & c & \end{matrix}\right/ = \left\backslash\begin{matrix} b+d & & 0\\ & a+c & \end{matrix}\right/ ,$$

$$\left\backslash\begin{matrix} b & & 0\\ & a & \end{matrix}\right/ \circ_9 \left\backslash\begin{matrix} d & & 0\\ & c & \end{matrix}\right/ = \left\backslash\begin{matrix} ad+bc & & 0\\ & ac-bd & \end{matrix}\right/ ,$$

$$\alpha\left\backslash\begin{matrix} b & & 0\\ & a & \end{matrix}\right/ = \left\backslash\begin{matrix} \alpha b & & 0\\ & \alpha a & \end{matrix}\right/ ,$$

$$\left\backslash\begin{matrix} b & & 0\\ & a & \end{matrix}\right/^{-1} = \left\backslash\begin{matrix} \frac{-b}{a^2+b^2} & & 0\\ & \frac{a}{a^2+b^2} & \end{matrix}\right/ ,$$

$$\left\backslash\begin{matrix} b & & 0\\ & a & \end{matrix}\right/^{(n,\circ_9)} = \left\backslash\begin{matrix} \beta_n & & 0\\ & \alpha_n & \end{matrix}\right/ .$$

Second, we can represent the complex number $a + b\,\mathbf{i}$ with the V-tertion $\left\backslash\begin{matrix} 0 & & b\\ & a & \end{matrix}\right/$.

Now, the above formulas obtain, respectively, the forms

$$\left\backslash\begin{matrix} 0 & & b\\ & a & \end{matrix}\right/ + \left\backslash\begin{matrix} 0 & & d\\ & c & \end{matrix}\right/ = \left\backslash\begin{matrix} 0 & & b+d\\ & a+c & \end{matrix}\right/ ,$$

$$\left\backslash\begin{matrix} 0 & & b\\ & a & \end{matrix}\right/ \circ_9 \left\backslash\begin{matrix} 0 & & d\\ & c & \end{matrix}\right/ = \left\backslash\begin{matrix} 0 & & ad+bc\\ & ac-bd & \end{matrix}\right/ ,$$

$$\alpha\left\backslash\begin{matrix} 0 & & b\\ & a & \end{matrix}\right/ = \left\backslash\begin{matrix} 0 & & \alpha b\\ & \alpha a & \end{matrix}\right/ ,$$

$$\left\backslash\begin{matrix} 0 & & b\\ & a & \end{matrix}\right/^{-1} = \left\backslash\begin{matrix} 0 & & \frac{-b}{a^2+b^2}\\ & \frac{a}{a^2+b^2} & \end{matrix}\right/ ,$$

$$\left\backslash\begin{matrix} 0 & & b\\ & a & \end{matrix}\right/^{(n,\circ_9)} = \left\backslash\begin{matrix} 0 & & \beta_n\\ & \alpha_n & \end{matrix}\right/ .$$

Therefore, each one of the tertions from A- and V-types can represent the complex numbers. Let us call this special type of tertions respectively c_LA-, c_LA- (complex-left) and c_RA-, c_RA- (complex-right) tertions, and in general all of them – c-tertions.

As we will see in the next two chapters, these representations will be useful for representation of (2×2)-matrices and quaternions, too.

Let

$$
\begin{aligned}
c_LA_{2,\circ_9} &= \left\{ \diagup \begin{array}{c} \alpha \\ \beta \quad 0 \end{array} \diagdown \mid \alpha, \beta \in \mathcal{R}, \beta \neq 0 \right\}, \\
c_RA_{2,\circ_9} &= \left\{ \diagup \begin{array}{c} \alpha \\ 0 \quad \beta \end{array} \diagdown \mid \alpha, \beta \in \mathcal{R}, \beta \neq 0 \right\}, \\
c_LV_{2,\circ_9} &= \left\{ \diagdown \begin{array}{c} \beta \quad 0 \\ \alpha \end{array} \diagup \mid \alpha, \beta \in \mathcal{R}, \beta \neq 0 \right\}, \\
c_RV_{2,\circ_9} &= \left\{ \diagdown \begin{array}{c} 0 \quad \beta \\ \alpha \end{array} \diagup \mid \alpha, \beta \in \mathcal{R}, \beta \neq 0 \right\}.
\end{aligned}
$$

As we saw above, the elements of each one of the sets $c_LA_{2,\circ_9}$, $c_RA_{2,\circ_9}$, $c_LV_{2,\circ_9}$ and $c_RV_{2,\circ_9}$ generate all complex numbers and vice-versa. So, the following assertions hold.

Theorem 1. *$\langle c_LA_{2,\circ_9}, +, \circ_9, O, E\rangle$, $\langle c_RA_{2,\circ_9}, +, \circ_9, O, E\rangle$, $\langle c_LV_{2,\circ_9}, +, \circ_9, \overline{O}, \overline{E}\rangle$, and $\langle c_RV_{2,\circ_9}, +, \circ_9, \overline{O}, \overline{E}\rangle$ are fields.*

Really, as we checked above, all conditions that are valid for one set with two operations over it and with unit elements, associated with the operations to be a field, are valid for sets $cA_{2,\circ_9}, c_LV_{2,\circ_9}$ and $c_RV_{2,\circ_9}$ with operation "+", having a unit element $\diagup \begin{array}{c} 0 \\ 0 \quad 0 \end{array} \diagdown$ and operation $\circ_9$, having a unit element $\diagup \begin{array}{c} 1 \\ 0 \quad 0 \end{array} \diagdown$. To these checks we will add only the tertion representation of the complex number formula

$$\frac{p + q\mathbf{i}}{a + b\mathbf{i}} = \frac{ap + bq}{a^2 + b^2} + \frac{aq - bp}{a^2 + b^2}\mathbf{i},$$

which is equivalent to the formula

$$(a + b\mathbf{i}) \left(\frac{ap + bq}{a^2 + b^2} + \frac{aq - bp}{a^2 + b^2}\mathbf{i} \right) = p + q\mathbf{i}.$$

This representation has the form

$$\diagup \begin{array}{c} a \\ b \quad 0 \end{array} \diagdown \ \circ_9 \ \diagup \begin{array}{c} \frac{ap+bq}{a^2+b^2} \\ \frac{aq-bp}{a^2+b^2} \quad 0 \end{array} \diagdown \ = \ \diagup \begin{array}{c} p \\ q \quad 0 \end{array} \diagdown .$$

Similar results can be obtained for the three other forms of tertions.

6 On the representations of the standard quaternions by A- and V-tertions

When we have an A-tertion and a V-tertion, we can construct, by analogy to the (2×2)-matrices, the new object

$$\left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle$$

that we can call a "quaternion". It can be represented by an A- and V-tertions, e.g., as follows

$$\diagup \begin{matrix} & a & \\ b & & x \end{matrix} \diagdown \quad *_1 \diagdown \begin{matrix} x & & c \\ & d & \end{matrix} \diagup \quad = \quad \left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle ,$$

$$\diagup \begin{matrix} & a & \\ b & & c \end{matrix} \diagdown \quad *_2 \diagdown \begin{matrix} b & & c \\ & d & \end{matrix} \diagup \quad = \quad \left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle ,$$

$$\diagup \begin{matrix} & a & \\ b & & c \end{matrix} \diagdown \quad *_3 \diagdown \begin{matrix} d & & e \\ & f & \end{matrix} \diagup \quad = \quad \left\langle \begin{matrix} & a(d+e) & \\ be & & cd \\ & (b+c)f & \end{matrix} \right\rangle .$$

Now, we can define

$$\left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle + \left\langle \begin{matrix} & e & \\ f & & g \\ & h & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a+e & \\ b+f & & c+g \\ & d+h & \end{matrix} \right\rangle ,$$

and

$$\left\langle \begin{matrix} & a & \\ b & & c \\ & d & \end{matrix} \right\rangle \#_1 \left\langle \begin{matrix} & e & \\ f & & g \\ & h & \end{matrix} \right\rangle = \left\langle \begin{matrix} & ae - bf - cg - dh & \\ af + be + ch + dg & & ag - bh + ce + dfc \\ & ah + bg - cf + de & \end{matrix} \right\rangle .$$

Let

$$E^* = \left\langle \begin{matrix} & 1 & \\ 0 & & 0 \\ & 0 & \end{matrix} \right\rangle , \ I^* = \left\langle \begin{matrix} & 0 & \\ 1 & & 0 \\ & 0 & \end{matrix} \right\rangle , \ J^* = \left\langle \begin{matrix} & 0 & \\ 0 & & 1 \\ & 0 & \end{matrix} \right\rangle , \ K^* = \left\langle \begin{matrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{matrix} \right\rangle .$$

Therefore,

$$\begin{aligned} E^* &= E *_1 \overline{O} = E *_2 \overline{O} = E *_3 \overline{I} = E *_3 \overline{J} = E *_3 \overline{K}, \\ I^* &= I *_1 \overline{O} = I *_2 \overline{I} = I *_3 \overline{J}, \\ J^* &= O *_1 \overline{J} = J *_2 \overline{J} = J *_3 \overline{I}, \\ O^* &= O *_1 \overline{O} = J *_1 \overline{I} = O *_2 \overline{O} = O *_3 \overline{O}, \end{aligned}$$

Then

$\#_1$	E^*	I^*	J^*	K^*
E^*	E^*	I^*	J^*	K^*
I^*	I^*	$-E^*$	${}^*K$	$-J^*$
J^*	J^*	$-K^*$	$-E^*$	I^*
K^*	K^*	J^*	$-I^*$	$-E^*$

Therefore,

$$(I^*)^2 = (J^*)^2 = (K^*)^2 = -E^*,$$

$$I^*J^*K^* = J^*K^*I^* = K^*I^*J^* = -E^*,$$

$$I^*K^*J^* = K^*J^*I^* = J^*I^*K^* = E^*.$$

Now, we see that

$$\left\langle \begin{array}{c} a \\ b \quad c \\ d \end{array} \right\rangle = aE^* + bI^* + cJ^* + dK^*$$

and

$$\left\langle \begin{array}{c} a \\ b \quad c \\ d \end{array} \right\rangle \#_1 \left\langle \begin{array}{c} e \\ f \quad g \\ h \end{array} \right\rangle$$

$$= \left\{ \begin{array}{l} a \left\langle \begin{array}{c} e \\ f \quad g \\ h \end{array} \right\rangle + b \left\langle \begin{array}{c} -f \\ e \quad -h \\ g \end{array} \right\rangle + c \left\langle \begin{array}{c} -g \\ h \quad e \\ -f \end{array} \right\rangle + d \left\langle \begin{array}{c} -h \\ -g \quad f \\ -e \end{array} \right\rangle \\ \\ e \left\langle \begin{array}{c} a \\ b \quad c \\ d \end{array} \right\rangle + f \left\langle \begin{array}{c} -b \\ a \quad d \\ -c \end{array} \right\rangle + g \left\langle \begin{array}{c} -c \\ d \quad a \\ b \end{array} \right\rangle + f \left\langle \begin{array}{c} -d \\ c \quad -b \\ -a \end{array} \right\rangle \end{array} \right. .$$

7 On the representations of some non-standard quaternions by *A*- and *V*-tertions

The form of operation $\#_1$ and the equalities $(I^*)^2 = (J^*)^2 = (K^*)^2 = -E^*$ lead to the idea of defining other types of quaternions. Now, we can change the condition $(K^*)^2 = -E^*$ with $(K^*)^2 = E^*$, but $K^* \neq E^*$. In this case, we can change operation $\#_1$ with the following one

$$\left\langle \begin{array}{c} a \\ b \quad c \\ d \end{array} \right\rangle \#_2 \left\langle \begin{array}{c} e \\ f \quad g \\ h \end{array} \right\rangle = \left\langle \begin{array}{c} ae - bf - cg + dh \\ af + be + ch - dg \qquad ag - bh + ce + df \\ ah + bg - cf + de \end{array} \right\rangle .$$

Then

$\#_2$	E^*	I^*	J^*	K^*
E^*	E^*	I^*	J^*	K^*
I^*	I^*	$-E^*$	K^*	$-J^*$
J^*	J^*	$-K^*$	$-E^*$	I^*
K^*	K^*	J^*	$-I^*$	E^*

.

and

$$(I^*)^2 = (J^*)^2 = -E^*,$$

$$J^*K^*I^* = K^*I^*J^* = J^*I^*K^* = -E^*,$$

$$I^*J^*K^* = I^*K^*J^* = K^*J^*I^* = E^*.$$

Moreover, if we like to have two different units J^* and K^*, for which

$$(J^*)^2 = (K^*)^2 = E^*,$$

but $J^* \neq E^*$ and $K^* \neq E^*$, and to keep the well-known equality

$$(I^*)^2 = -E^*,$$

then the formula for the new operation is

$$\left\langle \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \right\rangle \#_3 \left\langle \begin{array}{ccc} & e & \\ f & & g \\ & h & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & ae - bf + cg + dh & \\ af + be + ch - dg & & ag - bh + ce + dfc \\ & ah + bg - cf + de & \end{array} \right\rangle$$

and then

$\#_3$	E^*	I^*	J^*	K^*
E^*	E^*	I^*	J^*	K^*
I^*	I^*	$-E^*$	K^*	$-J^*$
J^*	J^*	$-K^*$	E^*	I^*
K^*	K^*	J^*	$-I^*$	E^*

,

and

$$I^*K^*J^* = J^*I^*K^* = J^*K^*I^* = -E^*,$$

$$I^*J^*K^* = K^*J^*I^* = K^*I^*J^* = E^*.$$

Finally, if we like to have in some sense dual formulas to the first ones, we can change operation $\#_1$ with the following one

$$\left\langle \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array} \right\rangle \#_4 \left\langle \begin{array}{ccc} & e & \\ f & & g \\ & h & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & -ae + bf + cg + dh & \\ af + be + ch - dg & & ag - bh + ce + df \\ & ah + bg - cf + de & \end{array} \right\rangle .$$

Now, we obtain

$\#_4$	E^*	I^*	J^*	K^*
E^*	$-E^*$	I^*	J^*	K^*
I^*	I^*	E^*	$^*K$	$-J^*$
J^*	J^*	$-K^*$	E^*	I^*
K^*	K^*	J^*	$-I^*$	E^*

.

and

$$(I^*)^2 = (J^*)^2 = (K^*)^2 = E^*,$$

while

$$(E^*)^2 = -E^*,$$

and

$$I^*K^*J^* = J^*I^*K^* = J^*K^*I^* = -E^*,$$

$$I^*J^*K^* = K^*I^*J^* = K^*J^*I^* = E^*.$$

Therefore, using A- and V-tertions we can construct different forms of quaternions.

8 Conclusion

As it was mentioned in the Introduction, the first paper of Anthony Shannon and me [17], based on [16] lead to appearance of a series of papers [1–7, 19–55] in which the concept of a tertion was extended. These papers are not discussed here, because I hope that the colleagues, who developed the idea of tertions will collect and publish their research, too. So, I will only give short remarks on the possible future research over tertions, and will formulate them as open problems.

Open Problem 1: What other operations can be defined over tertions from A_2, V_2, A_3, $\mathrm{V}_3, \ldots$?

Open Problem 2: What other generating operations of complex numbers, matrices and quaternions can be defined and which properties they will have?

Open Problem 3: What other objects can be represented by tertions?

Open Problem 4: What other constructions of tertions can be introduced and which operations over them can be defined?

Finally, we can mention that there is a difference in the approaches about existing of the objects, that are called here "tertions". In the literature, it is proved (correctly) that such objects cannot exist, but in respect to the existing operations over them. Here, for the tertions there are no restrictions to construct objects in their present forms. We introduced *new* operations and for them we checked that the properties which are impossible in the standard case, here are valid.

The author hopes that in future, the idea for tertions will be developed and they will find interesting applications in different areas of the science.

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