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Some geometric properties of the Padovan vectors in Euclidean 3-space

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Abstract: Padovan numbers were defined by Stewart (1996) in honor of the modern architect Richard Padovan (1935) and were first discovered in 1924 by Gerard Cordonnier. Padovan numbers are a special status of Tribonacci numbers with initial conditions and general terms. The ratio between Padovan numbers is one of the important algebraic numbers because it produces plastic numbers. Up to now, various studies have been conducted on Padovan numbers and Padovan polynomial sequences. In this study, Padovan vectors are defined for the first time by using the Padovan Binet-like formula and reduction relation. Then, geometric properties of Padovan vectors such as inner product, norm, and vector products are analyzed. In the last part of the study, Padovan vectors were calculated with Binet formulas in the *Geogebra* program. In addition, the first ten Padovan numbers and Padovan vectors were calculated using the Binet formulas and shown as points and vectors in three-dimensional space. According to the Padovan vectors found, the Padovan curve was drawn in space for the first time by using the curve fitting feature of the *Geogebra* program. Thus, with our study, a geometric approach to Padovan number sequences was brought for the first time.

Keywords: Padovan numbers, Padovan vectors, Inner product, Vectorial product, *Geogebra*.
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1 Introduction

In mathematics, the word “Calculus” corresponds to the term calculation, while in Latin it means “pebble-stone”. At the first age, people added meanings to numbers by using pebbles. The terms “Calcule” or “Pebble” are etymologically at the root of calculation. Early civilizations used different counting systems. Pythagoras discovered the power of numbers and said that the laws of nature are mathematical, and he understood that the notes in music depend on the speed of vibrations. Pythagoreans, on the other hand, have been interested in numbers corresponding to sounds. It is known that the science of numbers was interested in Egypt before Pythagoras. Hermetica, one of the ancient Egyptian sages; said the statement “The perfectly functioning universe is regulated by the power of numbers”. Since the first civilizations to the present, numbers have added meaning to our lives in many fields such as engineering, software, etc. The generator functions of Padovan numbers, which are a third-order sequence, were obtained by Shannon and Horadam (1971) [18]. Stewart (1996) geometrically has shown Padovan numbers with spiral curves drawn on the corners of conjoined triangles [20]. R. Padovan (2002) is one of those who undertake the biggest role in the formation of plastic numbers and Dom Hans Van der Laan numbers [15]. Shannon et al. (2006) defined the polynomial sequences of Padovan numbers and investigated the similarities and differences of these sequences with the Fibonacci sequences [17]. Kaygisiz and Sahin (2011) have worked on the k-series of generalized Van der Laan and generalized Perrin polynomials [14]. Voet et al. (2012) were inspired by the structure of Van der Laan numbers for space design in the 20th century [23]. Yilmaz and Bozkurt (2012) discussed some new features of the Padovan sequence using generator matrices [26]. Yazlik, Tollu, and Taskara (2013) used Padovan numbers to solve second-order difference equations in their study [25]. Sokhuma (2013) obtained the Padovan Q-matrix and has done studies on the generalization of this matrix [19]. Bilgici (2013) by defining Pell–Padovan-like numbers and examined some properties of these numbers [3]. Coskun and Taskara (2014) gave information about the eigenvalues, determinants, and norms of circular matrices of third-order number sequences such as Padovan and Perrin [5]. Seenukul et al. (2015) took the Padovan Q-matrix and examined similar properties related to the generalization of this matrix [16]. Deveci (2015) examined the finite groups of Pell–Padovan and Jacobsthal–Padovan sequences [7]. Goy (2018) examined Padovan numbers Toeplitz–Hessenberg matrices and determinant properties [13]. Cerda-Morales (2019) has handled new equations for Padovan numbers [4]. Faisant (2019) discussed the matrix properties of Padovan number sequences [11]. Diskaya et al. (2019) defined the Split (s, t) Padovan and Perrin quaternions and examined some properties [8]. Ddamulira (2020) obtained the Diophantine and Pillai equations as the sum of three Padovan numbers and investigated the properties of these equations [6]. Vieira et al. (2020) conducted a study on the historical analysis of the Padovan sequence and handled the $(s; t)$-Padovan quaternion matrix [21, 22]. Erdağ and Deveci (2021), have examined the representation and finite sums of Padovan-p Jacobsthal numbers [9]. Akbıyık and Yamaç (2021) developed a method to configure Perrin and Padovan sequences and found De Moivre-type properties for Padovan numbers. In addition, they defined a Padovan sequence with new initial conditions and examined the properties between formed these new sequences [1]. García Lomeli et al. (2022), have worked on with the Padovan numbers the Diophantine equations [12]. Erdağ and Halıcı (2022), define complex typed

p-Padovan numbers, and the equations between p-Padovan sequences and complex-typed p-Padovan sequences are discussed [10]. Anatriello et al. (2022) have obtained generalized Pascal triangles and associated k-Padovan-like sequences [2]. Yayga et al. (2022), examined the area of the Padovan q-difference matrix in sequence spaces [24]. In our study, we defined Padovan vectors for the first time by using a Padovan Binet-like formula and reduction relation. The inner product, norm, vector products, and triple products of Padovan vectors were calculated and the geometric properties of Padovan numbers were examined for the first time. In the last part of the study, the first ten Padovan vectors were obtained by calculating Padovan vectors with Binet formulas in the *Geogebra* program. In addition, the first ten Padovan numbers and Padovan vectors were calculated and shown as points and vectors in three-dimensional space. By using the curve fitting feature of the *Geogebra* program, the Padovan curve was drawn in space for the first time with the help of the Padovan vectors found. Thus, with our study, a geometric approach to Padovan number sequences was brought for the first time.

2 Materials and methods

It is understood that many events can be filtered through logic and reason in the intuitive or experimental research made about the order of the universe since the existence of humanity is interestingly connected to numbers. It has been seen that the ratios formed in some structures we see in nature are related to the number 1.618… and this ratio has been called the golden ratio. For example, it has been observed that the golden ratio is surprisingly obtained in the leaves of trees, the Aloe Vera plant, the spiral structure of the sunflower, the sea shells, the spiral structure of the Nautilus shell, the multiply of rabbits, and the proportions of the human face. The golden ratio, which can be observed in many places in nature, has also started to be used in art. In the art of painting, the golden ratio can be obtained in Leonardo Da Vinci’s Monalisa, Annunciation, The Last Supper, Michelangelo’s Creation of Adam, and Botticelli’s The Birth of Venus the tables named. In addition, the golden ratio is obtained by the ratios of the consecutive terms of the Fibonacci number sequences [25]. The mysterious effect of numbers in most observable events in the universe has attracted quite the attention of researchers. For this reason, studies on different types of numbers continue to increase and develop. Another remarkable ratio is the plastic ratio. The “Plastic Ratio”, which was defined with the ratio of

$$\rho = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}} = 1.3247179...$$

for the first time, was examined by Gérard Cordonnier in 1924 [15]. In addition, in 1958, Cordonnier gave lectures on the usage areas of plastic ratio in architectural structures for the first time. Hans Van Der Laan (1904–1991) from the Netherlands, who conducted the architecture course at Technische, is also an architect who uses the plastic ratio in his works. Hans Van Der Laan used the monastery’s primitive basilica as an example to train architects in the rebuilding of churches after the Second World War at the Hogeschool in Delf. Laan and his brother searched for patterns for architecture by experimenting with stones and later with building materials, eventually discovering a geometric scale in which its construction occurs via an irrational number and a new measurement pattern ideal for working with space objects. It has been understood that the plastic ratio studied by Padovan, Cordonier, and Van Der Laan are ratios of numbers obtained from the solution of the $x^3 - x - 1 = 0$ cubic equation. Thus, in 1996, these numbers were named Padovan numbers by Ian Stewart in honor of Richard Padovan.

Definition 2.1. *The string of numbers* 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, … *numbers formed by the formula of* $P_{n+3} = P_{n+1} + P_{n,}\ n \geq 0$ *iterations, the first four terms of which are given as* $P_0 = P_1 = P_2 = 1$ *and* $P_3 = P_4 = 2$, *is called the Padovan number sequence. From the solution of the third-degree polynomial* $x^3 - x - 1 = 0$, *the plastic ratio* $\rho = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}} = 1.3247179...$ *is obtained. The third-order roots of algebraic expressions* $x^3 - x - 1 = 0$ *are calculated*

$$\alpha = \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}},$$

$$\beta = \sqrt[3]{\frac{1}{16} + \frac{1}{48}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{16} - \frac{1}{48}\sqrt{\frac{23}{3}}} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}\right),$$

$$\gamma = -\sqrt[3]{\frac{1}{16} + \frac{1}{48}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{16} - \frac{1}{48}\sqrt{\frac{23}{3}}} - i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} - \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}}\right),$$

respectively with α, β *and* γ *open. In addition, display implicit representation of these polynomial roots can be represented by the ratios*

$$p_1 = \frac{(\beta-1)(\gamma-1)}{(\alpha-\beta)(\alpha-\gamma)}, \quad p_2 = \frac{(\alpha-1)(\gamma-1)}{(\beta-\alpha)(\beta-\gamma)} \quad \text{and} \quad p_3 = \frac{(\alpha-1)(\beta-1)}{(\gamma-\alpha)(\gamma-\beta)}. \tag{2.1}$$

The implicit algebraic form of Padovan numbers with α, β *and* γ *roots is given by the expression*

$$P_n = p_1\alpha^n + p_2\beta^n + p_3\gamma^n. \tag{2.2}$$

This expression is called the Binet-like formula of the Padovan sequence. The generator function of the Padovan sequence is

$$\sum_{n=0}^{\infty} P_n y^{n-1} = \frac{1+y}{1-y^2-y^3}$$

(see [15, 17]).

Lemma 2.2. [11] *Some properties of* α, β *and* γ *roots of Padovan numbers are given by*

1) $\alpha + \beta + \gamma = 0$,
2) $\alpha\beta + \alpha\gamma + \beta\lambda = -1$,
3) $\alpha\beta\gamma = 1$,
4) $\alpha^3 = \alpha + 1$,
5) $(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha) = 1$.

Lemma 2.3. [11] *Let $i \in N$ and P_n be the terms of the Padovan sequence, some properties of the Padovan number sequence are calculated by the equations*

1) $\sum_{n=0}^{i} P_n = P_{i+5} - 2$

2) $\sum_{n=0}^{i} P_n^2 = P_{i+2}^2 - P_{i-1}^2 - P_{i-3}^2$,

3) $\sum_{n=0}^{m} P_{2n} = P_{(2m+3)} - 1$

4) $\sum_{n=0}^{m} P_{(2n+1)} = P_{(2m+4)} - 1$

5) $\sum_{n=0}^{m} P_{3n} = P_{(3m+2)}$

6) $\sum_{n=0}^{m} P_{(3n+1)} = P_{(3m+3)} - 1$

7) $\sum_{n=0}^{m} P_{(3n+2)} = P_{(3m+4)} - 1$

8) $\sum_{n=0}^{m} P_{(5n)} = P_{(5m+1)}$.

Lemma 2.4. [11] *Let $i \in N$ and P_n be the terms of the Padovan sequence. The sums giving the product of terms in the Padovan sequence are given with the equations*

1) $\sum_{n=0}^{m} P_{(n)^2} P_{(n+1)} = P_{(m)} P_{(m+1)} P_{(m+2)}$

2) $\sum_{n=0}^{m} P_{(n)} P_{(n+2)} = P_{(m+2)} P_{(m+3)} - 1$

3) $\sum_{n=0}^{m} P_n^2 P_{n+1} = P_m P_{m+1} P_{m+2}$.

3 Main results

Padovan numbers have been examined as a number sequence and algebraically in many studies. However, the relationship of Padovan numbers with geometry has not been examined yet. For this reason, in this section, after defining the Padovan vectors, which were not previously found in the literature, the inner product and vector products of the Padovan vectors are calculated. Let us consider the vectors

$$\vec{a} = \left[1, \alpha, \alpha^2, ..., \alpha^{m-1}\right]^T,\ \vec{b} = \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T,\ \vec{c} = \left[1, \gamma, \gamma^2, ..., \gamma^{m-1}\right]^T \tag{3.1}$$

in n-dimensional Euclidean space, with third-order Padovan polynomial roots α, β and γ. Now we will calculate the double dot product of these vectors:

Lemma 3.1. *Let the implicit form of the Padovan number sequence for the roots $\alpha, \beta,$ and γ be given by the $P_n = p_1\alpha^n + p_2\beta^n + p_3\gamma^n$ equation given in Equation (2.2). The inner product of the vectors $\vec{a}$, $\vec{b}$, $\vec{c} \in E^n$ with themselves, is obtained as*

$$i)\ \vec{a}.\vec{a} = \frac{\alpha^{2m} - 1}{\alpha^2 - 1}, \quad ii)\ \vec{b}.\vec{b} = \frac{\beta^{2m} - 1}{\beta^2 - 1}, \quad iii)\ \vec{c}.\vec{c} = \frac{\gamma^{2m} - 1}{\gamma^2 - 1},$$

respectively.

Proof. i) The inner product of the vector $\vec{a} = \left[1, \alpha, \alpha^2, ..., \alpha^{m-1}\right]^T$ taken in Euclidean *n*-space is

$$\vec{a}.\vec{a} = \left\langle \left(1, \alpha, \alpha^2, ..., \alpha^{m-1}\right), \left(1, \alpha, \alpha^2, ..., \alpha^{m-1}\right) \right\rangle = \sum_{j=0}^{m-1} (\alpha^j)^2. \tag{3.2}$$

From the Lemma 2.2., if the $\alpha = \frac{1}{\beta\gamma}$ ratio (3.2) is written instead of one of the expressions α, it is obtained as $\vec{a}.\vec{a} = \sum_{j=0}^{m-1} \alpha^j . \left(\frac{1}{\beta\gamma}\right)^j = \sum_{j=0}^{m-1} \left(\frac{\alpha}{\beta\gamma}\right)^j$. When received the equality of $r = \frac{\alpha}{\beta\gamma} \neq 1$ for the abbreviation becomes $\vec{a}.\vec{a} = \sum_{j=0}^{m-1} (r)^j$. Equation (3.2) is calculated as

$$\vec{a}.\vec{a} = \frac{r^m - 1}{r - 1} = \frac{\left(\frac{\alpha}{\beta\gamma}\right)^m - 1}{\frac{\alpha}{\beta\gamma} - 1} = \frac{\frac{\alpha^m - \beta^m\gamma^m}{\beta^m\gamma^m}}{\frac{\alpha - \beta\gamma}{\beta\gamma}} = \frac{\alpha^m - \beta^m\gamma^m}{\beta^m\gamma^m} . \frac{\beta\gamma}{\alpha - \beta\gamma}, \tag{3.3}$$

since $\sum_{j=0}^{m-1} r^j = \frac{r^m - 1}{r - 1}$ equality is obtained for all real numbers satisfying the $r \neq 1$ equality from the binomial expansion. If the equation $\beta\gamma = \frac{1}{\alpha}$ is written instead of Equation (3.3), the result

$$\vec{a}.\vec{a} = \left(\frac{\alpha^m - \frac{1}{\alpha^m}}{\frac{1}{\alpha^m}}\right) . \left(\frac{\alpha - \frac{1}{\alpha}}{\frac{1}{\alpha}}\right) = \frac{\alpha^{2m} - 1}{\alpha^2 - 1}$$

is obtained.

ii) Similarly, let's find the dot product of $\vec{b}.\vec{b}$. The dot product of the vector $\vec{b} = \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T$ taken in Euclidean n-space with itself is

$$\vec{b}.\vec{b} = \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T = \sum_{j=0}^{m-1} (\beta^j)^2. \tag{3.4}$$

If the ratio of $\beta = \frac{1}{\alpha\gamma}$ in the Lemma 2.2. equation is substituted for one of the β expressions in equation (3.4), it is obtained as $\vec{b}.\vec{b} = \sum_{j=0}^{m-1} \beta^j . \left(\frac{1}{\alpha\gamma}\right)^j = \sum_{j=0}^{m-1} \frac{\beta^j}{\alpha^j\gamma^j} = \sum_{j=0}^{m-1} \left(\frac{\beta}{\alpha\gamma}\right)^j$. Taking the equality of $r = \frac{\beta}{\alpha\gamma} \neq 1$ for the abbreviation becomes $\vec{b}.\vec{b} = \sum_{j=0}^{m-1} (r)^j$. Equation (3.4) is calculated as

$$\vec{b}.\vec{b} = \frac{r^m - 1}{r - 1} = \frac{\left(\frac{\beta}{\alpha\gamma}\right)^m - 1}{\frac{\beta}{\alpha\gamma} - 1} = \frac{\frac{\beta^m - \alpha^m\gamma^m}{\alpha^m\gamma^m}}{\frac{\beta - \alpha\gamma}{\alpha\gamma}} = \frac{\beta^m - \alpha^m\gamma^m}{\alpha^m\gamma^m} . \frac{\alpha\gamma}{\beta - \alpha\gamma} \tag{3.5}$$

from the binomial expansion. If the equation $\alpha\gamma = \frac{1}{\beta}$ is written instead of equation (3.5), the result $\vec{b}.\vec{b} = \left(\frac{\beta^m - \frac{1}{\beta^m}}{\frac{1}{\beta^m}}\right).\left(\frac{\beta - \frac{1}{\beta}}{\frac{1}{\beta}}\right) = \frac{\beta^{2m}-1}{\beta^2-1}$ is obtained.

iii) Finally, let us calculate the product $\vec{c}.\vec{c}$. With the vector $\vec{c} = \left[1,\gamma,\gamma^2,...,\gamma^{m-1}\right]^T$ being a vector taken in Euclidean m-space, the dot product of this vector by itself is

$$\vec{c}.\vec{c} = \left[1,\gamma,\gamma^2,...,\gamma^{m-1}\right]^T\left[1,\gamma,\gamma^2,...,\gamma^{m-1}\right]^T = \sum_{j=0}^{m-1}(\gamma^j)^2. \tag{3.6}$$

If the ratio of $\gamma = \frac{1}{\beta\alpha}$ from the Lemma 2.2.(3) equation is substituted for one of the α expressions in the Equation (3.6), it is obtained as $\vec{c}.\vec{c} = \sum_{j=0}^{m-1}\gamma^j.\left(\frac{1}{\alpha\beta}\right)^j = \sum_{j=0}^{m-1}\left(\frac{\gamma}{\alpha\beta}\right)^j$. Using the equation $r = \frac{\gamma}{\alpha\beta} \neq 1$ for the abbreviation, it becomes $\vec{c}.\vec{c} = \sum_{j=0}^{m-1}(r)^j$.

$$\vec{c}.\vec{c} = \frac{r^m-1}{r-1} = \frac{\left(\frac{\gamma}{\alpha\beta}\right)^m - 1}{\frac{\gamma}{\alpha\beta}-1} = \frac{\frac{\gamma^m-\alpha^m\beta^m}{\alpha^m\beta^m}}{\frac{\gamma-\alpha\beta}{\alpha\beta}} = \frac{\gamma^m-\alpha^m\beta^m}{\alpha^m\beta^m}.\frac{\alpha\beta}{\gamma-\alpha\beta} \tag{3.7}$$

is obtained when using the binomial expansion in Equation (3.6). Substituting the equation $\alpha\beta = \frac{1}{\gamma}$ in equation (3.7), the result $\vec{c}.\vec{c} = \left(\frac{\gamma^m - \frac{1}{\gamma^m}}{\frac{1}{\gamma^m}}\right).\left(\frac{\gamma - \frac{1}{\gamma}}{\frac{1}{\gamma}}\right) = \frac{\gamma^{2m}-1}{\gamma^2-1}$ is obtained. □

Lemma 3.2. *Let α,β and γ be the polynomial roots of the Padovan number sequence. The double dot products of the vectors $\vec{a}$, $\vec{b}$, $\vec{c} \in E^m$ are calculated by the equations*

$$i)\ \vec{a}.\vec{c} = \left(\frac{\beta^m-1}{\beta^{m-1}.(\beta-1)}\right),\quad ii)\ \vec{a}.\vec{b} = \left(\frac{\gamma^m-1}{\gamma^{m-1}.(\gamma-1)}\right),\quad iii)\ \vec{b}.\vec{c} = \left(\frac{\alpha^m-1}{\alpha^{m-1}.(\alpha-1)}\right),$$

respectively.

Proof. i) The dot product of vectors $\vec{a} = \left[1,\alpha,\alpha^2,...,\alpha^{m-1}\right]^T$ and $\vec{c} = \left[1,\gamma,\gamma^2,...,\gamma^{m-1}\right]^T$ taken in Euclidean m-space is

$$\vec{a}.\vec{c} = \left[1,\alpha,\alpha^2,...,\alpha^{m-1}\right]^T\left[1,\gamma,\gamma^2,...,\gamma^{m-1}\right]^T = \sum\alpha^j\gamma^j. \tag{3.8}$$

If the ratio of $\alpha\gamma = \frac{1}{\beta}$ is substituted as equivalent to the expression $\alpha\gamma$ in the equation (3.8), it is obtained as $\vec{a}.\vec{c} = \sum\alpha^j\gamma^j = \sum\gamma^j\alpha^j = \sum\left(\frac{1}{\beta}\right)^j$. For easy operation, when $r = \frac{1}{\beta}$ is taken, equation (3.8) is calculated as

$$\vec{a}.\vec{c} = \left(\frac{1-\frac{1}{\beta^m}}{1-\frac{1}{\beta}} \right) = \left(\frac{\frac{\beta^m - 1}{\beta^m}}{\frac{\beta-1}{\beta}} \right) = \left(\frac{\beta^m - 1}{\beta^m} . \frac{\beta}{\beta-1} \right) = \left(\frac{\beta^m - 1}{\beta^{m-1}.(\beta-1)} \right) \tag{3.9}$$

from the binomial expansion.

ii) The dot product of two vectors $\vec{a} = \left[1, \alpha, \alpha^2, ..., \alpha^{m-1}\right]^T$, and $b = \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T$ taken in Euclidean n-space is

$$\vec{a}.\vec{b} = \left[1, \alpha, \alpha^2, ..., \alpha^{m-1}\right]^T \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T = \sum \alpha^j \beta^j. \tag{3.10}$$

If the ratio $\alpha\beta = \frac{1}{\gamma}$ is written against the expression $\alpha\beta$ in the equation (3.10), it is obtained as $\vec{a}.\vec{b} = \sum \alpha^j \beta^j = \sum \beta^j \alpha^j = \sum \left(\frac{1}{\gamma} \right)^j$. When the equation is taken by writing $r = \frac{1}{\gamma}$ in this equation, the equation (3.10) is calculated as

$$\vec{a}.\vec{b} = \left(\frac{1-\frac{1}{\gamma^m}}{1-\frac{1}{\gamma}} \right) = \left(\frac{\frac{\gamma^m - 1}{\gamma^m}}{\frac{\gamma-1}{\gamma}} \right) = \left(\frac{\gamma^m - 1}{\gamma^m} . \frac{\gamma}{\gamma-1} \right) = \frac{\gamma^m - 1}{(\gamma-1)\gamma^{m-1}} \tag{3.11}$$

from the binomial expansion.

iii) The dot product of vectors $\vec{b} = \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T$ and $\vec{c} = \left[1, \gamma, \gamma^2, ..., \gamma^{m-1}\right]^T$ taken in the Euclidean m-space is

$$\vec{b}.\vec{c} = \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T \left[1, \gamma, \gamma^2, ..., \gamma^{m-1}\right]^T = \sum \beta^j \gamma^j. \tag{3.12}$$

If the ratio $\beta\gamma = \frac{1}{\alpha}$ is written as the equivalent of the expression $\beta\gamma$ in equation (3.12), it is obtained as $\vec{b}.\vec{c} = \sum \gamma^j \beta^j = \sum \beta^j \gamma^j = \sum \left(\frac{1}{\alpha} \right)^j$. When written as $r = \frac{1}{\alpha}$ for abbreviation, we can conclude the Equation (3.12) as

$$\vec{b}.\vec{c} = \left(\frac{1-\frac{1}{\alpha^m}}{1-\frac{1}{\alpha}} \right) = \left(\frac{\frac{\alpha^m - 1}{\alpha^m}}{\frac{\alpha-1}{\alpha}} \right) \cong \left(\frac{\alpha^m - 1}{\alpha^m} . \frac{\alpha}{\alpha-1} \right) = \frac{\alpha^m - 1}{(\alpha-1)\alpha^{m-1}}, \tag{3.13}$$

since $\sum_{j=0}^{m-1} r^j = \frac{r^m - 1}{r-1}$ equality is provided for all real numbers that validate the $r \neq 1$ equation from the binomial expansion. □

Lemma 3.3. *Let α, β and γ be the polynomial roots of the Padovan number sequence. The norms of the $\vec{a}, \vec{b}, \vec{c}$ vectors taken in the three-dimensional Euclidean space consisting of the roots of the polynomial forming the Padovan number sequence are found with the equations*

$$i) \left\|\vec{a}\right\| = \sqrt{\frac{\alpha^{2m}-1}{\alpha^2-1}}, \quad ii) \left\|\vec{b}\right\| = \sqrt{\frac{\beta^{2m}-1}{\beta^2-1}}, \quad iii) \left\|\vec{c}\right\| = \sqrt{\frac{\gamma^{2m}-1}{\gamma^2-1}}.$$

Proof. Since the norm of any vector $\vec{v} = (v_1, v_2, ..., v_n)$ is $\|v\| = \sqrt{\vec{v}.\vec{v}}$, the $\|a\| = \sqrt{\langle \vec{a}.\vec{a} \rangle} = \sqrt{\frac{\alpha^{2m}-1}{\alpha^2-1}}$ norm is obtained by using the $\vec{a}.\vec{a} = \frac{\alpha^{2m}-1}{\alpha^2-1}$ dot product in Lemma 3.1. Similarly, using $\vec{b}.\vec{b} = \frac{\beta^{2m}-1}{\beta^2-1}$ and $\vec{c}.\vec{c} = \frac{\gamma^{2m}-1}{\gamma^2-1}$ products in Lemma 3.1 results $\|b\| = \sqrt{\langle \vec{b}.\vec{b} \rangle} = \sqrt{\frac{\beta^{2m}-1}{\beta^2-1}}$ and $\|c\| = \sqrt{\langle \vec{c}.\vec{c} \rangle} = \sqrt{\frac{\gamma^{2m}-1}{\gamma^2-1}}$ are obtained. □

Definition 3.4. (Padovan vectors) *Assume that the, $P_n, P_{n+1}, ..., P_{n+m-1}$ are the Padovan numbers, the matrix of m-dimensional Padovan vector is represented by*

$$\vec{P}_n^{\,m} = \begin{bmatrix} P_n \\ P_{n+1} \\ \vdots \\ P_{n+m-1} \end{bmatrix} = \begin{bmatrix} P_n & P_{n+1} & \dots & P_{n+m-1} \end{bmatrix}^T \tag{3.14}$$

Therefore, in two-dimensional Euclidean space, the Padovan vector is defined as $\vec{P}_n^{\,2} = \begin{bmatrix} P_n & P_{n+1} \end{bmatrix}$, and in three-dimensional Euclidean space, the Padovan number is defined as $\vec{P}_n^{\,3} = \begin{bmatrix} P_n & P_{n+1} & P_{n+2} \end{bmatrix}^T$.

Theorem 3.5. *Let α, β and γ be the roots of the third-degree polynomial belonging to the Padovan number sequence, $P_n, P_{n+1}, ..., P_{n+m-1}$ including the Padovan numbers. The vectors $\vec{a}$, $\vec{b}$, $\vec{c} \in E^n$ given in equation* (1) *and the coefficients $p_1, p_2, p_3 \in R$ given in* (2.1) *and the α, β and γ polynomial roots, and the m-dimensional Padovan vector can also be represented by the equation*

$$\vec{P}_n^{\,m} = p_1\alpha^n \vec{a} + p_2\beta^n \vec{b} + p_3\gamma^n \vec{c}. \tag{3.15}$$

Proof. When the *m*-dimensional Padovan vector is taken as $\vec{P}_n^{\,m} = \begin{bmatrix} P_n & P_{n+1} & \dots & P_{n+m-1} \end{bmatrix}^T$ for $P_n, P_{n+1}, ..., P_{n+m-1}$ Padovan numbers, the implicit form of the Padovan numbers in (2.2) defined by the p_1, p_2, p_3 coefficients given in Equation (2.1) is given by $P_n = p_1\alpha^n + p_2\beta^n + p_3\gamma^n$. When this implicit form definition is applied instead of each term in the matrix (3.14),

$$\vec{P}_n^{\,m} = \begin{bmatrix} p_1\alpha^n + p_2\beta^n + p_3\gamma^n \\ p_1\alpha^{n+1} + p_2\beta^{n+1} + p_3\gamma^{n+1} \\ \vdots \\ p_1\alpha^{n+m+1} + p_2\beta^{n+m+1} + p_3\gamma^{n+m+1} \end{bmatrix}$$

is obtained. By arranging this matrix,

$$\vec{P}_n^{\,m} = p_1\alpha^n \begin{bmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{m+1} \end{bmatrix} + p_2\beta^n \begin{bmatrix} 1 \\ \beta \\ \vdots \\ \beta^{m+1} \end{bmatrix} + p_3\gamma^n \begin{bmatrix} 1 \\ \gamma \\ \vdots \\ \gamma^{m+1} \end{bmatrix}$$

is obtained by placing $\alpha^n, \beta^n, \gamma^n$ in the equation

$$\vec{P}_n^{\,m} = p_1 \begin{bmatrix} \alpha^n \\ \alpha^{n+1} \\ \vdots \\ \alpha^{n+m+1} \end{bmatrix} + p_2 \begin{bmatrix} \beta^n \\ \beta^{n+1} \\ \vdots \\ \beta^{n+m+1} \end{bmatrix} + p_3 \begin{bmatrix} \gamma^n \\ \gamma^{n+1} \\ \vdots \\ \gamma^{n+m+1} \end{bmatrix}.$$

By using equation (3.1), it is seen that the equality of the Padovan vector

$$\vec{P}_n^{\,m} = p_1\alpha^n \vec{a} + p_2\beta^n \vec{b} + p_3\gamma^n \vec{c}$$

is provided. □

Conclusion 3.6. *Let α, β and γ be the roots of the third-degree polynomial belonging to the Padovan number sequence, where P_n, P_{n-1}, P_{n-2} is the Padovan numbers equations*

$$\alpha^n = \left(\alpha^2 - 1\right)P_n + P_{n-1} + \left(1 + \alpha - \alpha^2\right)P_{n-2}$$

$$\beta^n = \left(\beta^2 - 1\right)P_n + P_{n-1} + \left(1 + \beta - \beta^2\right)P_{n-2}$$

$$\gamma^n = \left(\gamma^2 - 1\right)P_n + P_{n-1} + \left(1 + \gamma - \gamma^2\right)P_{n-2}$$

are obtained by using the Equation (3.15), [11].

Lemma 3.7. *Let the coefficients α, β and γ be the roots of the third-order polynomial belonging to the Padovan number sequence, and the $\vec{a} = \left[1, \alpha, \alpha^2, ..., \alpha^{m-1}\right]^T$, $\vec{b} = \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T$, $\vec{c} = \left[1, \gamma, \gamma^2, ..., \gamma^{m-1}\right]^T$ vectors consisting of these roots be n-dimensional vectors in Euclidean 3-space. Alternatively, the m-dimensional P_n^m Padovan vector can be represented by the equation*

$$\vec{P}_n^{\,m} = \frac{(\beta-1).(\gamma-1)}{(\alpha-\beta).(\alpha-\lambda)}\alpha^n \vec{a} + \frac{(\alpha-1).(\gamma-1)}{(\beta-\alpha).(\beta-\gamma)}\beta^n \vec{b} + \frac{(\alpha-1).(\beta-1)}{(\gamma-\alpha).(\gamma-\beta)}\gamma^n \vec{c}\,.$$

Proof. When the α^n, β^n and γ^n terms in result 3.6 are multiplied by the $\vec{a} = \left[1, \alpha, \alpha^2, ..., \alpha^{m-1}\right]^T$, $\vec{b} = \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T$, $\vec{c} = \left[1, \gamma, \gamma^2, ..., \gamma^{m-1}\right]^T$ vectors taken in Euclidean space,

$$\alpha^n \vec{a} = \left(\alpha^2 - 1\right) P_n \vec{a} + P_{n-1} \vec{a} + \left(1 + \alpha - \alpha^2\right) P_{n-2} \vec{a}$$

$$\beta^n \vec{b} = \left(\beta^2 - 1\right) P_n \vec{b} + P_{n-1} \vec{b} + \left(1 + \beta - \beta^2\right) P_{n-2} \vec{b}$$

$$\gamma^n \vec{c} = \left(\gamma^2 - 1\right) P_n \vec{c} + P_{n-1} \vec{c} + \left(1 + \gamma - \gamma^2\right) P_{n-2} \vec{c}$$

is obtained. Since the $p_1, p_2, p_3 \in R$ coefficients given in the vector $\vec{P}_n^{\,m} = \begin{bmatrix} P_n & P_{n+1} & \ldots & P_{n+m-1} \end{bmatrix}^T$ from Equation (2.1) are substituted in Equation (2.2) $p_1, p_2, p_3 \in R$, the matrix

$$\vec{P}_n^{\,m} = \begin{bmatrix} \frac{(\beta-1).(\gamma-1)}{(\alpha-\beta).(\alpha-\lambda)}\alpha^n + \frac{(\alpha-1).(\gamma-1)}{(\beta-\alpha).(\beta-\gamma)}\beta^n + \frac{(\alpha-1).(\beta-1)}{(\gamma-\alpha).(\gamma-\beta)}\gamma^n \\ \frac{(\beta-1).(\gamma-1)}{(\alpha-\beta).(\alpha-\lambda)}\alpha^{n+1} + \frac{(\alpha-1).(\gamma-1)}{(\beta-\alpha).(\beta-\gamma)}\beta^{n+1} + \frac{(\alpha-1).(\beta-1)}{(\gamma-\alpha).(\gamma-\beta)}\gamma^{n+1} \\ \vdots \\ \frac{(\beta-1).(\gamma-1)}{(\alpha-\beta).(\alpha-\lambda)}\alpha^{n+m-1} + \frac{(\alpha-1).(\gamma-1)}{(\beta-\alpha).(\beta-\gamma)}\beta^{n+m-1} + \frac{(\alpha-1).(\beta-1)}{(\gamma-\alpha).(\gamma-\beta)}\gamma^{n+m-1} \end{bmatrix}$$

is obtained. Since this matrix is arranged as

$$\vec{P}_n^{\,m} = \frac{(\beta-1).(\gamma-1)}{(\alpha-\beta).(\alpha-\lambda)} \begin{bmatrix} \alpha^n \\ \alpha^{n+1} \\ \vdots \\ \alpha^{n+m-1} \end{bmatrix} + \frac{(\alpha-1).(\gamma-1)}{(\beta-\alpha).(\beta-\gamma)} \begin{bmatrix} \beta^n \\ \beta^{n+1} \\ \vdots \\ \beta^{n+m-1} \end{bmatrix} + \frac{(\alpha-1).(\beta-1)}{(\gamma-\alpha).(\gamma-\beta)} \begin{bmatrix} \gamma^n \\ \gamma^{n+1} \\ \vdots \\ \gamma^{n+m-1} \end{bmatrix},$$

the result

$$\vec{P}_n^{\,m} = \frac{(\beta-1).(\gamma-1)}{(\alpha-\beta).(\alpha-\lambda)}\alpha^n \vec{a} + \frac{(\alpha-1).(\gamma-1)}{(\beta-\alpha).(\beta-\gamma)}\beta^n \vec{b} + \frac{(\alpha-1).(\beta-1)}{(\gamma-\alpha).(\gamma-\beta)}\gamma^n \vec{c}$$

is obtained. □

Definition 3.8. (Inner Product) *Let two Padovan vectors of size m be taken by* $\vec{P}_{n_1}^{\,m} = \begin{bmatrix} P_{n_1} & P_{n_1+1} \ldots & P_{n_1+m-1} \end{bmatrix}^T$, *whose components consist of* $P_{n_i}, P_{n_i+1}, \ldots, P_{n_i+m-1}$ *Padovan numbers for i* = 1, 2. *The inner product of* $P_{n_1}^{\,m}$ *and* $P_{n_2}^{\,m}$ *Padovan vectors is defined by*

$$\left\langle \vec{P}_{n_1}^{\,m}, \vec{P}_{n_2}^{\,m} \right\rangle = P_{n_1} P_{n_2} + P_{n_1+1} P_{n_2+1} + \ldots + P_{n_1+m-1} P_{n_2+m-1} .$$

Theorem 3.9. *Let the coefficients* α, β *and* γ *be the roots of the third-order polynomial of the Padovan number sequence* $\delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$. *The dot product of two m-dimensional Padovan vectors* $P_{n_1}^{\,m}$ *and* $P_{n_2}^{\,m}$ *with roots* α, β *and* γ *is represented by the equations*

$$\begin{aligned} \left\langle \vec{P}_{n_1}^{\,m}, \vec{P}_{n_2}^{\,m} \right\rangle &= \left\langle \vec{a}, \vec{a} \right\rangle p_1^{\,2} \alpha^{n_1+n_2} + \left\langle \vec{b}, \vec{b} \right\rangle p_2^{\,2} \beta^{n_1+n_2} + \left\langle \vec{c}, \vec{c} \right\rangle p_3^{\,2} \gamma^{n_1+n_2} \\ &\quad + \left\langle \vec{a}, \vec{b} \right\rangle \left[p_1 p_2 \left(\alpha^{n_1} \beta^{n_2} + \alpha^{n_2} \beta^{n_1} \right) \right] + \left\langle \vec{a}, \vec{c} \right\rangle \left[p_1 p_3 \left(\alpha^{n_1} \gamma^{n_2} + \alpha^{n_2} \gamma^{n_1} \right) \right] \\ &\quad + \left\langle \vec{b}, \vec{c} \right\rangle \left[p_2 p_3 \left(\beta^{n_1} \gamma^{n_2} + \beta^{n_2} \gamma^{n_1} \right) \right] \end{aligned}$$

or

$$\left\langle \vec{P}_{n_1}{}^{m}, \vec{P}_{n_2}{}^{m} \right\rangle = \frac{(\alpha^{2m}-1)(\beta-\gamma)^2}{\delta^2(\alpha-1)^3}\alpha^{n_1+n_2-3} + \frac{(\beta^{2m}-1)(\gamma-\alpha)^2}{\delta^2(\beta-1)^3}\beta^{n_1+n_2-3} + \frac{(\gamma^{2m}-1)(\alpha-\beta)^2}{\delta^2(\gamma-1)^3}\gamma^{n_1+n_2-3}$$
$$-\frac{\left(\gamma^m-1\right)}{\gamma^{m-1}}\frac{(\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1})}{(\alpha-\beta)\delta} - \frac{\left(\beta^m-1\right)}{\beta^{m-1}}\frac{(\alpha^{n_1}\gamma^{n_2}+\alpha^{n_2}\gamma^{n_1})}{(\gamma-\alpha)\delta} - \frac{\left(\alpha^m-1\right)}{\alpha^{m-1}}\frac{(\beta^{n_1}\gamma^{n_2}+\beta^{n_2}\gamma^{n_1})}{(\beta-\gamma)\delta}.$$

Proof. Let us take two Padovan vectors P_{n_1} and P_{n_2} by using the equation $P_n = p_1\alpha^n\vec{a} + p_2\beta^n\vec{b} + p_3\gamma^n\vec{c}$ in equation (3.15). The dot product of these two vectors is obtained by the equation

$$\left\langle \vec{P}_{n_1}, \vec{P}_{n_2} \right\rangle = \left\langle \left(p_1\alpha^{n_1}\vec{a} + p_2\beta^{n_1}\vec{b} + p_3\gamma^{n_1}\vec{c}\right), \left(p_1\alpha^{n_2}\vec{a} + p_2\beta^{n_2}\vec{b} + p_3\gamma^{n_2}\vec{c}\right) \right\rangle$$
$$= \frac{\alpha^{2m}-1}{\alpha^2-1}\frac{\frac{1}{(\alpha-1)^2}}{\frac{\delta^2}{(\beta-\lambda)^2}}\alpha^{n_1+n_2} + \frac{\beta^{2m}-1}{\beta^2-1}\frac{\frac{1}{(\beta-1)^2}}{\frac{\delta^2}{(\gamma-\alpha)^2}}\beta^{n_1+n_2} + \frac{\gamma^{2m}-1}{\gamma^2-1}\frac{\frac{1}{(\gamma-1)^2}}{\frac{\delta^2}{(\alpha-\beta)^2}}\gamma^{n_1+n_2}$$
$$-\frac{\gamma^m-1}{\gamma^{m-1}}\frac{\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1}}{(\alpha-\beta)-\delta} - \frac{\beta^m-1}{\beta^{m-1}}\frac{\alpha^{n_1}\lambda^{n_2}+\alpha^{n_2}\lambda^{n_1}}{(\gamma-\alpha)^2-\delta} - \frac{\alpha^m-1}{\alpha^{m-1}}\frac{\beta^{n_1}\lambda^{n_2}+\beta^{n_2}\lambda^{n_1}}{(\beta-\lambda)-\delta}$$
$$= \frac{(\alpha^{2m}-1)(\beta-\gamma)^2}{\delta^2(\alpha-1)^3}\alpha^{n_1+n_2-3} + \frac{(\beta^{2m}-1)(\gamma-\alpha)^2}{\delta^2(\beta-1)^3}\beta^{n_1+n_2-3} + \frac{(\gamma^{2m}-1)(\alpha-\beta)^2}{\delta^2(\gamma-1)^3}\gamma^{n_1+n_2-3}$$
$$-\frac{\left(\gamma^m-1\right)}{\gamma^{m-1}}\frac{(\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1})}{(\alpha-\beta)\delta} - \frac{\left(\beta^m-1\right)}{\beta^{m-1}}\frac{(\alpha^{n_1}\gamma^{n_2}+\alpha^{n_2}\gamma^{n_1})}{(\gamma-\alpha)\delta} - \frac{\left(\alpha^m-1\right)}{\alpha^{m-1}}\frac{(\beta^{n_1}\gamma^{n_2}+\beta^{n_2}\gamma^{n_1})}{(\beta-\gamma)\delta}.$$

This completes the proof. □

Result 3.10. (Norm) *Let the coefficients α, β and γ be the roots of the third-order polynomial belonging to the Padovan number sequence, and the vectors $\vec{a} = \left[1, \alpha, \alpha^2, ..., \alpha^{m-1}\right]^T$, $\vec{b} = \left[1, \beta, \beta^2, ..., \beta^{m-1}\right]^T$, $\vec{c} = \left[1, \gamma, \gamma^2, ..., \gamma^{m-1}\right]^T$ consisting of these roots be n-dimensional vectors in Euclidean 3-space. The norm of the m-dimensional Padovan vector P_n^m is calculated by*

$$\left\|\vec{P}_n{}^m\right\|^2 = \left\|\vec{a}\right\|^2 p_1{}^2\alpha^{2n} + \left\|\vec{b}\right\|^2 p_2{}^2\beta^{2n} + \left\|\vec{c}\right\|^2 p_3{}^2\gamma^{2n}$$
$$+2\left\langle \vec{a}, \vec{b}\right\rangle p_1p_2\gamma^{-n} + 2\left\langle \vec{a}, \vec{c}\right\rangle p_1p_3\beta^{-n} + 2\left\langle \vec{b}, \vec{c}\right\rangle p_2p_3\alpha^{-n}$$

or

$$\left\|\vec{P}_n{}^m\right\|^2 = \frac{(\alpha^{2m}-1)(\beta-\gamma)^2}{\delta^2(\alpha-1)^3}\alpha^{2n-3} + \frac{(\beta^{2m}-1)(\gamma-\alpha)^2}{\delta^2(\beta-1)^3}\beta^{2n-3} + \frac{(\gamma^{2m}-1)(\alpha-\beta)^2}{\delta^2(\gamma-1)^3}\gamma^{2n-3}$$
$$-\frac{2\left(\gamma^m-1\right)}{\gamma^{m-1}}\frac{(\alpha\beta)^n}{(\alpha-\beta)\delta} - \frac{2\left(\beta^m-1\right)}{\beta^{m-1}}\frac{(\alpha\gamma)^n}{(\gamma-\alpha)\delta} - \frac{2\left(\alpha^m-1\right)}{\alpha^{m-1}}\frac{(\beta\gamma)^n}{(\beta-\gamma)\delta},$$

where $\delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$.

Theorem 3.11. (Vector product) *Let the coefficients* α, β *and* γ *be the roots of the third-degree polynomial belonging to the Padovan number sequence. The norm of the two Padovan vectors* $P_{n_1}{}^m$ *and* $P_{n_2}{}^m$ *with m dimensions is calculated as*

$$\begin{aligned}\vec{P}_{n_1}^m \times \vec{P}_{n_2}^m = &\left(\vec{a}\times\vec{b}\right)\left[\frac{(\gamma-1)}{\delta(\alpha-\beta)}\left(\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1}\right)\right]+\left(\vec{a}\times\vec{c}\right)\left[\frac{(\beta-1)}{\delta(\gamma-\alpha)}\left(\alpha^{n_1}\gamma^{n_2}+\alpha^{n_2}\gamma^{n_1}\right)\right]\\ &+\left(\vec{b}\times\vec{c}\right)\left[\frac{(\alpha-1)}{\delta(\beta-\gamma)}\left(\beta^{n_1}\gamma^{n_2}+\beta^{n_2}\gamma^{n_1}\right)\right],\end{aligned}$$

with $\delta=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ for the vectors $\vec{a}=\left[1,\alpha,\alpha^2,...,\alpha^{m-1}\right]^T$, $\vec{b}=\left[1,\beta,\beta^2,...,\beta^{m-1}\right]^T$, $\vec{c}=\left[1,\gamma,\gamma^2,...,\gamma^{m-1}\right]^T$ *consisting of the roots* α, β *and* γ.

Proof. By using the equation (3.15) in Theorem 3.5, the vector product of Padovan vectors $P_{n_1}^m = p_1\alpha^{n_1}\vec{a}+p_2\beta^{n_1}\vec{b}+p_3\gamma^{n_1}\vec{c}$ and $\vec{P}_{n_1}^m = p_1\alpha^{n_2}\vec{a}+p_2\beta^{n_2}\vec{b}+p_3\gamma^{n_2}\vec{c}$ is found by the equation

$$\begin{aligned}\vec{P}_{n_1}^m \times \vec{P}_{n_2}^m &= \left(p_1\alpha^{n_1}\vec{a}+p_2\beta^{n_1}\vec{b}+p_3\gamma^{n_1}\vec{c}\right)\times\left(p_1\alpha^{n_2}\vec{a}+p_2\beta^{n_2}\vec{b}+p_3\gamma^{n_2}\vec{c}\right)\\ &=\left(\vec{a}\times\vec{a}\right)p_1{}^2\alpha^{n_1+n_2}+\left(\vec{b}\times\vec{b}\right)p_2{}^2\beta^{n_1+n_2}+\left(\vec{c}\times\vec{c}\right)p_3{}^2\gamma^{n_1+n_2}\\ &\quad+\left(\vec{a}\times\vec{b}\right)\left[p_1p_2\left(\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1}\right)\right]+\left(\vec{a}\times\vec{c}\right)\left[p_1p_3\left(\alpha^{n_1}\gamma^{n_2}+\alpha^{n_2}\gamma^{n_1}\right)\right]\\ &\quad+\left(\vec{b}\times\vec{c}\right)\left[p_2p_3\left(\beta^{n_1}\gamma^{n_2}+\beta^{n_2}\gamma^{n_1}\right)\right]\end{aligned}$$

In this equation, if $\vec{a}\times\vec{a}=0$, $\vec{b}\times\vec{b}=0$, $\vec{c}\times\vec{c}=0$ is written and edited instead of vector products, we get the result

$$\begin{aligned}\vec{P}_{n_1}^m \times \vec{P}_{n_2}^m = &\left(\vec{a}\times\vec{b}\right)\left[p_1p_2\left(\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1}\right)\right]+\left(\vec{a}\times\vec{c}\right)\left[p_1p_3\left(\alpha^{n_1}\gamma^{n_2}+\alpha^{n_2}\gamma^{n_1}\right)\right]\\ &+\left(\vec{b}\times\vec{c}\right)\left[p_2p_3\left(\beta^{n_1}\gamma^{n_2}+\beta^{n_2}\gamma^{n_1}\right)\right].\end{aligned}$$

The desired result is obtained when Equation (2.1) is written instead of p_1, p_2 and p_3 terms and abbreviated with the equation $(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)=1$ in Lemma 2.2.(5). □

Theorem 3.12. (Triple product) *Let the coefficients* α, β *and* γ *be the roots of the third-degree polynomial belonging to the Padovan number sequence. The triple product of the two Padovan vectors* $P_{n_1}{}^m$ *and* $P_{n_2}{}^m$ *of m size is calculated as*

$$\vec{P}_{n_1}^m \times \vec{P}_{n_2}^m = \frac{-1}{\delta^2}\left\{\begin{array}{l}\left\langle\vec{b}\times\vec{c},\vec{a}\right\rangle\alpha^{n_3}\left(\beta^{n_1}\gamma^{n_2}+\beta^{n_2}\gamma^{n_1}\right)+\left\langle\vec{a}\times\vec{c},\vec{b}\right\rangle\beta^{n_3}\left(\alpha^{n_1}\gamma^{n_2}+\alpha^{n_2}\gamma^{n_1}\right)\\ +\left\langle\vec{a}\times\vec{b},\vec{c}\right\rangle\gamma^{n_3}\left(\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1}\right)\end{array}\right\},$$

with $\delta=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ *for vectors* $\vec{a}=\left[1,\alpha,\alpha^2,...,\alpha^{m-1}\right]^T$, $\vec{b}=\left[1,\beta,\beta^2,...,\beta^{m-1}\right]^T$, $\vec{c}=\left[1,\gamma,\gamma^2,...,\gamma^{m-1}\right]^T$ *consisting of the roots* α, β *and* γ.

Proof. Using the equation (3.15) in Theorem 3.5, Equation (2.1) is written in place of the p_1, p_2 and p_3 terms in the

$$\left\langle \vec{P}_{n_1}^m \times \vec{P}_{n_2}^m, \vec{P}_{n_3}^m \right\rangle = \left\langle \left(p_1\alpha^{n_1}\vec{a} + p_2\beta^{n_1}\vec{b} + p_3\gamma^{n_1}\vec{c} \right) \times \left(p_1\alpha^{n_2}\vec{a} + p_2\beta^{n_2}\vec{b} + p_3\gamma^{n_2}\vec{c} \right), \left(p_1\alpha^{n_3}\vec{a} + p_2\beta^{n_3}\vec{b} + p_3\gamma^{n_3}\vec{c} \right) \right\rangle$$

equation to calculate the mixed product of the Padovan vectors $P_{n_1}^m = p_1\alpha^{n_1}\vec{a} + p_2\beta^{n_1}\vec{b} + p_3\gamma^{n_1}\vec{c}$ and $\vec{P}_{n_1}^m = p_1\alpha^{n_2}\vec{a} + p_2\beta^{n_2}\vec{b} + p_3\gamma^{n_2}\vec{c}$. Then, the desired result is obtained when the equations $(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)=1$ and $\delta=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ are substituted in the obtained equation. □

Corollary 3.13. *Let be taken three-dimensional Padovan vectors* $\vec{P}_{n_1}^{\ 3} = \begin{bmatrix} P_{n_1} & P_{n_1+1} & P_{n_1+2} \end{bmatrix}^T$ *and* $\vec{P}_{n_2}^{\ 3} = \begin{bmatrix} P_{n_2} & P_{n_2+1} & P_{n_2+2} \end{bmatrix}^T$ *included the Padovan numbers* $P_{n_i}, P_{n_i+1}, P_{n_i+2}$. *The inner product of the Padovan vectors* $P_{n_1}^{\ 3}$ *and* $P_{n_2}^{\ 3}$ *are defined by*

$$\left\langle \vec{P}_{n_1}^{\ 3}, \vec{P}_{n_2}^{\ 3} \right\rangle = P_{n_1}P_{n_2} + P_{n_1+1}P_{n_2+1} + P_{n_1+2}P_{n_2+2}.$$

Theorem 3.14. *Let the coefficients* α,β *and* γ *be the root of the third order polynomial belonging to the Padovan number sequence, and let be* $\delta=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$. *The expression of the inner product of the Padovan vectors* $P_{n_1}^{\ 3}$ *and* $P_{n_2}^{\ 3}$ *with the roots* α,β *and* γ *are computed by*

$$\begin{aligned}\left\langle \vec{P}_{n_1}^{\ 3}, \vec{P}_{n_2}^{\ 3} \right\rangle &= \left\langle \vec{a},\vec{a} \right\rangle p_1^{\ 2}\alpha^{n_1+n_2} + \left\langle \vec{b},\vec{b} \right\rangle p_2^{\ 2}\beta^{n_1+n_2} + \left\langle \vec{c},\vec{c} \right\rangle p_3^{\ 2}\gamma^{n_1+n_2} \\ &+ \left\langle \vec{a},\vec{b} \right\rangle \left[p_1p_2\left(\alpha^{n_1}\beta^{n_2} + \alpha^{n_2}\beta^{n_1} \right) \right] + \left\langle \vec{a},\vec{c} \right\rangle \left[p_1p_3\left(\alpha^{n_1}\gamma^{n_2} + \alpha^{n_2}\gamma^{n_1} \right) \right] \\ &+ \left\langle \vec{b},\vec{c} \right\rangle \left[p_2p_3\left(\beta^{n_1}\gamma^{n_2} + \beta^{n_2}\gamma^{n_1} \right) \right]\end{aligned}$$

or

$$\begin{aligned}\left\langle \vec{P}_{n_1}^{\ 3}, \vec{P}_{n_2}^{\ 3} \right\rangle =& \frac{(\alpha^6-1)(\beta-\gamma)^2}{\delta^2(\alpha-1)^3}\alpha^{n_1+n_2-3} + \frac{(\beta^6-1)(\gamma-\alpha)^2}{\delta^2(\beta-1)^3}\beta^{n_1+n_2-3} + \frac{(\gamma^6-1)(\alpha-\beta)^2}{\delta^2(\gamma-1)^3}\gamma^{n_1+n_2-3} \\ &- \frac{\left(\gamma^3-1\right)}{\gamma^2}\frac{(\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1})}{(\alpha-\beta)\delta} - \frac{\left(\beta^3-1\right)}{\beta^2}\frac{(\alpha^{n_1}\gamma^{n_2}+\alpha^{n_2}\gamma^{n_1})}{(\gamma-\alpha)\delta} - \frac{\left(\alpha^3-1\right)}{\alpha^2}\frac{(\beta^{n_1}\gamma^{n_2}+\beta^{n_2}\gamma^{n_1})}{(\beta-\gamma)\delta}\end{aligned}$$

where $\vec{a} = \left[1,\alpha,\alpha^2\right]^T$, $\vec{b} = \left[1,\beta,\beta^2\right]^T$, $\vec{c} = \left[1,\gamma,\gamma^2\right]^T$ *are the three-dimensional vectors.*

Corollary 3.15. *Let the coefficients* α, β *and* γ *be the root of the third order polynomial belonging to the Padovan number sequence, and let be* $\delta=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$. *The norm of the three-dimensional Padovan vector* P_n^3 *is calculated by*

$$\begin{aligned}\left\| \vec{P}_n^{\ 3} \right\|^2 &= \left\| \vec{a} \right\|^2 p_1^{\ 2}\alpha^{2n} + \left\| \vec{b} \right\|^2 p_2^{\ 2}\beta^{2n} + \left\| \vec{c} \right\|^2 p_3^{\ 2}\gamma^{2n} \\ &+ 2\left\langle \vec{a},\vec{b} \right\rangle p_1p_2\gamma^{-n} + 2\left\langle \vec{a},\vec{c} \right\rangle p_1p_3\beta^{-n} + 2\left\langle \vec{b},\vec{c} \right\rangle p_2p_3\alpha^{-n}\end{aligned}$$

or

$$\left\|\vec{P}_n^{\,3}\right\|^2=\frac{(\alpha^6-1)(\beta-\gamma)^2}{\delta^2(\alpha-1)^3}\alpha^{2n-3}+\frac{(\beta^6-1)(\gamma-\alpha)^2}{\delta^2(\beta-1)^3}\beta^{2n-3}+\frac{(\gamma^6-1)(\alpha-\beta)^2}{\delta^2(\gamma-1)^3}\gamma^{2n-3}$$
$$-\frac{2\left(\gamma^3-1\right)}{\gamma^2}\frac{(\alpha\beta)^n}{(\alpha-\beta)\delta}-\frac{2\left(\beta^3-1\right)}{\beta^2}\frac{(\alpha\gamma)^n}{(\gamma-\alpha)\delta}-\frac{2\left(\alpha^3-1\right)}{\alpha^2}\frac{(\beta\gamma)^n}{(\beta-\gamma)\delta}$$

where $\vec{a}=\left[1,\alpha,\alpha^2\right]^T$, $\vec{b}=\left[1,\beta,\beta^2\right]^T$, $\vec{c}=\left[1,\gamma,\gamma^2\right]^T$ *are the three-dimensional vectors.*

Given the three-dimensional vectors $\vec{a}=\left[1,\alpha,\alpha^2\right]^T$, $\vec{b}=\left[1,\beta,\beta^2\right]^T$, $\vec{c}=\left[1,\gamma,\gamma^2\right]^T$ consisting of the roots α, β and γ, the vectorial products and triple products can be calculated similarly in Theorems 3.11 and 3.12.

Corollary 3.16. *Let two-dimensional Padovan vectors* $\vec{P}_{n_1}^{\,2}=\left[P_{n_1}\quad P_{n_1+1}\right]^T$ *and* $\vec{P}_{n_2}^{\,2}=\left[P_{n_2}\quad P_{n_2+1}\right]^T$ *included the Padovan numbers* P_{n_i} *and* P_{n_i+1} *be taken. The inner product of the Padovan vectors* $P_{n_1}^{\,2}$ *and* $P_{n_2}^{\,2}$ *are defined by*

$$\left\langle \vec{P}_{n_1}^{\,2},\vec{P}_{n_2}^{\,2}\right\rangle=P_{n_1}P_{n_2}+P_{n_1+1}P_{n_2+1}.$$

Theorem 3.17. *Let the coefficients* α, β *and* γ *be the root of the third order polynomial belonging to the Padovan number sequence, and let be* $\delta=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$. *The expression of the inner product of the Padovan vectors* $P_{n_1}^{\,2}$ *and* $P_{n_2}^{\,2}$ *with the roots* α, β *and* γ *are computed by*

$$\left\langle \vec{P}_{n_1}^{\,2},\vec{P}_{n_2}^{\,2}\right\rangle=\left\langle\vec{a},\vec{a}\right\rangle p_1^{\,2}\alpha^{n_1+n_2}+\left\langle\vec{b},\vec{b}\right\rangle p_2^{\,2}\beta^{n_1+n_2}+\left\langle\vec{c},\vec{c}\right\rangle p_3^{\,2}\gamma^{n_1+n_2}$$
$$+\left\langle\vec{a},\vec{b}\right\rangle\left[p_1p_2\left(\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1}\right)\right]+\left\langle\vec{a},\vec{c}\right\rangle\left[p_1p_3\left(\alpha^{n_1}\gamma^{n_2}+\alpha^{n_2}\gamma^{n_1}\right)\right]$$
$$+\left\langle\vec{b},\vec{c}\right\rangle\left[p_2p_3\left(\beta^{n_1}\gamma^{n_2}+\beta^{n_2}\gamma^{n_1}\right)\right]$$

or

$$\left\langle \vec{P}_{n_1}^{\,2},\vec{P}_{n_2}^{\,2}\right\rangle=\frac{(\alpha^4-1)(\beta-\gamma)^2}{\delta^2(\alpha-1)^3}\alpha^{n_1+n_2-3}+\frac{(\beta^4-1)(\gamma-\alpha)^2}{\delta^2(\beta-1)^3}\beta^{n_1+n_2-3}+\frac{(\gamma^4-1)(\alpha-\beta)^2}{\delta^2(\gamma-1)^3}\gamma^{n_1+n_2-3}$$
$$-\frac{\left(\gamma^2-1\right)}{\gamma}\frac{(\alpha^{n_1}\beta^{n_2}+\alpha^{n_2}\beta^{n_1})}{(\alpha-\beta)\delta}-\frac{\left(\beta^2-1\right)}{\beta}\frac{(\alpha^{n_1}\gamma^{n_2}+\alpha^{n_2}\gamma^{n_1})}{(\gamma-\alpha)\delta}-\frac{\left(\alpha^2-1\right)}{\alpha}\frac{(\beta^{n_1}\gamma^{n_2}+\beta^{n_2}\gamma^{n_1})}{(\beta-\gamma)\delta}$$

where $\vec{a}=[1,\alpha]^T$, $\vec{b}=[1,\beta]^T$, $\vec{c}=[1,\gamma]^T$ *are the two-dimensional vectors.*

Corollary 3.18. *Let* $\vec{a}=[1,\alpha]^T$, $\vec{b}=[1,\beta]^T$, $\vec{c}=[1,\gamma]^T$ *be two-dimensional Padovan vectors with the Padovan polynomial coefficients* α, β *and* γ. *The norm of the two-dimensional Padovan vectors*

$$\left\|\vec{P}_n^{\,m}\right\|^2=\left\|\vec{a}\right\|^2p_1^{\,2}\alpha^{2n}+\left\|\vec{b}\right\|^2p_2^{\,2}\beta^{2n}+\left\|\vec{c}\right\|^2p_3^{\,2}\gamma^{2n}$$
$$+2\left\langle\vec{a},\vec{b}\right\rangle p_1p_2\gamma^{-n}+2\left\langle\vec{a},\vec{c}\right\rangle p_1p_3\beta^{-n}+2\left\langle\vec{b},\vec{c}\right\rangle p_2p_3\alpha^{-n}$$

or

$$\left\|\vec{P}_n^{\,2}\right\|^2 = \frac{(\alpha^4-1)(\beta-\gamma)^2}{\delta^2(\alpha-1)^3}\alpha^{2n-3} + \frac{(\beta^4-1)(\gamma-\alpha)^2}{\delta^2(\beta-1)^3}\beta^{2n-3} + \frac{(\gamma^4-1)(\alpha-\beta)^2}{\delta^2(\gamma-1)^3}\gamma^{2n-3}$$
$$-\frac{2\left(\gamma^2-1\right)}{\gamma}\frac{(\alpha\beta)^n}{(\alpha-\beta)\delta} - \frac{2\left(\beta^2-1\right)}{\beta}\frac{(\alpha\gamma)^n}{(\gamma-\alpha)\delta} - \frac{2\left(\alpha^2-1\right)}{\alpha}\frac{(\beta\gamma)^n}{(\beta-\gamma)\delta}$$

where $\delta = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$.

Additionally, the vectorial and triple product of the Padovan vectors $P_{n_1}^{\ 2}$ and $P_{n_2}^{\ 2}$ can be similarly calculated in Theorems 3.11 and 3.12.

4 Numerical example

In this section, calculations of Padovan vectors and their geometric properties have been made using the Padovan–Binet formulas, and a three-dimensional Padovan curve has been obtained with the help of the Padovan vectors obtained. Firstly, calculations were made by following the steps given in the *Geogebra* algorithm. In the first step, the coefficients α, β and γ of the given formula in Definition 2.1 were calculated. In the second step, the coefficients p_1, p_2, p_3 were calculated using Equation (2.1). In the third step, ten Padovan numbers were computed $P_1 = 1$, $P_2 = 1$, $P_3 = 2$, $P_4 = 2$, $P_5 = 3$, $P_6 = 4$, $P_7 = 5$, $P_8 = 7$, $P_9 = 9$, $P_{10} = 12$ by substituting the terms $n = 1, 2, ..., 10$ in the Equation (2.2).

In the fourth step, the Padovan vectors $\vec{P}_n^{\,m} = p_1\alpha^n\vec{a} + p_2\beta^n\vec{b} + p_3\gamma^n\vec{c}$ in the Equation (3.15) are calculated as

$$\vec{P}_1^{\,3} = (1,1,2),\ \vec{P}_2^{\,3} = (1,2,2),\ \vec{P}_3^{\,3} = (2,2,3),\ \vec{P}_4^{\,3} = (2,3,4),\quad \vec{P}_5^{\,3} = (3,4,5),\ \vec{P}_6^{\,3} = (4,5,7),$$
$$\vec{P}_7^{\,3} = (5,7,9),\ \ \vec{P}_8^{\,3} = (7,9,12),\ \vec{P}_9^{\,3} = (9,12,16),\ \vec{P}_{10}^{\,3} = (12,16,21)$$

for the values n = 1, 2, …, 10 and m = 3. Then with the initial point origin, the Padovan vectors were drawn in Figure 1.

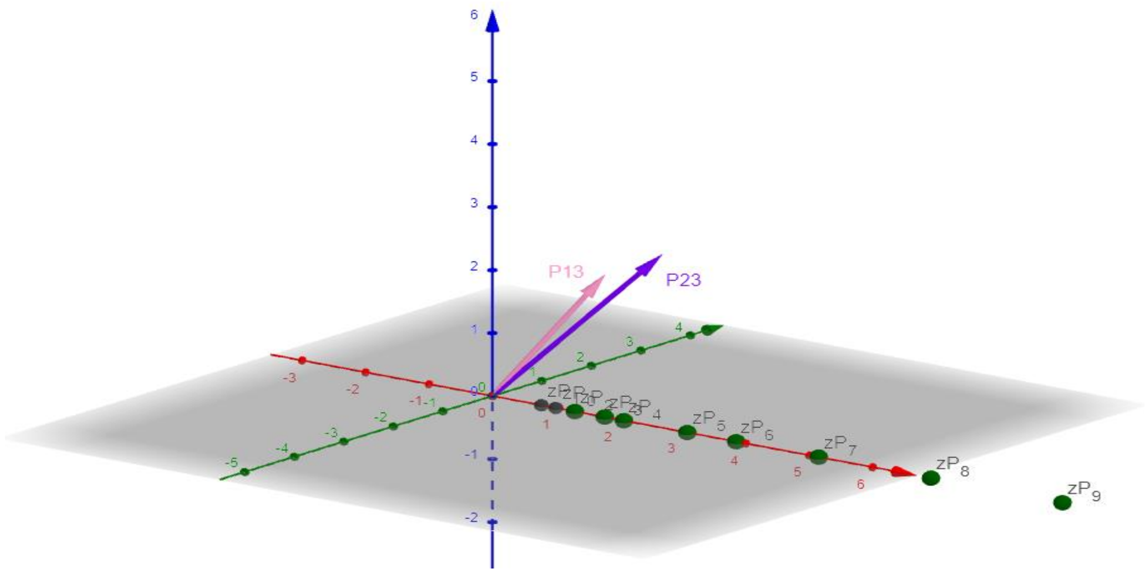

Figure 1. The Padovan numbers, the Padovan vectors $\vec{P}_1^{\,3}$ and $\vec{P}_2^{\,3}$

If we will take the Padovan vectors $\vec{P}_1^3 = (1,1,2)$ and $\vec{P}_2^3 = (1,2,2)$ in Figure 1, the inner product, and the vectorial product are obtained by $\left\langle \vec{P}_1^3, \vec{P}_2^3 \right\rangle = 7$ and $\vec{P}_1^3 \times \vec{P}_2^3 = (-2,0,1)$. Additionally, the norms are computed by $\left\| \vec{P}_1^3 \right\| = 6$ $\left\| \vec{P}_2^3 \right\| = 3$.

In Step 6, a three-dimensional Padovan curve was obtained with the curve fitting feature in the *Geogebra* program according to these Padovan vectors, see Figure 2.

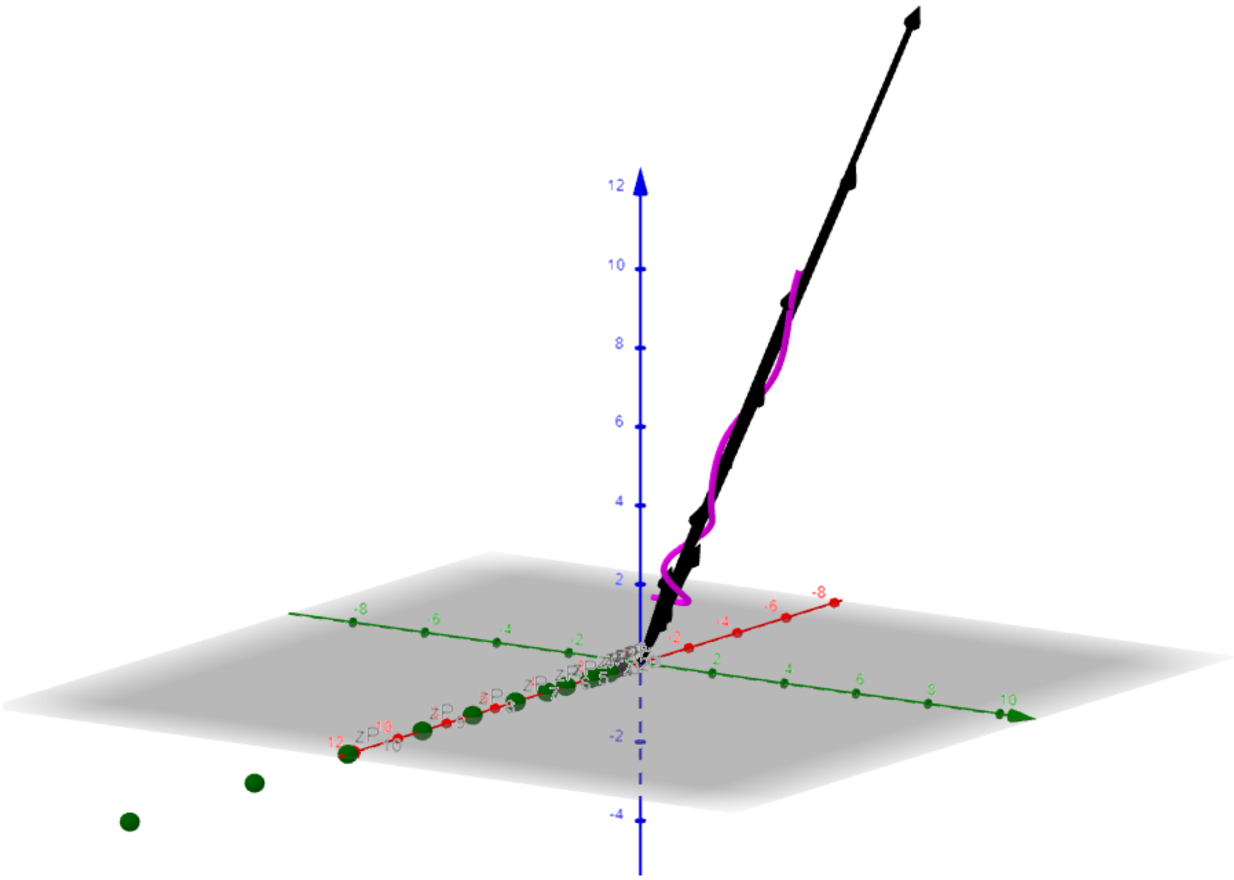

Figure 2. Padovan numbers, Padovan vectors and Padovan curve

***Geogebra* Algorithm**

Step 1: Calculate the coefficients α, β and γ with the values given in Definition 2.1.

Step 2: Compute the coefficients p_1, p_2 and p_3 given in equation (2.1).

Step 3: Obtain the Padovan numbers for n = 1, 2, ...,10 in the Padovan number sequence given in (2.2).

Step 4: Compute the Padovan vectors P_n^m in the equation of Padovan vectors given in (3.15) for the values n = 1, 2, ...,10.

Step 5: Calculate the inner products, norms, vectorial products of the Padovan vectors $\vec{P}_1^3$ and $\vec{P}_2^3$.

Step 6: Obtain a three-dimensional Padovan curve by using the curve fitting command for the Padovan vectors $t = \{\mathrm{P}_1^3, \mathrm{P}_2^3, \mathrm{P}_3^3, \mathrm{P}_4^3, \mathrm{P}_5^3, \mathrm{P}_6^3, \mathrm{P}_7^3, \mathrm{P}_8^3, \mathrm{P}_9^3, \mathrm{P}_{10}^3\}$.

4 Conclusion

Our study introduced Padovan vectors for the first time using a Padovan Binet-like formula and reduction relation. We computed the inner product, norm, vector products, and triple products of Padovan vectors and investigated the geometric properties of Padovan numbers. To generate the first ten Padovan vectors, we utilized Binet formulas in the *Geogebra* program. Moreover, we calculated the first ten Padovan numbers and vectors and presented them as points and vectors in three-dimensional space. By utilizing the *Geogebra* program's curve fitting feature, we drew the Padovan curve in space for the first time using the discovered Padovan vectors. As a result, our study introduced a novel geometric approach to Padovan number sequences.

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