Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 4, 827–841
DOI: 10.7546/nntdm.2023.29.4.827-841

The t-Fibonacci sequences in the 2-generator p-groups of nilpotency class 2

Elahe Mehraban[1], Ömur Deveci[2] and Evren Hincal[3]

[1] Department of Mathematics, Near East University TRNC
Mersin 10, Nicosia, 99138, Turkey
e-mail: e.mehraban.math@gmail.com

[2] Department of Mathematics, Faculty of Science and Letters
Kafkas University, 36100, Turkey
e-mail: odeveci36@hotmail.com

[3] Department of Mathematics, Near East University TRNC
Mersin 10, Nicosia, 99138, Turkey
e-mail: evren.hincal@neu.edu.tr

Received: 28 September 2023 **Revised:** 28 November 2023
Accepted: 4 December 2023 **Online First:** 23 December 2023

Abstract: In this paper, we consider the 2-generator p-groups of nilpotency class 2. We will discuss the lengths of the periods of the t-Fibonacci sequences in these groups.
Keywords: Period, Nilpotent group, t-Fibonacci sequence, p-group.
2020 Mathematics Subject Classification: 20F05, 11B39, 20D60.

1 Introduction

Fibonacci sequence and its generalization t-Fibonacci sequence are famous sequences in mathematics. Many authors studied on these sequences (for example [1, 2, 5, 6, 9, 12, 18, 19]). For $t \geq 2$, the t–Fibonacci number sequence, $\{F_n^t\}_{n=0}^{\infty}$, is defined by

$$F_n^t = F_{n-1}^t + F_{n-2}^t + \cdots + F_{n-t}^t, \quad n \geq t,$$

and we seed the sequence with $F_0^t = 0,\ F_1^t = 0, \ldots,\ F_{t-2}^t = 0, F_{t-1}^t = 1$. Let $K_t(m)$ denote the minimal lenght of the period of the series $\{F_n^t (\bmod m)\}_{n=0}^{\infty}$. We call it wall number of m with respective to t-Fibonacci number sequence (see [20]).

Definition 1.1. *A t-Fibonacci sequence in a finite group $G = \langle X \rangle$ is a sequence of group elements $x_1, x_2, \ldots, x_n, \ldots,$for which, given an intial (seed) set $X = \{a_1, \ldots, a_j\}$, each element is definted by*

$$x_n = \begin{cases} a_n, & for\ \ n \leq j, \\ x_1 x_2 \ldots (x_{n-1}), & for\ \ j < n \leq t, \\ x_{n-k} \ldots (x_{n-1}), & for\ \ n > t. \end{cases}$$

The t-Fibonacci sequence of the group $G = \langle X \rangle$ and its period are denoted by $F_t(G; X)$ and $L_t(G; X)$, respectively (see [17]).

By [3], for the 2-generator p-groups of nilpotency class 2 where p is an odd prime, we have:

- $G_1 \cong \langle a \rangle \rtimes \langle b \rangle$ where $[a, b] = a^{p^{\alpha-\gamma}}, |a| = p^\alpha, |b| = p^\beta, |[a, b]| = p^\gamma, \alpha, \beta, \gamma \in \mathbb{N}, \alpha = 2\gamma$ and $\beta = \gamma$.
- $G_2 \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$ where $[a, b] = c, [a, c] = [b, c] = 1, |a| = p^\alpha, |b| = p^\beta, |c| = p^\gamma, \alpha, \beta, \gamma \in \mathbb{N}$ and $\alpha \geq \beta \geq \gamma$.
- $G_3 \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$ where $[a, b] = a^{p^{\alpha-\gamma}} c, [c, b] = a^{-p^{2(\alpha-\gamma)}} c^{-p^{\alpha-\gamma}}, |a| = p^\alpha, |b| = p^\beta, |c| = p^\sigma, |[a, b]| = p^\gamma, \alpha, \beta, \gamma, \sigma \in \mathbb{N}, \alpha \geq \beta \geq \sigma \geq 1$ and $\alpha + \sigma \geq 2\gamma$.

Note that $\rtimes$ is simidirect product in groups. In group theory, a semidirect product is a generalization of the direct product which expresses a group as a product of subgroups.
From the above, we have the following lemmas.

Lemma 1.1. *(i) Every element of G_1 can be written uniquely in the form $a^j b^k$, where $1 \leq j \leq p^\alpha$ and $1 \leq k \leq p^\beta$. Then, $|G_1| = p^{\alpha+\beta}$.*
(ii) Every element of G_2 may be uniquely presented by $c^i a^j b^k$, where $1 \leq i \leq p^\gamma,\ 1 \leq j \leq p^\alpha$ and $1 \leq k \leq p^\beta$. So that $|G_2| = p^{\alpha+\beta+\gamma}$.
(iii) Every element of G_3 may be uniquely presented by $c^i a^j b^k$, where $1 \leq i \leq p^\sigma,\ 1 \leq j \leq p^\alpha$ and $1 \leq k \leq p^\beta$. Hence, $|G_3| = p^{\alpha+\beta+\gamma}$.

Lemma 1.2. *For every integer n and $m \geq 2$, if*

$$\begin{cases} F_n^t \equiv 0 & (\bmod m), \\ F_{n+1}^t \equiv 0 & (\bmod m), \\ \quad \vdots & \\ F_{n+t-2}^t \equiv 0 & (\bmod m), \\ F_{n+t-1}^t \equiv 0 & (\bmod m). \end{cases}$$

Then $K_t(m) \mid n$ (see [14]).

Sections $2, 3$ and 4 are devoted to studying the t-Fibonacci sequence in the groups G_1, G_2, and G_3, respectively.

2 The t-Fibonacci sequence in the group G_1

In this section, we discuss the t-Fibonacci sequence in $G_1 \cong \langle a \rangle \rtimes \langle b \rangle$ where $[a,b] = a^{p^{\alpha-\gamma}}$, $|a| = p^{\alpha}, |b| = p^{\beta}, |[a,b]| = p^{\gamma}, \alpha, \beta, \gamma \in \mathbb{N}, \alpha = 2\gamma$, $\beta = \gamma$ and get the period of G_1 with respect to $X = \{a, b\}$. First, we find a standard form of 3-Fibonacci sequence $x_3,\ x_4, \ldots$ of G_1. For this, we need the following sequence:

$$h_1(3) = 1,\ \ h_2(3) = 0, \quad h_3(3) = p^{\gamma} + 1,$$
$$h_n(3) = h_{n-3}(3) + h_{n-2}(3) + h_{n-1}(3) + p^{\gamma}(h_{n-3}(3)(F^3_{n-2} + F^3_{n-1}) + h_{n-2}(3)F^3_{n-1}),\ \ n \geq 4.$$

Lemma 2.1. *Every element of $F_3(G_1; X)$ may be presented by $x_n = b^{F^3_n} a^{h_n(3)},\ \ n \geq 4$.*

Proof. By the relation $[a,b] = a^{p^{\gamma}}$ of G_1, we get $ab = ba^{(p^{\gamma}+1)}$. For $n = 3, n = 4$ and $n = 5$, we have $x_3 = ab = ba^{(p^{\gamma}+1)}$, $x_4 = ab(ba^{(p^{\gamma}+1)}) = b^2a^{(2+3p^{\gamma})}$ and $x_5 = bba^{(p^{\gamma}+1)}b^2a^{(2+3p^{\gamma})} = b^4a^{(3+6p^{\gamma})}$. Then by induction method on n, we get

$$\begin{aligned} x_n &= x_{n-3}x_{n-2}x_{n-1} = b^{F^3_{n-3}}a^{h_{n-3}(3)}b^{F^3_{n-2}}a^{h_{n-2}(3)}b^{F^3_{n-1}}a^{h_{n-1}(3)} \\ &= b^{F^3_{n-3}+F^3_{n-2}}a^{h_{n-3}(3)(1+F^3_{n-2}p^{\gamma})}a^{h_{n-2}(3)}b^{F^3_{n-1}}a^{h_{n-1}(3)} \\ &= b^{F^3_{n-3}+F^3_{n-2}+F^3_{n-1}}a^{h_{n-3}(3)+h_{n-2}(3)+h_{n-1}(3)+p^{\gamma}(h_{n-3}(3)(F^3_{n-2}+F^3_{n-1})+h_{n-2}(3)F^3_{n-1})} \\ &= b^{F^3_n}a^{h_n(3)}. \end{aligned}$$

Then the assertion holds. □

Example 2.1. *For integers $\alpha = 2,\ \beta = \gamma = 1$, $p = 3$, by Lemma 2.1 and the relations of G_1, we have*

$$x_1 = a, \quad x_2 = b, \quad x_3 = ab = ba^{(3+1)} = ba^4 = b^{F^3_3}a^{h_3(3)} \equiv ba^1,\ x_4 = ab(ba^4) = b^2a^11 \equiv b^2a^2,$$
$$x_5 = b^4a^{21} \equiv b^1, \ldots,\ x_{40} = a,\ x_{41} = b,\ x_{42} = ba^1 \ldots.$$

Consequently,

$$x_{40} = x_{39+1} = x_1, \quad x_{41} = x_{39+2} = x_2, \quad x_{42} = x_{39+3} = x_3.$$

Then $L_3(G_1; X) = 39$.

Lemma 2.2. *If $L_3(G_1; X) = t$ then t is the Least integer such that all of the equations*

$$\begin{cases} h_{t+1}(3) \equiv 1 \pmod{p^{\alpha}}, \\ h_{t+2}(3) \equiv 0 \pmod{p^{\alpha}}, \\ h_{t+3}(3) \equiv 1 \pmod{p^{\alpha}}, \\ F^3_{t+1} \equiv 0 \pmod{p^{\beta}}, \\ F^3_{t+2} \equiv 1 \pmod{p^{\beta}}, \\ F^3_{t+3} \equiv 1 \pmod{p^{\beta}}, \end{cases}$$

hold. Moreover, $K_3(m)$ divides $L_3(G_1; X)$.

Proof. By Lemma 2.1, we get $x_n = b^{F^3_n}a^{h_n(3)}$. On the other hand, $x_{t+1} = a,\ x_{t+2} = b$ and $x_{t+3} = ab$.

Every element of G_1 can be written uniquely in the form a^jb^k, where $1 \leq j \leq p^{\alpha}$ and $1 \leq k \leq p^{\beta}$. So we have

$$\begin{cases} h_{t+1}(3) \equiv 1 \pmod{p^{\alpha}}, \\ h_{t+2}(3) \equiv 0 \pmod{p^{\alpha}}, \\ h_{t+3}(3) \equiv 1 \pmod{p^{\alpha}}, \\ F^3_{t+1} \equiv 0 \pmod{p^{\beta}}, \\ F^3_{t+2} \equiv 1 \pmod{p^{\beta}}, \\ F^3_{t+3} \equiv 1 \pmod{p^{\beta}}. \end{cases}$$

So, Lemma 1.2 yield that $K_3(m) \mid t$. □

In Table 1, by using the software Maple 18, we calculate $K_3(n), h_{K_3(n^2)+1}(3),\ h_{K_3(n^2)+2}(3)$ and $h_{K_3(n^2)+3}(3)$ for $n = p < 50$.

Table 1. $hP_3(n),\ h_{K_3(n^2)+1}(3),\ h_{K_3(n^2)+2}(3)$ and $h_{K_3(n^2)+3}(3)$

$n\!=\!p$	$K_3(n)$	$h_{K_3(n^2)+1}(3) \pmod{p^2}$	$h_{K_3(n^2)+2}(3) \pmod{p^2}$	$h_{K_3(n^)+3}(3) \pmod{p^2}$
3	13	$h_{39+1}(3) \equiv 1$	$h_{39+2}(3) \equiv 0$	$h_{39+3}(3) \equiv 4$
5	31	$h_{155+1}(3) \equiv 1$	$h_{155+2}(3) \equiv 0$	$h_{155+3}(3) \equiv 6$
7	48	$h_{326+1}(3) \equiv 1$	$h_{326+2}(3) \equiv 0$	$h_{326+3}(3) \equiv 8$
11	110	$h_{1210+1}(3) \equiv 1$	$h_{1210+2}(3) \equiv 0$	$h_{1210+3}(3) \equiv 12$
13	168	$h_{2184+1}(3) \equiv 1$	$h_{2184+2}(3) \equiv 0$	$h_{2184+3}(3) \equiv 14$
17	96	$h_{1632+1}(3) \equiv 1$	$h_{1632+2}(3) \equiv 0$	$h_{1632+3}(3) \equiv 18$
19	360	$h_{6840+1}(3) \equiv 1$	$h_{6840+2}(3) \equiv 0$	$h_{6840+3}(3) \equiv 20$
23	553	$h_{12719+1}(3) \equiv 1$	$h_{12791+2}(3) \equiv 0$	$h_{12791+3}(3) \equiv 24$
29	280	$h_{8120+1}(3) \equiv 1$	$h_{8120+2}(3) \equiv 0$	$h_{8120+3}(3) \equiv 30$
31	331	$h_{10261+1}(3) \equiv 1$	$h_{10261+2}(3) \equiv 0$	$h_{10261+3}(3) \equiv 32$
37	469	$h_{17353+1}(3) \equiv 1$	$h_{17353+2}(3) \equiv 0$	$h_{17353+3}(3) \equiv 38$
41	560	$h_{22960+1}(3) \equiv 1$	$h_{22960+2}(3) \equiv 0$	$h_{22960+3}(3) \equiv 42$
43	1232	$h_{52976+1}(3) \equiv 1$	$h_{52976+2}(3) \equiv 0$	$h_{52976+3}(3) \equiv 44$
47	46	$h_{2162+1}(3) \equiv 1$	$h_{2162+2}(3) \equiv 0$	$h_{2162+3}(3) \equiv 48$

We are now in a position to state the following important Theorem

Theorem 2.1. *For integer $t \geq 1$ and p is a prime. If $\beta = \gamma = 1$ and $\alpha = 2$, Then*

$$L_3(G_1; X) = K_3(p^2) = pK_3(p), \quad p < 50.$$

Proof. Let $p < 50$ and $m = p$. Then,

$$x_{K_3(m)+1} = b^{F^3_{K_3(m)+1}} a^{h_{K_3(m)+1}(3)} = a$$
$$x_{K_3(m)+2} = b^{F^3_{K_3(m)+2}} a^{h_{K_3(m)+2}(3)} = b,$$
$$x_{K_3(m)+3} = b^{F^3_{K_3(m)+3}} a^{h_{K_3(m)+3}(3)} = ba^{p+1}.$$

By Lemma 1.2 and Table 1, we have

$F^3_{K_3(m)+1} \equiv F^3_1 \equiv 0(\bmod m), F^3_{K_3(m)+2} \equiv F^3_2 \equiv 1(\bmod m)$ and $F^3_{K_3(m)+3} \equiv F^3_3 \equiv 1(\bmod m)$.
$h_{K_3(m)+1}(3) \equiv 1(\bmod m^2), h_{K_3(m)+2}(3) \equiv 0(\bmod m^2)$ and $h_{K_3(m)+3}(3) \equiv 1+m(\bmod m^t)$.

Therefore, $x_{n+1} = a,\ x_{n+2} = b,\ x_{n+3} = ba^{p+1},$ i.e. $L_3(G_1; X) \mid K_3(p^2)$.
Let $l = L_3(G_1, X)$ then we get

$$\begin{cases} h_{l+1}(3) \equiv 1 & (\bmod m^2), \\ h_{l+2}(3) \equiv 0 & (\bmod m^2), \\ h_{l+3}(3) \equiv 1+m & (\bmod m^2), \\ F^3_{l+1} \equiv 0 & (\bmod m), \\ F^3_{l+2} \equiv 1 & (\bmod m), \\ F^3_{l+3} \equiv 1 & (\bmod m). \end{cases}$$

Hence, suppose $l = s \times K_3(m)$. By Lemma 1.2 and Table 1, we have

$$\begin{cases} h_{l+1}(3) \equiv 1 & (\bmod m^2), \\ h_{l+2}(3) \equiv 0 & (\bmod m^2), \\ h_{l+3}(3) \equiv 1+m & (\bmod m^2). \end{cases}$$

So, $K_3(m^2)$ is a divisor of $L_3(G_1; X)$. Then we obtain $L_3(G_1; X) = L_3(p^2)$. □

Here, we discuss the period of t-Fibonacci sequence in the group G_1. First, we need the following sequnces:

$h_1(t) = h_1(t-1), h_2(t) = h_2(t-1), \ldots, h_t(t) = h_t(t-1), h_{t+1}(t) = h_{t+1}(t-1) + (1+p^\gamma)^{F^{t-1}_{n+t-3}}$
$h_n(t) = h_{n-1}(t) + h_{n-2}(t) + h_{n-3}(t) + \cdots + h_{n-t}(t) + p^\gamma(h_{n-t}(t)(F^t_{n-2} + F^t_{n-1} + \cdots + F^t_{n+t-4})$
$+ h_{n-t+1}(t)(F^t_{n-1} + F^t_n + \cdots + F^t_{n+t-4}) + \cdots + h_{n-1}(t)(F^t_{n+t-4})$ for $n \geq 4,\ t \geq 3$.

Lemma 2.3. *Every element of $F_t(G_1; X)$ may be represented by $x_n(t) = b^{F^t_{n+t-3}} a^{h_n(t)}$ for $n \geq 4$, $t \geq 3$.*

Proof. We use two dimensional induction method on k and n. Indeed, by Lemma 2.1, we have $x_n(3) = b^{F^3_n} a^{h_n(3)}$ when $x_n(s) = b^{F^s_n} a^{h_n(s)}$ $(4 \leq s \leq t)$, it is sufficient to show that

$$x_n(t+1) = b^{F^{t+1}_{n+t-2}} a^{h_n(t+1)}.$$

For this, we use the induction method on n:
If $3 \leq s \leq t$, from definitions of F^t_n and $h_n(t)$, we get $F^{t+1}_s = F^t_s$ and $h_s(t+1) = h_s(t)$, then $x_s(t+1) = x_s(t)$. By this and the induction hypothesis on t, we obtain

$$x_s(t+1) = b^{F^{t+1}_{s+t-2}} a^{h_s(t+1)}.$$

Now we suppose that the hypothesis of induction holds for all $s \leq n-1$, by definition of $x_n(t+1)$, if $F_n := F^t_n$ and $x_n := x_n(t)$, then

$$
\begin{aligned}
x_n &= x_{n-t}x_{n-t+1}x_{n-t+2}x_{n-t+3}\cdots x_{n-2}(x_{n-1})\\
&= b^{F_{n-3}}a^{h_{n-t}}b^{F_{n-2}}a^{h_{n-t+1}}b^{F_{n-1}}a^{h_{n-t+2}}b^{F_n}a^{h_{n-t+3}}\times\cdots\times b^{F_{n+t-4}}a^{h_{n-1}}\\
&= b^{F_{n-3}+F_{n-2}}a^{h_{n-t}+h_{n-t+1}+p^{\gamma}(F_{n-2})}a^{h_{n-t+1}}b^{F_{n-1}}a^{h_{n-t+2}}b^{F_n}a^{h_{n-t+3}}\times\cdots\times b^{F_{n+t-4}}a^{h_{n-1}}\\
&= \cdots\\
&= b^{F_{n-3}+F_{n-2}+\cdots+F_{n+t-4}}a^{h_{n-1}(t)+h_{n-2}(t)+h_{n-3}(t)+\cdots+h_{n-t}(t)}\\
&\qquad a^{p^{\gamma}(h_{n-t}(t)(F^t_{n-2}+F^t_{n-1}+\cdots+F^t_{n+t-4})+h_{n-t+1}(t)(F^t_{n-1}+F^t_{n}+\cdots+F^t_{n+t-4})+\cdots+h_{n-1}(t)(F^t_{n+t-4})}\\
&= b^{F_{n+t-3}}a^{h_n(t)}.
\end{aligned}
$$

This completes the proof. □

Theorem 2.2. *Let p be a prime and let $t \geq 1$ be a positive integer. Then*

$$K_t(p)|L_t(G_1;X).$$

Proof. Similar to the proof of Lemma 2.2, it can be easily proved. □

In Table 2, we obtain $L_4(G_1;X)$ where $\alpha = 2, \beta = \gamma = 1$ and $p < 50$.

Table 2. The period of the t-Fibonacci sequences of group G_1.

p	$L_3(G_1;X)$	$K_3(p)$	$L_4(G_1;X)$	$K_4(p)$
3	39	13	78	26
5	155	31	1560	312
7	326	48	2394	342
11	1210	110	1320	120
13	2184	168	1092	84
17	1632	96	43504	4912
19	6840	360	130302	6858
23	12719	553	279818	12166
29	8120	280	812	280
31	10261	331	868620	28020
37	17353	496	50616	1368
41	22960	560	9840	240
43	52976	1232	7000400	162800
47	2162	46	4879634	103822

We finish this section with an open question as follows: *Prove or disprove whether for the group G_1 and $\beta = \gamma$ and $\alpha = 2\gamma$, we have*

$$L_t(G_1;X) = K_t(p^2) = pK_t(p).$$

3 The t-Fibonacci length of the group G_2

We consider the group $G2$ as follows:

$$G_2 \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle,$$

where $[a,b]=c, [a,c]=[b,c]=1, |a|=p^{\alpha}, |b|=p^{\beta}, |c|=p^{\gamma}, \alpha,\beta,\gamma\in\mathbb{N}, \alpha\geq\beta\geq\gamma$.

In this section, we find the t-Fibonacci sequence in G_2 with respect to to $X=\{c,a,b\}$. First, we study the 4–Fibonacci sequence of these group.

$$\begin{aligned} g_n(4) &= F^4_{n-1}+F^4_{n-2}+F^4_{n-3},\\ u_n(4) &= u_{n-4}(4)+u_{n-3}(4)+u_{n-2}(4)+u_{n-1}(4)-(F^4_{n-4}\times u_{n-3}(4)\\ &\quad +(F^4_{n-4}+F^4_{n-3})\times u_{n-2}(4)+(F^4_{n-4}+F^4_{n-3}+F^4_{n-2})\times u_{n-1}(4)). \end{aligned}$$

By using the relation $[a,b]=c$ in G_2, we can write $[b^i,a^j]=c^{-ij}$, where $i,j\in\mathbb{N}$. Hence, by the relations of $[a,c]=[b,c]=1$ and Lemma 1.2, for every $x\in G_2$, we have $[x,c]=1$.

Lemma 3.1. *Every element of $F_4(G_2;X)$ may be presented by $x_n=c^{u_n(4)}a^{g_n(4)}b^{F^4_n},\ n\geq 4$.*

Proof. We have $x_4=cab, x_5=c^1a^2b^2$, $x_6=ab(cab)(c^1a^2b^2)=c^{-3}a^4b^4$ and $x_7=c^{-22}a^7b^8$. Then by induction on n, we get

$$\begin{aligned} x_n &= x_{n-4}x_{n-3}x_{n-2}x_{n-1} = c^{u_{n-4}(4)}a^{t_{n-4}(4)}b^{F^4_{n-4}}c^{u_{n-3}(4)}a^{t_{n-3}(4)}b^{F^4_{n-3}}c^{u_{n-2}(4)}a^{t_{n-2}(4)}b^{F^4_{n-2}}\\ &\quad c^{u_{n-1}(4)}a^{t_{n-1}(4)}b^{F^4_{n-1}}\\ &= c^{u_{n-4}(4)+u_{n-3}(4)+u_{n-2}(4)+u_{n-1}(4)}a^{t_{n-4}(4)}b^{F^4_{n-4}}a^{t_{n-3}(4)}b^{F^4_{n-3}}a^{t_{n-2}(4)}b^{F^4_{n-2}}a^{t_{n-1}(4)}b^{F^4_{n-1}}\\ &= c^{u_{n-4}(4)+u_{n-3}(4)+u_{n-2}(4)+u_{n-1}(4)-F^4_{n-4}\times u_{n-3}(4)}a^{t_{n-4}(4)+t_{n-3}(4)}b^{F^4_{n-4}+F^4_{n-3}}a^{t_{n-2}(4)}b^{F^4_{n-2}}a^{t_{n-1}(4)}b^{F^4_{n-1}}\\ &= c^{u_{n-4}(4)+u_{n-3}(4)+u_{n-2}(4)+u_{n-1}(4)-F^4_{n-4}\times u_{n-3}(4)}-(F^4_{n-4}+F^4_{n-3})\times u_{n-3}(4)a^{t_{n-4}(4)+t_{n-3}(4)+t_{n-2}(4)}\\ &\quad b^{F^4_{n-4}+F^4_{n-3}+F^4_{n-2}}a^{t_{n-1}(4)}b^{F^4_{n-1}}\\ &= c^{u_{n-4}(4)+u_{n-3}(4)+u_{n-2}(4)+u_{n-1}(4)-(F^4_{n-4}\times u_{n-3}(4)+(F^4_{n-4}+F^4_{n-3})\times u_{n-2}(4)+(F^4_{n-4}+F^4_{n-3}+F^4_{n-2})\times u_{n-1}(4))}\\ &\quad a^{F^4_{n-1}+F^4_{n-2}+F^4_{n-3}}b^{F^4_{n-4}+F^4_{n-3}+F^4_{n-2}+F^4_{n-1}}\\ &= c^{u_n(4)}a^{t_n(4)}b^{F^4_n}. \end{aligned}$$

Thus the result holds. □

Example 3.1. *For the group G_2 and integers $\alpha=\beta=\gamma=1$, $p=5$, by Lemma 3.1, we have*

$$x_1=c, x_2=a, x_3=b, x_4=cab, x_5=abcab=c^1a^2b^2=c^{g_5}a^{u_5}b^{F_5}, x_6=c^{-3}a^3b^4, x_7=c^3a^2b^3\ldots,$$
$$x_{313}=x_{312+1}\equiv c^1a^0b^0=c, x_{314}=x_{312+2}=a,\ x_{315}=x_{312+3}=b,\ldots.$$

Consequently, $x_{313}=x_{312+1}=x_1, x_{314}=x_{312+2}=x_2, x_{315}=x_{312+3}=x_3$.
Therefore, $L(G_2;X)=K_4(5)$.

In Table 3, using the software Maple 18, we calculate $K^4_n, u_{F^4_n+1}(4),\ u_{F^4_n+2}(4)$ and $u_{F^4_n+3}(4)$ for $n=p<50$.

Table 3. $K_4(n), u_{F_n^4+1}(4), u_{F_n^4+2}(4), u_{F_n^4+3}(4)$ and $u_{F_n^4+4}(4)$ for $n = p < 50$.

$n = p$	K_n^4	$u_{F_n^4+1}(4)(\text{mod } p)$	$u_{F_n^4+2}(4)(\text{mod } p)$	$u_{F_n^4+3}(4)(\text{mod } p)$	$u_{F_n^4+4}(4)(\text{mod } p)$
3	26	$u_{78+1}(4) \equiv 1$	$u_{78+2}(4) \equiv 0$	$u_{78+3}(4) \equiv 0$	$u_{78+4}(4) \equiv 1$
5	312	$u_{312+1}(4) \equiv 1$	$u_{312+2}(4) \equiv 0$	$u_{312+3}(4) \equiv 0$	$u_{312+4}(4) \equiv 1$
7	342	$u_{342+1}(4) \equiv 1$	$u_{342+2}(4) \equiv 0$	$u_{342+3}(4) \equiv 0$	$u_{342+4}(4) \equiv 1$
11	120	$u_{120+1}(4) \equiv 1$	$u_{120+2}(4) \equiv 0$	$u_{120+3}(4) \equiv 0$	$u_{120+4}(4) \equiv 1$
13	84	$u_{84+1}(4) \equiv 1$	$u_{84+2}(4) \equiv 0$	$u_{84+3}(4) \equiv 0$	$u_{84+4}(4) \equiv 1$
17	4912	$u_{4912+1}(4) \equiv 1$	$u_{4912+2}(4) \equiv 0$	$u_{4912+3}(4) \equiv 0$	$u_{4912+4}(4) \equiv 1$
19	6858	$u_{6858+1}(4) \equiv 1$	$u_{6858+2}(4) \equiv 0$	$u_{6858+3}(4) \equiv 0$	$u_{6858+4}(4) \equiv 1$
23	12166	$u_{12166+1}(4) \equiv 1$	$u_{12166+2}(4) \equiv 0$	$u_{12166+3}(4) \equiv 0$	$u_{12166+4}(4) \equiv 1$
29	280	$u_{280+1}(4) \equiv 1$	$u_{280+2}(4) \equiv 0$	$u_{280+3}(4) \equiv 0$	$u_{280+4}(4) \equiv 1$
31	28020	$u_{28020+1}(4) \equiv 1$	$u_{28020+2}(4) \equiv 0$	$u_{28020+3}(4) \equiv 0$	$u_{28020+4}(4) \equiv 1$
37	13688	$u_{1368+1}(4) \equiv 1$	$u_{1368+2}(4) \equiv 0$	$u_{1368+3}(4) \equiv 0$	$u_{1368+4}(4) \equiv 1$
41	240	$u_{240+1}(4) \equiv 1$	$u_{240+2}(4) \equiv 0$	$u_{240+3}(4) \equiv 0$	$u_{240+4}(4) \equiv 1$
43	162800	$u_{162800+1}(4) \equiv 1$	$u_{162800+2}(4) \equiv 0$	$u_{162800+3}(4) \equiv 0$	$u_{162800+4}(4) \equiv 1$
47	103822	$u_{103822+1}(4) \equiv 1$	$u_{103822+2}(4) \equiv 0$	$u_{103822+3}(4) \equiv 0$	$u_{103822+4}(4) \equiv 1$

Lemma 3.2. *If $L_3(G_2; X) = l$, then l is the Least integer such that all of the following equations hold.*

$$\begin{cases} g_{l+1}(4) \equiv 0 & (\text{mod } p^\alpha), \\ g_{1+2}(4) \equiv 1 & (\text{mod } p^\alpha), \\ g_{l+3}(4) \equiv 0 & (\text{mod } p^\alpha), \\ g_{l+4}(4) \equiv 1 & (\text{mod } p^\alpha), \\ u_{l+1}(4) \equiv 1 & (\text{mod } p^\gamma), \\ u_{l+2}(4) \equiv 0 & (\text{mod } p^\gamma), \\ t_{l+3}(4) \equiv 0 & (\text{mod } p^\gamma), \\ t_{l+4}(4) \equiv 1 & (\text{mod } p^\gamma), \\ F_{l+1}^4 \equiv 0 & (\text{mod } p^\beta), \\ F_{l+2}^4 \equiv 0 & (\text{mod } p^\beta), \\ F_{l+3}^4 \equiv 1 & (\text{mod } p^\beta), \\ F_{l+4}^4 \equiv 1 & (\text{mod } p^\beta). \end{cases}$$

Moreover, $K_4(m)$ where $m = p^\beta$ divides $L_4(G_2; X)$.

Proof. By Lemma 3.1, we get $x_n = c^{u_n(4)} a^{g_n(4)} b^{F_n^4}$. Since $x_{l+1} = c,\ x_{l+2} = a,\ x_{l+3} = b,\ x_{l+4} = cab$ and from Lemma 1.1 and 1.2, the results are obtained immediately. □

Theorem 3.1. *For the group G_2, $\alpha = \beta = \gamma = 1$ and $5 \leq p < 50$, we have $L_4(G_2; X) = K_4(p)$.*

Proof. For $5 \leq p < 50$, we have

$$\begin{aligned} x_{K_4(p)+1} &= c^{u_{K_4(p)+1}(4)} a^{g_{K_4(p)+1}(4)} b^{F^4_{K_4(p)+1}} = c, \\ x_{K_4(p)+2} &= c^{u_{K_4(p)+2}(4)} a^{g_{K_4(p)+2}(4)} b^{F^4_{K_4(p)+2}} = a, \\ x_{K_4(p)+3} &= c^{u_{K_4(p)+3}(4)} a^{g_{K_4(p)+3}(4)} b^{F^4_{K_4(p)+3}} = b, \\ x_{K_4(p)+4} &= c^{u_{K_4(p)+4}(4)} a^{g_{K_4(p)+4}(4)} b^{F^4_{K_4(p)+4}} = cab. \end{aligned}$$

By Lemma 1.2 and Table 3, we may write

$$\begin{aligned} &F^4_{K_4(p)+1} \equiv F_1 \equiv 0 \pmod p, && F^4_{K_4(p)+2} \equiv F_2 \equiv 0 \pmod p, \\ &F^4_{K_4(p)+3} \equiv F_3 \equiv 1 \pmod p, && F^4_{K_4(p)+4} \equiv F_4 \equiv 1 \pmod p, \\ &g_{K_4(p)+1}(4) \equiv 0 \pmod p, && g_{K_4(p)+2}(4) \equiv 1 \pmod p, \\ &g_{K_4(p)+3}(4) \equiv 0 \pmod p, && g_{K_4(p)+4}(4) \equiv 1 \pmod p, \\ &u_{K_4(p)+1}(4) \equiv 1 \pmod p, && u_{K_4(p)+2}(4) \equiv 0 \pmod p, \\ &u_{K_4(p)+3}(4) \equiv 0 \pmod p, && u_{K_4(p)+3}(4) \equiv 1 \pmod p. \end{aligned}$$

Therefore, $L_4(G_2; X) \mid K_4(p)$. Using Lemma 3.2, shows that $K_4(p)$ is a divisor of $L_4(G_2; X)$. According to these results, it is seen that

$$L_4(G_2; X) = K_4(p).$$

This completes the proof. □

$$\begin{aligned} &g_n(t) = F^t_{n-3} + F^t_{n-2} + \cdots + F^t_{n+t-5}, \\ &u_1(t) = u_1(t-1), \ldots, u_t(t) = u_t(t-1), u_{t+1}(t) = F^t_{n+t-5} + u_{t+1}(t-1). \\ &u_n(t) = u_{n-t}(t) + u_{n-t+1}(t) + \cdots + u_{n-1}(t) - (F^t_{n-4} \times u_{n-t+1}(t)) + (F^t_{n-4} + F^t_{n-3}) \times u_{n-t+2}(t)) \\ &\qquad + \cdots + (F^t_{n-4} + F^t_{n-3} + \cdots + F^t_{n+t-6}) \times u_{n-1}(t). \end{aligned}$$

Lemma 3.3. *Every element of $F_t(G_2; X)$ may be represented by $x_n(t) = c^{u_n(t)} a^{g_n(t)} b^{F_{n+t-4}(t)}$ for $n, k \geq 4$.*

Proof. We use two dimensional induction method on t and n. Indeed, by Lemma 3.1, we have $x_n(4) = c^{u_n(4)} a^{g_n(4)} b^{F^4_n}$ and if $x_n(s) = c^{u_n(s)} a^{g_n(s)} b^{F^s_n}$ $(5 \leq s \leq t)$, it is sufficient to show that

$$x_n(t+1) = c^{u_n(t+1)} a^{g_n(t+1)} b^{F^{t+1}_n}.$$

For this, we use the induction method on n:
If $3 \leq s \leq t$, from definitions of $F^t_n, g_n(t)$ and $u_n(t)$ we get $F^{t+1}_s = F^t_s, g_n(t+1) = g_n(t)$ and $u_s(t+1) = h_s(t)$, then $x_s(t+1) = x_s(t)$. By this and the induction hypothesis on t, we have

$$x_s(t+1) = c^{u_s(t+1)} a^{g_s(t+1)} b^{F^{t+1}_s}.$$

Now we suppose that the hypothesis of induction holds for all $s \leq n-1$, by definition of $x_n(t+1)$, if $F_n := F_n^t$ and $x_n := x_n(t)$, then,

$$\begin{aligned}
x_n &= x_{n-(t)}x_{n-t+1}x_{n-t+2}x_{n-t+3}\cdots x_{n-2}(x_{n-1})\\
&= c^{u_{n-k}}a^{g_{n-k}}b^{F_{n-4}}c^{u_{n-k+1}}a^{g_{n-k+1}}b^{F_{n-3}} \times \cdots \times c^{u_{n-1}}a^{g_{n-1}}b^{F_{n+k-5}}\\
&= c^{u_{n-k}+u_{n-k+1}}a^{g_{n-k}}b^{F_{n-4}}a^{g_{n-k+1}}b^{F_{n-3}} \times \cdots \times c^{u_{n-1}}a^{g_{n-1}}b^{F_{n+k-5}}\\
&= c^{u_{n-k}+u_{n-k+1}-(F_{n-4}\times g_{n-k+1})}a^{g_{n-k}+g_{n-k+1}}b^{F_{n-4}+F_{n-3}} \times \cdots \times c^{u_{n-1}}a^{g_{n-1}}b^{F_{n+k-5}}\\
&= \cdots\\
&= c^{u_{n-t}(t)+u_{n-t+1}(t)+\cdots+u_{n-1}(t)-(F_{n-4}\times u_{n-t+1}(t))+(F_{n-4}+F_{n-3})\times u_{n-t+2}(t))+\cdots+(F_{n-4}+F_{n-3}+\cdots+F_{n+t-6})\times u_{n-1}(t)}\\
&\quad a^{F_{n-3}+F_{n-2}+\cdots+F_{n+t-5}}b^{F_{n-4}+F_{n-5}+\cdots+F_{n+k-5}}\\
&= c^{u_n(t)}a^{g_n(t)}b^{F_{n+t-4}}.
\end{aligned}$$

This completes the proof. □

In Table 4, we obtain $L_4(G_2;X)$ where $\alpha = 2, \beta = \gamma = 1$ and $p < 50$.

Table 4. The period of t-Fibonacci sequence of the group G_2.

p	$L_4(G_2;X)$	$K_4(p)$	p	$L_4(G_2;X)$	$K_4(p)$
3	78	26	23	12166	12166
5	312	312	29	280	280
7	342	342	31	28020	28020
11	120	120	37	1368	1368
13	84	84	41	240	240
17	4912	4912	43	162800	162800
19	6858	6858	47	103822	103822

We finish this section with an open question as follows: *For the group G_2 and $\alpha = \beta = \gamma = 1$, we have*
(i) *If $p \mid t-1$, we have $L_t(G_2;X) = pK_t(p)$.*
(ii) *Otherwise, we have $L_t(G_2;X) = K_t(p)$.*

4 The t-Fibonacci length of the group G_3

Now, we consider $G_3 \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$ where $[a,b] = a^{p^{\alpha-\gamma}}c, [c,b] = a^{-p^{2(\alpha-\gamma)}}c^{-p^{\alpha-\gamma}}$, $|a| = p^{\alpha}, |b| = p^{\beta}, |c| = p^{\sigma}, |[a,b]| = p^{\gamma}, \alpha,\beta,\gamma,\sigma \in \mathbb{N}, \alpha \geq \beta \geq \sigma \geq 1, \alpha+\sigma \geq 2\gamma$. We define the sequences $\{e_n\}_1^{\infty}$ and $\{g_n\}_1^{\infty}$ of integers as follows

$$e_1(4)=1,\ e_2(4)=e_3(4)=0,\ e_4(4)=1, g_1(4)=0,\ g_2(4)=1,\ g_3(4)=0, g_4(4)=1,$$
$$\begin{aligned}e_n(4) &= e_{n-1}(4)+e_{n-2}(4)+e_{n-3}(4)+e_{n-4}(4)+p^{\alpha-\gamma}(F^4_{n-4}e_{n-3}(4)+e_{n-2}(F^4_{n-4}+F^4_{n-3})\\ &\quad +e_{n-1}(F^4_{n-4}+F^4_{n-3}+F^4_{n-2}))-(g_{n-3}(4)F^4_{n-4}\\ &\quad +g_{n-2}(F^4_{n-3}+F^4_{n-4})+g_{n-1}(F^4_{n-4}+F^4_{n-3}+F^4_{n-2}))\ \ n\geq 5,\\ g_n(4) &= g_{n-1}(4)+g_{n-2}(4)+g_{n-3}(4)+g_{n-4}(4)+p^{2(\alpha-\gamma)}(F^4_{n-4}e_{n-3}(4)+e_{n-2}(4)(F^4_{n-4}\\ &\quad +F^4_{n-3}+e_{n-1}(4)(F^4_{n-4}+F^4_{n-3}+F^4_{n-2}))-p^{\alpha-\gamma}(g_{n-3}(4)F^4_{n-4}\\ &\quad +g_{n-2}(F^4_{n-3}+F^4_{n-4})+g_{n-1}(F^4_{n-4}+F^4_{n-3}+F^4_{n-2})),\ \ n\geq 5.\end{aligned}$$

We are in a position to find a standard form of the 4-Fibonacci sequence $x_4, x_5, \ldots$ of $G_3,\ n\geq 5$. First, by the relations of G_3, we obtain $ba=a[b,a]b=c^{-1}a^{1-p^{\alpha-\gamma}}b$ and $bc=c[b,c]b=c^{p^{(\alpha-\gamma)}}a^{p^{2(\alpha-\gamma)}}b$.

Lemma 4.1. *Every element of $F_4(G_3;X)$ may be presented by $x_n(4)=c^{e_n(4)}a^{g_n(4)}b^{F^4_n},\ n\geq 4$.*

Proof. We have $x_4(4)=cab, x_5(5)=cabcab=c^{1+p^{\alpha-\gamma}}a^{2+p^{2(\alpha-\gamma)}-p^{\alpha-\gamma}+2}b^2$,
$x_6(4)=c^{-4+5p^{\alpha-\gamma}+p^{2(\alpha-\gamma)}}a^{2-2p^{3(\alpha-\gamma)}+5p^{2(\alpha-\gamma)}-3p^{\alpha-\gamma}+2}b^4$ and
$x_7(4)=c^{-14-4p^{3(\alpha-\gamma)}+p^{2(\alpha-\gamma)}+7p^{\alpha-\gamma}}a^{4+12p^{3(\alpha-\gamma)}-2p^{3(\alpha-\gamma)}+7p^{2(\alpha-\gamma)}-17p^{\alpha-\gamma}}b^7$. Then by induction method on n, we get

$$\begin{aligned}x_n(4) &= x_{n-4}(4)x_{n-3}(4)x_{n-2}(4)x_{n-1}(4)\\ &= c^{e_{n-4}(4)}a^{g_{n-4}(4)}b^{F^4_{n-4}}c^{e_{n-3}(4)}a^{g_{n-3}(4)}b^{F^4_{n-3}}c^{e_{n-2}(4)}a^{g_{n-2}(4)}b^{F^4_{n-2}}c^{e_{n-1}(4)}a^{g_{n-1}(4)}b^{F^4_{n-1}}\\ &= c^{e_{n-4}(4)}a^{g_{n-4}(4)}c^{e_{n-3}(4)}[b,c]^{F^4_{n-4}e_{n-3}}b^{F^4_{n-3}}a^{g_{n-3}(4)}b^{F^4_{n-3}}c^{e_{n-2}(4)}a^{g_{n-2}(4)}b^{F^4_{n-2}}c^{e_{n-1}(4)}a^{g_{n-1}(4)}b^{F^4_{n-1}}\\ &= c^{e_{n-4}(4)+e_{n-3}(4)+p^{\alpha-\gamma}(F^4_{n-4}e_{n-3}(4))}a^{g_{n-4}(4)+g_{n-3}(4)+p^{2(\alpha-\gamma)}(F^4_{n-4}e_{n-3}(4))}\\ &\quad b^{F^4_{n-3}}a^{g_{n-3}(4)}b^{F^4_{n-3}}c^{e_{n-2}(4)}a^{g_{n-2}(4)}b^{F^4_{n-2}}c^{e_{n-1}(4)}a^{g_{n-1}(4)}b^{F^4_{n-1}}\\ &= c^{e_{n-4}(4)+e_{n-3}(4)+p^{\alpha-\gamma}(F^4_{n-4}e_{n-3}(4))-F^4_{n-4}e_{n-3}(4)}a^{g_{n-4}(4)+g_{n-3}(4)+p^{2(\alpha-\gamma)}(F^4_{n-4}e_{n-3}(4))-p^{\alpha-\gamma}(F^4_{n-4}e_{n-3}(4))}\\ &\quad b^{F^4_{n-4}+F^4_{n-3}}c^{e_{n-2}(4)}a^{g_{n-2}(4)}b^{F^4_{n-2}}c^{e_{n-1}(4)}a^{g_{n-1}(4)}b^{F^4_{n-1}}\\ &= c^{e_{n-1}(4)+e_{n-2}(4)+e_{n-3}(4)+e_{n-4}(4)+p^{\alpha-\gamma}(F^4_{n-4}e_{n-3}(4)+e_{n-2}(F^4_{n-4}+F^4_{n-3})+e_{n-1}(F^4_{n-4}+F^4_{n-3}+F^4_{n-2}))}\\ &\quad c^{-(g_{n-3}(4)F^4_{n-4}+g_{n-2}(F^4_{n-3}+F^4_{n-4})+g_{n-1}(F^4_{n-4}+F^4_{n-3}+F^4_{n-2}))}\\ &\quad a^{g_{n-1}(4)+g_{n-2}(4)+g_{n-3}(4)+g_{n-4}(4)+p^{2(\alpha-\gamma)}(F^4_{n-4}e_{n-3}(4)+e_{n-2}(4)(F^4_{n-4}+F^4_{n-3}+e_{n-1}(4)(F^4_{n-4}+F^4_{n-3}+F^4_{n-2}))}\\ &\quad a^{-p^{\alpha-\gamma}(g_{n-3}(4)f^4_{n-4}+g_{n-2}(F^4_{n-3}+F^4_{n-4})+g_{n-1}(f^4_{n-4}+F^4_{n-3}+F^4_{n-2}))}b^{F^4_{n-4}+F^4_{n-3}+F^4_{n-2}+F^4_{n-1}}\\ &= c^{e_n(4)}a^{g_n(4)}b^{F^4_n}.\end{aligned}$$

Then, the result is as follows. □

Example 4.1. *For integers $\beta=\gamma=2, \sigma=1, \alpha=3$, $p=3$, by Lemma 4.1 and the relations of G_3, we have:*
$x_1=c,\ x_2=a,\ x_3=b,\ x_4=cab,\ x_5=abcab=c^4a^8b^2=c^{e_5(4)}a^{g_5(4)}b^{F^4_5},\ x_6=c^6a^{13}b^1,\ x_7=c^5a^4b^2,\ x_8=c^4a^{17}b^0,\ x_9=c^7a^6b^2,\ x_{10}=c^6a^{19}b^2,\ \ldots,\ x_{235}=c,\ x_{236}=a,\ x_{237}=b,\ x_{238}=cab\ldots$.

Consequently, $x_{235} = x_{234+1} = x_1, x_{236} = x_{234+2} = x_2, x_{237} = x_{234+3} = x_3, x_{238} = x_{234+4} = x_4$. *Therefore,*

$$L_4(G_3; X) = K_4(3^3).$$

In Table 5, we obtain $L_4(G_3; X)$ where $\alpha = 3, \beta = \gamma = 2, \sigma = 1$ and $p < 50$.

Table 5. The period of 4-Fibonacci sequence of the group G_3.

p	$L_4(G_3;X)$	$K_4(p^3)$	p	$L_4(G_3;X)$	$K_4(p^3)$
3	234	234	23	6435814	6435814
5	7800	7800	29	23548	23548
7	16758	16458	31	26927220	26927220
11	14520	14520	37	1872792	1872792
13	14196	14196	41	403440	403440
17	1419568	1419568	43	301017200	301017200
19	2475738	2475738	47	4879634	4879634

$$\begin{aligned}
e_n(t) &= e_{n-t}(t) + e_{n-t+1}(t) + \cdots + e_{n-1}(t) + p^{\alpha-\gamma}(F^t_{n-t}e_{n-t+1}(t) + (F^t_{n-t} + F^t_{n-t+1})e_{n-t+2} \\
&\quad + \cdots + (F^t_{n-t} + F^t_{n-t+1} + \cdots + F^t_{n+t-3})e_{n-1}) - (F^t_{n-t}g_{n-t+1}(t) + (F^t_{n-t} + F^t_{n-t+1})g_{n-t+2} \\
&\quad + \cdots + (F^t_{n-t} + F^t_{n-t+1} + \cdots + F^t_{n+t-3})g_{n-1}) \\
g_n(t) &= g_{n-t}(t) + g_{n-t+1}(t) + \cdots + g_{n-1}(t) + p^{2(\alpha-\gamma)}(F^t_{n-t}e_{n-t+1}(t) + (F^t_{n-t} + F^t_{n-t+1})e_{n-t+2} \\
&\quad + \cdots + (F^t_{n-t} + F^t_{n-t+1} + \cdots + F^t_{n+t-3})e_{n-1}) - p^{\alpha-\gamma}(F^t_{n-t}g_{n-t+1}(t) + (F^t_{n-t} + F^t_{n-t+1})g_{n-t+2} \\
&\quad + \cdots + (F^t_{n-t} + F^t_{n-t+1} + \cdots + F^t_{n+t-3})g_{n-1})
\end{aligned}$$

Lemma 4.2. *Every element of* $F_t(G_3; X)$ *may be represented by* $x_n(t) = c^{e_n(t)}a^{g_n(t)}b^{F^t_{n+t-4}}$, $n,\ t \geq 4$.

Proof. We use two dimensional induction method on t and n. Indeed, by Lemma 4.1, we have $x_n(4) = c^{e_n(4)}a^{g_n(4)}b^{F^4_{n+t-4}}$ and if $x_n(s) = c^{e_n(s)}a^{g_n(s)}b^{F^s_{n+t-4}}$ $(5 \leq s \leq t)$, it is sufficient to show that

$$x_n(t+1) = c^{e_n(t+1)}a^{g_n(t+1)}b^{F^{t+1}_{n+t-3}}.$$

For this, we use an induction mehod on n:
If $4 \leq s \leq t$, from definitions of $F^t_n, e_n(t)$ and $h_n(t)$ we get $F^{t+1}_s = F^t_s, e_s(t+1) = e_s(t)$ and $h_s(t+1) = h_s(t)$, then $x_s(t+1) = x_s(t)$. By this and the induction hypothesis on t, we have

$$x_s(t+1) = c^{e_s(t+1)}a^{g_s(t+1)}b^{F^{t+1}_{s+t-3}}.$$

Now we suppose that the hypothesis of induction holds for all $s \leq n-1$, by definition of $x_n(t+1)$, if $F_n := F^t_n$ and $x_n := x_n(t)$, then

$$
\begin{aligned}
x_n &= x_{n-t}x_{n-t+1}x_{n-t+2}x_{n-t+3}\cdots x_{n-2}(x_{n-1})\\
&= c^{e_{n-t}}a^{g_{n-t}}b^{F_{n-4}}c^{e_{n-t+1}}a^{g_{n-t+1}}b^{F_{n-3}}c^{e_{n-t+2}}a^{g_{n-t+2}}b^{F_{n-2}}\cdots c^{e_{n-1}}a^{g_{n-1}}b^{F_{n+t-5}}\\
&= c^{e_{n-t}}a^{g_{n-t}}c^{e_{n-t+1}}[b,c]^{F^4_{n-4}e_{n-t+1}}b^{F_{n-3}}c^{e_{n-t+2}}a^{g_{n-t+2}}b^{F_{n-2}}\cdots c^{e_{n-1}}a^{g_{n-1}}b^{F_{n+t-5}}\\
&= c^{e_{n-t}+e_{n-t+1}+p^{\alpha-\gamma}(F_{n-4}e_{n-t+1})}a^{g_{n-t}+g_{n-t+1}+p^{2(\alpha-\gamma)}(F_{n-4}e_{n-t+1})}b^{F_{n-3}}c^{e_{n-t+2}}a^{g_{n-t+2}}b^{F_{n-2}}\\
&\quad\cdots c^{e_{n-1}}a^{g_{n-1}}b^{F_{n+t-5}}\\
&= c^{e_{n-t}+e_{n-t+1}+p^{\alpha-\gamma}(F_{n-4}e_{n-t+1})-F_{n-4}e_{n-t+1}}a^{g_{n-t}+g_{n-t+1}+p^{2(\alpha-\gamma)}(F_{n-4}e_{n-t+1})-p^{\alpha-\gamma}(F_{n-4}e_{n-t+1})}b^{F_{n-3}}\\
&\quad c^{e_{n-t+2}}a^{g_{n-t+2}}b^{F_{n-2}}\cdots c^{e_{n-1}}a^{g_{n-1}}b^{F_{n+t-5}}\\
&= \cdots\\
&= c^{t_1}a^{t_2}b^{F_{n-t}+F_{n-t+1}+\cdots+F_{n+t-5}} = c^{e_n(t)}a^{g_n(t)}b^{F^t_{n+t-4}}.
\end{aligned}
$$

where,

$$
\begin{aligned}
t_1 &= e_{n-t}+e_{n-t+1}+\cdots+e_{n-1}+p^{\alpha-\gamma}(F_{n-t}e_{n-t+1}+(F_{n-t}+F_{n-t+1})e_{n-t+2}\\
&\quad+\cdots+(F_{n-t}+F_{n-t+1}+\cdots+F_{n+t-3})e_{n-1})-(F_{n-t}g_{n-t+1}+(F_{n-t}+F_{n-t+1})g_{n-t+2}+\cdots+\\
&\quad(F_{n-t}+F_{n-t+1}+\cdots+F_{n+t-3})g_{n-1},\\
t_2 &= g_{n-t}+g_{n-t+1}+\cdots+g_{n-1}+p^{2(\alpha-\gamma)}\\
&\quad(F_{n-t}e_{n-t+1}+(F_{n-t}+F_{n-t+1})e_{n-t+2}+\cdots+(F_{n-t}+F_{n-t+1}+\cdots+F_{n+t-3})e_{n-1})\\
&\quad-p^{\alpha-\gamma}(F_{n-t}g_{n-t+1}+(F_{n-t}+F_{n-t+1})g_{n-t+2}+\cdots+(F_{n-t}+F_{n-t+1}+\cdots+F_{n+t-3})g_{n-1}),
\end{aligned}
$$

the result is now immediate. □

Lemma 4.3. *For every $m = p^{\beta}$, we have $K_4(m)|L_4(G_3; X)$.*

Proof. The proof is similar to that of Lemma 3.2. □

We end this section by an open question as follows: *Prove or disprove whether for $\beta = \gamma = \sigma + 1$ and $\alpha \geq \sigma + 2$, we have*

$$L_t(G_3; X) = K_t(p^3) = p^2K_t(p).$$

5 Conclusion

Here, we study the lengths of the periods of the t-Fibonacci sequences in the 2-generator p-groups of nilpotency class 2. We show that the minimal lenght of the period of t-Fibonacci divide the lengths of the periods of the t-Fibonacci sequences in the 2-generator p-groups of nilpotency class 2.

References

[1] Akuzum, Y., & Deveci, Ö. (2020). The Hadamard-type k-step Fibonacci sequences in groups. *Communications in Algebra*, 48(7), 2844–2856.

[2] Atanassov, K., Dimitrov, D., & Shannon, A. (2009). A remark on ψ-function and Fibonacci sequence. *Notes on Number Theory and Discrete Mathematics*, 15(1), 1–11.

[3] Bacon, M. R., & Kappe, L. C. (1993). The nonabelian tensor square of a 2-generator p-group of class 2. *Archiv der Mathematik*, 61(6), 508–516.

[4] Bradie, B. (2010). Extension and refinements of some properties of sums involing Pell number. *Missouri Journal of Mathematical Sciences*, 22(1), 37–43.

[5] Bravo, J., & Luca, F. (2012). Powers of two in generalized Fibonacci sequences. *Revista Colombiana de Matematicas*, 46(1), 67–79.

[6] Bugeaud, Y., Luca, F., Mignotte, M., & Siksek, S. (2008). Fibonacci numbers at most one away from a perfect power. *Elemente der Mathematik*, 63(2), 65–75.

[7] Campbell, C. M., & Campbell, P. (2005). The Fibonacci length of certain centro-polyhedral groups. *Journal of Applied Mathematics and Computing*, 19, 231–240.

[8] Campbell, C., Cample, P., Doostie, H., & Robertson, E. (2004). Fibonacci length for metacyclic groups. *Algebra Colloquium*, 11(2), 215–229.

[9] Deveci, Ö. (2013). The $k-$nacci sequences and the generalized order k-Pell sequences in the semi-direct product of finite cyclice groups. *Chiang Mai Journal of Science*, 40(1), 89–98.

[10] Deveci, Ö., & Karaduman, E., (2015). The Pell sequences in finite groups. *Utilitas Mathematica*, 96, 263–279.

[11] Deveci, Ö., & Shannon, A. (2018). The quaternion-Pell sequence. *Communications in Algebra*, 49(2), 1–7.

[12] Deveci, Ö., Tas, S., & Kilicman, A. (2017). On the $2k$-step Jordan–Fibonacci sequence. *Advances in Difference Equations*, 121.

[13] Doostie, H., & Hashemi, M. (2006). Fibonacci lengths involving the Wall number $K(n)$. *Journal of Applied Mathematics*, 20, 171–180.

[14] Hashemi, M., & Pirzadeh, M. (2018). t-nacci sequences in some special groups of order m^3. *Mahematical Reports*, 20(4), 389–399.

[15] Hashemi, M., & Mehraban, E. (2018). On the generalized order 2-Pell sequence of some classes of groups. *Communications in Algebra*, 46(9), 4104–4119.

[16] Kilic, E., & Tasci, D. (2006). The generalized Binet formula, representation and sum of the generalized order k-Pell numbers. *Taiwanese Journal of Mathematics*, 10(6), 1661–1670.

[17] Knox, S. (1992). Fibonacci sequences in finite groups. *The Fibonacci Quarterly*, 30(2), 116–120.

[18] Mehraban, E., & Hashemi, M. (2023). Fibonacci length and the generalized order k-Pell sequences of the 2-generator p-groups of nilpotency class 2. *Journal of Algebra and Its Applications*, 22(3), Article ID 2350061.

[19] Soykan, Y. (2019). On a generalized Pentanacci sequence. *Asian Reserach Journal of Mathematics*, 14(3), 1–9.

[20] Wall, D. D. (1960). Fibonacci series modulo m. *American Mathematical Monthly*, 67(6), 525–532.