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# Nontrivial lower bounds for the $p$-adic valuations of some type of rational numbers and an application for establishing the integrality of some rational sequences

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**Abstract:** In this note, based on a certain functional equation of the dilogarithm function, we establish nontrivial lower bounds for the $p$-adic valuation (where $p$ is a given prime number) of some type of rational numbers involving harmonic numbers. Then we use our estimate to derive the integrality of some sequences of rational numbers, which cannot be seen directly from their definitions.
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## 1 Introduction and notation

Throughout this paper, we let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the set of nonnegative integers. For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ denote the integer part of $x$. For a given prime number $p$ and a given nonzero rational number $r$, we let $\vartheta_p(r)$ denote the usual $p$-adic valuation of $r$; if in addition $r$ is positive then we let $\log_p(r)$ denote its logarithm to the base $p$ (i.e., $\log_p(r) := \frac{\log r}{\log p}$). For a given prime number $p$ and a given positive integer $n$, we let $s_p(n)$ denote the sum of base-$p$ digits of $n$. Next, the least common multiple of given positive integers $u_1, u_2, \dots, u_n$ ($n \in \mathbb{N}$) is denoted by $\mathrm{lcm}(u_1, u_2, \dots, u_n)$. At some places of this paper, we need to use the well-known formulas:

$$\vartheta_p\left(\mathrm{lcm}(1, 2, \dots, n)\right) = \left\lfloor \log_p(n) \right\rfloor, \tag{1.1}$$

$$\vartheta_p(n!) = \frac{n - s_p(n)}{p - 1} \tag{1.2}$$

(which are valid for any prime $p$ and any positive integer $n$). Note that only the second one is nontrivial; it is known as *the Legendre formula*, a proof of which can be found in [6, Theorem 2.6.4, page 77]). Furthermore, we let $(H_n)_{n \in \mathbb{N}_0}$ denote the sequence of harmonic numbers, defined by $H_0 = 0$ and $H_n := \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ ($\forall n \in \mathbb{N}$). We let also $\mathrm{Li}_2$ denote the dilogarithm function, defined by:

$$\mathrm{Li}_2(X) := \sum_{n=1}^{+\infty} \frac{X^n}{n^2} \qquad (\forall X \in \mathbb{C}, |X| \leq 1).$$

It is known (see e.g., [5]) that $\mathrm{Li}_2$ satisfies the functional equation:

$$\mathrm{Li}_2(X) + \mathrm{Li}_2\left(\frac{X}{X-1}\right) = -\frac{1}{2}\log^2(1 - X) \tag{1.3}$$

(for $X$ in the neighborhood of $0$). If $f$ is an analytic function at $0$ (or simply a formal power series), we define the *order* of $f$, which we denote by $\mathrm{Ord}(f)$, the multiplicity of $0$ in $f$. Actually, the map $f \mapsto \mathrm{Ord}(f)$ constitutes a discrete valuation on the ring $\mathbb{C}[[X]]$ of formal power series (see e.g., [7, Chap VII]); namely, it satisfies

$$\begin{array}{r} \mathrm{Ord}(f + g) \geq \min\left(\mathrm{Ord}(f), \mathrm{Ord}(g)\right), \\ \mathrm{Ord}(fg) = \mathrm{Ord}(f) + \mathrm{Ord}(g) \end{array} \qquad (\forall f, g \in \mathbb{C}[[X]]).$$

Besides, if $f, g \in \mathbb{C}[[X]]$ with $g(0) = 0$, then it is easily checked that $f \circ g \in \mathbb{C}[[X]]$ and that

$$\mathrm{Ord}(f \circ g) = \mathrm{Ord}(f) \cdot \mathrm{Ord}(g).$$

Using this concept of order, the $n$-th degree Taylor polynomial of $f \in \mathbb{C}[[X]]$ (where $n \in \mathbb{N}_0$) is the unique complex polynomial $f_n$ of degree $\leq n$, satisfying $\mathrm{Ord}(f - f_n) > n$. Consequently, for $n \in \mathbb{N}_0$, two analytic functions $f$ and $g$ at zero have the same $n$-th degree Taylor polynomial if and only if $\mathrm{Ord}(f - g) > n$.

Very recently, the author [4] has obtained, by different methods, a lower bound for the $p$-adic valuations of the rational numbers of the form $\sum_{k=1}^{n}\left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right)\frac{p^k}{k}$ ($n \in \mathbb{N}$, $p$ a prime, $a \in \mathbb{Z}$ with $a \not\equiv 0 \pmod p$) (generalizing the earlier results of [1–3] which uniquely concern the case $p = 2$). One of these methods exploits the functional equation (analogous to (1.3)): $\mathrm{Li}_1(X) + \mathrm{Li}_1\left(\frac{X}{X-1}\right) = 0$ (where $\mathrm{Li}_1(X) := \sum_{n=1}^{+\infty}\frac{X^n}{n} = -\log(1 - X)$, for all $X \in \mathbb{C}$ with $|X| < 1$).

Following the same method, that we adapt to the function $\mathrm{Li}_2$ and its functional equation (1.3), we will establish nontrivial lower bounds for the $p$-adic valuations of the rational numbers of the form

$$\sum_{k=1}^{n}\left(\frac{1}{a^k}+\frac{1}{(p-a)^k}+\frac{kH_{k-1}}{a^k}\right)\frac{p^k}{k^2}$$

($n \in \mathbb{N}$, $p$ a prime, $a \in \mathbb{Z}$ with $a \not\equiv 0 \pmod p$). Then by specializing $(p, a)$ to $(2, 1)$, we derive the integrality of the sequence of general term

$$\frac{n!^2}{4^n}\sum_{k=1}^{n}\left(2+kH_{k-1}\right)\frac{2^k}{k^2} \qquad (n \in \mathbb{N}\setminus\{3,5,7\})$$

(a fact which cannot be seen directly from the last expression). We conclude the note by deducing the integrality of another sequence of rational numbers (related to the preceding) and by some general remarks.

# 2 The results and the proofs

Our main result is the following:

**Theorem 2.1.** *Let $p$ be a prime number and $a$ be an integer not multiple of $p$. Then we have for all positive integer $n$:*

$$\vartheta_p\left(\sum_{k=1}^{n}\left(\frac{1}{a^k}+\frac{1}{(p-a)^k}+\frac{kH_{k-1}}{a^k}\right)\frac{p^k}{k^2}\right) \geq n+1-2\left\lfloor \log_p(n)\right\rfloor .$$

To prove Theorem 2.1, we need the following lemma:

**Lemma 2.2.** *Let $n \in \mathbb{N}_0$ and $f$ and $g$ be two analytic functions at $0$ with $g(0) = 0$. Let also $f_n$ denote the $n$-th degree Taylor polynomial of $f$. Then the $n$-th degree Taylor polynomial of $(f \circ g)$ is the same with the $n$-th degree Taylor polynomial of $(f_n \circ g)$.*

*Proof.* We have to show that $\mathrm{Ord}(f \circ g - f_n \circ g) > n$. We have

$$\mathrm{Ord}\left(f \circ g - f_n \circ g\right) = \mathrm{Ord}\left((f - f_n) \circ g\right) = \mathrm{Ord}(f - f_n) \cdot \mathrm{Ord}(g) > n$$

(since $\mathrm{Ord}(f - f_n) > n$ and $\mathrm{Ord}(g) \geq 1$). The lemma is proved. □

*Proof of Theorem 2.1.* By substituting into Equation (1.3) $X$ by $\frac{X}{a}$, we get

$$\mathrm{Li}_2\left(\frac{X}{a}\right)+\mathrm{Li}_2\left(\frac{X}{X-a}\right)+\frac{1}{2}\log^2\left(1-\frac{X}{a}\right)=0. \tag{2.1}$$

Now, let $n \in \mathbb{N}$. Since the $n$-th degree Taylor polynomials of the two functions $t \mapsto \mathrm{Li}_2(t)$ and $t \mapsto \frac{1}{2}\log^2(1-t)$ at $0$ are, respectively, $\sum_{k=1}^{n}\frac{t^k}{k^2}$ and $\sum_{k=1}^{n}\frac{H_{k-1}}{k}t^k$ and since the functions $t \mapsto \frac{t}{a}$ and $t \mapsto \frac{t}{t-a}$ both vanish at $0$, then (according to Lemma 2.2) the $n$-th degree Taylor polynomial of the function $X \stackrel{g}{\longmapsto} \mathrm{Li}_2\left(\frac{X}{a}\right) + \mathrm{Li}_2\left(\frac{X}{X-a}\right) + \frac{1}{2}\log^2\left(1 - \frac{X}{a}\right)$ is the same as the $n$-th degree Taylor polynomial of the rational function

$$R_n(X) := \sum_{k=1}^{n} \frac{\left(\frac{X}{a}\right)^k}{k^2} + \sum_{k=1}^{n} \frac{\left(\frac{X}{X-a}\right)^k}{k^2} + \sum_{k=1}^{n} \frac{H_{k-1}}{k}\left(\frac{X}{a}\right)^k.$$

But on the other hand, in view of (2.1), the $n$-th degree Taylor polynomial of $g$ at $0$ is zero. Comparing these two results, we deduce that the multiplicity of $0$ in $R_n$ is at least $(n+1)$. Consequently, $R_n(X)$ can be written as:

$$R_n(X) = X^{n+1} \cdot \frac{U_n(X)}{a^n(X-a)^n \mathrm{lcm}(1,2,\ldots,n)^2},$$

where $U_n \in \mathbb{Z}[X]$. In particular, we have

$$R_n(p) = p^{n+1} \cdot \frac{U_n(p)}{a^n(p-a)^n \mathrm{lcm}(1,2,\ldots,n)^2}.$$

Next, because $U_n(p) \in \mathbb{Z}$ (since $U_n \in \mathbb{Z}[X]$) and $a$ is not a multiple of $p$, then by taking the $p$-adic valuations in the two sides of the last identity, we derive that:

$$\begin{aligned} \vartheta_p\left(R_n(p)\right) &\geq n + 1 - 2\vartheta_p\left(\mathrm{lcm}(1,2,\ldots,n)\right) \\ &= n + 1 - 2\left\lfloor \log_p(n) \right\rfloor, \end{aligned}$$

as required. This achieves the proof. □

By taking $(p, a) = (2, 1)$ in Theorem 2.1, we derive the following important corollary from which we will deduce the integrality of a certain rational sequence.

**Corollary 2.3.** *For all positive integer $n$, we have*

$$\vartheta_2\left(\sum_{k=1}^{n} (2 + kH_{k-1})\frac{2^k}{k^2}\right) \geq n + 1 - 2\left\lfloor \log_2(n) \right\rfloor. \qquad \square$$

As an application of Corollary 2.3, we obtain the integrality of a particular rational sequence, which cannot be seen directly from its original expression.

**Theorem 2.4.** *For every $n \in \mathbb{N} \setminus \{3, 5, 7\}$, the rational number*

$$\frac{n!^2}{4^n} \sum_{k=1}^{n} (2 + kH_{k-1})\frac{2^k}{k^2}$$

*is in fact a positive integer.*

The proof of Theorem 2.4 needs the following lemma:

**Lemma 2.5.** *For all integer $n \geq 8$, we have*

$$s_2(n) + \lfloor \log_2(n) \rfloor \leq \frac{n+1}{2}.$$

*Proof.* Let $n \geq 8$ be an integer and let $\overline{1a_{k-1}\dots a_1a_0}_{(2)} = a_0 + 2a_1 + 2^2a_2 + \dots + 2^{k-1}a_{k-1} + 2^k$ be its representation in the binary system (with $k \in \mathbb{N}$, $k \geq 3$, and $a_0, a_1, \dots, a_{k-1} \in \{0, 1\}$). Then, we have

$$\begin{aligned} s_2(n) + \lfloor \log_2(n) \rfloor &= a_0 + a_1 + \dots + a_{k-1} + 1 + k \\ &= \frac{1}{2}\left(2a_0 + 2a_1 + \dots + 2a_{k-1} + 2(k+1)\right) \\ &\leq \frac{1}{2}\left(1 + a_0 + 2a_1 + 2^2a_2 + \dots + 2^{k-1}a_{k-1} + 2^k\right) \\ &\qquad (\textit{since } a_0 \leq 1 \textit{ and } 2(k+1) \leq 2^k) \\ &= \frac{1}{2}(n+1), \end{aligned}$$

as required. □

*Proof of Theorem 2.4.* For $n \in \{1, 2, 4, 6\}$, we verify the required result by hand. Take for the sequel $n \geq 8$. Since we have obviously

$$n!^2 \sum_{k=1}^{n} (2 + kH_{k-1}) \frac{2^k}{k^2} \in \mathbb{N},$$

then we have just to show that:

$$\vartheta_2\left(\frac{n!^2}{4^n} \sum_{k=1}^{n} (2 + kH_{k-1}) \frac{2^k}{k^2}\right) \geq 0.$$

By using Legendre's formula (1.2) for $p = 2$ together with Corollary 2.3, we have that:

$$\begin{aligned} \vartheta_2\left(\frac{n!^2}{4^n} \sum_{k=1}^{n} (2 + kH_{k-1}) \frac{2^k}{k^2}\right) &= 2\left(n - s_2(n)\right) - 2n + \vartheta_2\left(\sum_{k=1}^{n} (2 + kH_{k-1}) \frac{2^k}{k^2}\right) \\ &\geq 2\left(n - s_2(n)\right) - 2n + n + 1 - 2\lfloor \log_2(n) \rfloor \\ &= n + 1 - 2\left(s_2(n) + \lfloor \log_2(n) \rfloor\right) \\ &\geq 0 \qquad (\textit{according to Lemma 2.5}), \end{aligned}$$

as required. This completes the proof of Theorem 2.4. □

From Theorem 2.4, we derive the following corollary:

**Corollary 2.6.** *For every $n \in \mathbb{N} \setminus \{3, 5, 7\}$, the rational number*

$$\frac{(2n)!^2}{4^n} \sum_{k=1}^{n} \frac{2 + (n+k)H_{n+k-1}}{(n+k)^2 2^{n-k}}$$

*is in fact a positive integer.*

*Proof.* Let $n \in \mathbb{N} \setminus \{3,5,7\}$ and set

$$u_n := \frac{n!^2}{4^n} \sum_{k=1}^{n} \left(2 + kH_{k-1}\right) \frac{2^k}{k^2}.$$

Then we have

$$\begin{aligned}
\frac{(2n)!^2}{4^n} \sum_{k=1}^{n} \frac{2 + (n+k)H_{n+k-1}}{(n+k)^2 2^{n-k}} &= \frac{(2n)!^2}{4^{2n}} \sum_{k=1}^{n} \left(2 + (n+k)H_{n+k-1}\right) \frac{2^{n+k}}{(n+k)^2} \\
&= \frac{(2n)!^2}{4^{2n}} \sum_{k=n+1}^{2n} \left(2 + kH_{k-1}\right) \frac{2^k}{k^2} \\
&= \frac{(2n)!^2}{4^{2n}} \left[\sum_{k=1}^{2n} \left(2 + kH_{k-1}\right) \frac{2^k}{k^2} - \sum_{k=1}^{n} \left(2 + kH_{k-1}\right) \frac{2^k}{k^2}\right] \\
&= \frac{(2n)!^2}{4^{2n}} \sum_{k=1}^{2n} \left(2 + kH_{k-1}\right) \frac{2^k}{k^2} - \frac{(2n)!^2}{4^n n!^2} \cdot \frac{n!^2}{4^n} \sum_{k=1}^{n} \left(2 + kH_{k-1}\right) \frac{2^k}{k^2} \\
&= u_{2n} - \left(\frac{(2n)!}{2^n n!}\right)^2 u_n.
\end{aligned}$$

But since $u_n, u_{2n} \in \mathbb{Z}$ (according to Theorem 2.4) and

$$\frac{(2n)!}{2^n n!} = \frac{1 \times 2 \times \cdots \times (2n)}{2 \times 4 \times \cdots \times (2n)} = 1 \times 3 \times 5 \times \cdots \times (2n-1) \in \mathbb{Z},$$

we have that $u_{2n} - \left(\frac{(2n)!}{2^n n!}\right)^2 u_n \in \mathbb{Z}$. The required result of the corollary then follows. $\square$

**Remarks 2.7**

1. Using Theorem 2.2 of [4], we can easily verify that the lower bound of Theorem 2.1 is essentially optimal.

2. The rational sequence introduced in Theorem 2.4 can be alternatively defined by the recurrence:
$$\begin{cases} u_0 = 0 \\ u_n = \frac{n^2}{4} u_{n-1} + \frac{(n-1)!^2}{2^n} \left(2 + nH_{n-1}\right) \quad (\forall n \geq 1) \end{cases}.$$
Similarly, the rational sequence introduced in Corollary 2.6 can be alternatively defined by the recurrence:
$$\begin{cases} v_0 = 0 \\ v_n = \frac{n^2(2n-1)^2}{4} v_{n-1} + K_n \quad (\forall n \geq 1) \end{cases},$$
where $(K_n)_n$ is a sequence of rational numbers having a closed form in terms of harmonic numbers.

3. Using some more complicated functional equations of polylogarithms (such as Equation (6.108) of [5, page 178]), we can give other results similar to Theorem 2.1 and then establish the integrality of some other rational sequences similar to those of Theorem 2.4 and Corollary 2.6.

4. It is also possible to prove Theorem 2.1 by means of the $p$-adic dilogarithm function together with Theorem 2.2 of [4] (this non elementary method is detailed in [4] for the $p$-adic logarithm function).


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