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On a modification of $\underline{\mathrm{Set}}(n)$

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To Tony Shannon for his 85th anniversary!

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Abstract: A modification of the set $\underline{\mathrm{Set}}(n)$ for a fixed natural number n is introduced in the form: $\underline{\mathrm{Set}}(n, f)$, where f is an arithmetic function. The sets $\underline{\mathrm{Set}}(n, \varphi), \underline{\mathrm{Set}}(n, \psi), \underline{\mathrm{Set}}(n, \sigma)$ are discussed, where φ, ψ and σ are Euler's function, Dedekind's function and the sum of the positive divisors of n, respectively.
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1 Introduction

Let us, following [1], for a fixed natural number $n \geq 2$ having the canonical form

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where k, $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers and $p_1 < p_2 < \cdots < p_k$ are different prime numbers, define:

$$\underline{\text{set}}(n) = \{p_1, p_2, \ldots, p_k\}$$

$$\underline{\text{Set}}(n) = \{m \mid m = \prod_{i=1}^{k} p_i^{\beta_i} \ \& \ \delta(n) \leq \beta_i \leq \Delta(n)\},$$

where[1]

$$\delta(n) = \min(\alpha_1, \ldots, \alpha_k),$$

$$\Delta(n) = \max(\alpha_1, \ldots, \alpha_k).$$

Now, we can define a new set, subset of $\underline{\text{Set}}$, with the form

$$\underline{\text{Set}}(n, f) = \{m \mid m \in \underline{\text{Set}}(n) \ \& \ f(m) \in \underline{\text{Set}}(n)\}. \tag{1}$$

Here, we will show the conditions for an element $m \in \underline{\text{Set}}(n)$ to also satisfy $m \in \underline{\text{Set}}(n, f)$, where f is the Euler's totient function φ and the Dedekind's function ψ.

2 The case of Euler's totient function

Let f be the Euler's totient function φ. Therefore, below, 2 must be a divisor of n, because in $\underline{\text{Set}}(n, \varphi)$ for $n \geq 3$, all numbers must be even, i.e.,

$$n = 2^{\alpha} \cdot \prod_{i=1}^{k} p_i^{\alpha_i}, \tag{2}$$

where $k, \alpha, \alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers and $3 \leq p_1 < p_2 < \cdots < p_k$ are different prime numbers. Let

$$m = 2^{a} \cdot \prod_{i=1}^{k} p_i^{\beta_i} \in \underline{\text{Set}}(n, f).$$

Therefore,

$$\varphi(m) = 2^{a-1} \cdot \prod_{i=1}^{k} p_i^{\beta_i - 1} \cdot (p_i - 1) \in \underline{\text{Set}}(n).$$

Hence, $p_1 - 1 = 2^{b_1}$, because if $p_1 - 1$ has a divisor different of 2, it must be a divisor of n, while $p_1 \geq 3$ is the minimal one. By the same reason, for each i $(2 \leq i \leq k)$

$$p_i - 1 = 2^{b_i} \cdot \prod_{j=1}^{i-1} p_j^{\gamma_{i,j}},$$

where $\gamma_{i,j} \geq 0$, i.e., p_j cannot be a divisor of $p_i - 1$.

Therefore,

$$\varphi(m) = 2^{a-1+\sum\limits_{i=1}^{k} b_i} \cdot \prod_{i=1}^{k} \left(p_i^{\beta_i - 1} \prod_{j=1}^{i-1} p_i^{\gamma_{i,j}} \right) = 2^{a-1+\sum\limits_{i=1}^{k} b_i} \cdot \prod_{i=1}^{k} p_i^{\beta_i - 1 + \sum\limits_{j=1}^{i-1} \gamma_{i,j}}.$$

[1] Other authors (see, e.g. [2]) denote the functions δ and Δ by h and H, respectively.

Hence, $\varphi(m) \in \underline{\text{Set}}(n, \varphi)$ only if

$$\delta(n) \leq a - 1 + \sum_{i=1}^{k} b_i \leq \Delta(n),$$

i.e., $\Delta(n) \geq a + k - 1$ or $a \leq \Delta(n) - k + 1$, and for each i $(1 \leq i \leq k - 1)$

$$\delta(n) \leq \beta_i - 1 + \sum_{j=1}^{i-1} \gamma_{i,j} \leq \Delta(n),$$

and

$$\delta(n) \leq \beta_k - 1,$$

i.e., $\beta_k \geq 2$. Also, if for p_i there is no $p_s > p_i$ for which p_i is a divisor of $p_s - 1$, then β_i must be greater than 1.

For example, when $n = 24 = 2^3.3$, then

$$\underline{\text{Set}}(24) = \{2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3, 2 \cdot 3^2, 2^2 \cdot 3^2, 2^3 \cdot 3^2, 2 \cdot 3^3, 2^2 \cdot 3^3, 2^3 \cdot 3^3\}$$

and

$$\underline{\text{Set}}(24, \varphi) = \{2 \cdot 3^2, 2^2 \cdot 3^2, 2^3 \cdot 3^2, 2 \cdot 3^3, 2^2 \cdot 3^3, 2^3 \cdot 3^3\}.$$

When $n = 50 = 2 \cdot 5^2$, then

$$\underline{\text{Set}}(50) = \{2 \cdot 5, 2^2 \cdot 5, 2 \cdot 5^2, 2^2 \cdot 5^2\}$$

and

$$\underline{\text{Set}}(50, \varphi) = \{2^2 \cdot 5\}.$$

But when $n = 242 = 2 \cdot 11^2$, then

$$\underline{\text{Set}}(242) = \{2 \cdot 11, 2^2 \cdot 11, 2 \cdot 11^2, 2^2 \cdot 11^2\}$$

and

$$\underline{\text{Set}}(242, \varphi) = \varnothing,$$

because $\varphi(22) = 2 \cdot 5, \varphi(44) = 2^2 \cdot 5, \varphi(242) = 2 \cdot 5 \cdot 11, \varphi(484) = 2^2 \cdot 5 \cdot 11$, i.e., none of these numbers can be an element of $\underline{\text{Set}}(242, \varphi)$.

Now, on the basis of (1), we can define another set, subset of $\underline{\text{Set}}(n, f)$, with the form

$$\underline{\text{Set}}(n, f^2) = \{m \mid m \in \underline{\text{Set}}(n) \ \& \ f(m) \in \underline{\text{Set}}(n) \ \& \ f(f(m)) \in \underline{\text{Set}}(n)\}.$$

In this case,

$$\underline{\text{Set}}(24, \varphi^2) = \{2 \cdot 3^3, 2^2 \cdot 3^3, 2^3 \cdot 3^3\},$$

but

$$\underline{\text{Set}}(50, \varphi^2) = \varnothing.$$

Of course, we can give also the definition for each natural number $s \geq 1$

$$\underline{\text{Set}}(n, f^{s+1}) = \{m \mid m \in \underline{\text{Set}}(n) \ \& \ m \in \underline{\text{Set}}(n, f^s) \ \& \ f(f(m)) \in \underline{\text{Set}}(n, f^s)\}.$$

We see directly that

$$\underline{\text{Set}}(24, \varphi^3) = \varnothing.$$

More general, we see that

$$s \le \Delta(n).$$

Proposition 2.1. *Let* $p_1, \ldots, p_k$ *be the prime factors of* n. *If there exists a prime* $p \notin \underline{\text{set}}(n)$ *such that*

$$p \mid \prod_{i=1}^{k} (p_i - 1),$$

then

$$\underline{\text{Set}}(n, \varphi) = \varnothing.$$

Proof. The proof follows from the definition (1). □

For example, if $n = 2^a \cdot 3^b \cdot 11^c$, then $5 | (2-1) \cdot (3-1) \cdot (11-1)$ and $5 \ne 2, 3, 11$, so $\underline{\text{Set}}(n, \varphi) = \varnothing$.

We will discuss some particular cases.

I. Let $n = 2^\alpha \cdot p^\beta$ and $m = 2^a \cdot p^b$, where p is odd and

$$g = \max(\alpha, \beta) \ge a, b \ge \min(\alpha, \beta) = h \ge 1.$$

Then

$$\varphi(m) = 2^{a-1} \cdot p^{b-1} \cdot (p-1).$$

- Case 1. $a = 1$. Then $b \ne 1$ because $(p-1, p) = 1$. Now,

  $$\varphi(m) = p^{b-1} \cdot (p-1),$$

  so we must have $p - 1 = 2^s$ with $g \ge s \ge h$ and $p = 2^s + 1$ is a Fermat's prime. Thus,

  $$\varphi(m) = 2^s \cdot p^{b-1} = 2^s \cdot (2^s + 1)^{b-1}.$$

- Case 2. $a > 1$ and $b \ne 1$. Then $p - 1 = 2^r$, i.e., $p = 2^r + 1$ is a Fermat's prime, and

  $$\varphi(m) = 2^{a-1+r} \cdot p^{b-1}$$

  with $g \ge a - 1 + r \ge h$. Thus $g - a + 1 \ge r \ge h - a + 1$ and

  $$\varphi(m) = 2^{a-1+r} \cdot (2^r + 1)^{b-1}.$$

For example, if $\beta = 1$, then $h = 1, g = \alpha$, $\alpha \ge s \ge 1$, $a = 1, p = 2^s + 1$ and

$$m = 2 \cdot (2^s + 1)^b,$$

$$\varphi(m) = 2^s \cdot (2^s + 1)^{b-1}.$$

If $a > 1, h > 1$, then $p = 2^r + 1, m = 2^a \cdot (2^r + 1)^b$ and

$$\varphi(m) = 2^{a-1+r} \cdot (2^r + 1)^{b-1}$$

with $\alpha - 1 \ge r \ge 2 - a$.

II. Let $n = 2^{\alpha} \cdot p^{\beta} \cdot q^{\gamma}, m = 2^{a} \cdot p^{b} \cdot q^{c}$ with

$$k = \max(\alpha, \beta, \gamma) \geq a, b, c \geq \min(\alpha, \beta, \gamma) = h.$$

Then

$$\varphi(m) = 2^{a-1} \cdot p^{b-1} \cdot q^{c-1} \cdot (p-1) \cdot (q-1).$$

If $c = 1$, then since $q \not| \ q-1$ and $p < q$ (we may select in such a way), we get a contradiction. Thus, $c > 1$.

- Case 1. $a = 1$. Then

$$\varphi(m) = p^{b-1} \cdot q^{c-1} \cdot (p-1) \cdot (q-1).$$

If $b = 1$, then

$$\varphi(m) = q^{c-1} \cdot (p-1) \cdot (q-1).$$

So, we must have $q - 1 = 2^{x} \cdot p^{t}, p - 1 = 2^{y}$. Thus $p = 2^{y} + 1$ and $q = 2^{x} \cdot p^{t} + 1$ for $t \geq 1, x \geq 0$ and

$$q = 2^{x} \cdot (2^{y} + 1)^{t} + 1.$$

If $b > 1$, then

$$\varphi(m) = p^{b-1} \cdot q^{c-1} \cdot (p-1) \cdot (q-1)$$

and we must have $p - 1 = 2^{x}, q - 1 = 2^{y}$ for $y \geq 1$ or $q - 1 = 2^{y} \cdot p^{t}$ for $y \geq 1, t \geq 1$. In the first case, $p = 2^{x} + 1, q = 2^{y} + 1$ are Fermat primes, and

$$\varphi(m) = 2^{x+y} \cdot p^{b-1} \cdot q^{c-1}.$$

In the second case $p - 1 = 2^{x}, q - 1 = 2^{y} \cdot p^{t} = 2^{y} \cdot (2^{x} + 1)^{t}$, so $p = 2^{x} + 1$ is a Fermat prime and q is a prime of the form $q = 2^{y} \cdot (2^{x} + 1)^{t} + 1$.

- Case 2. $a > 1 (c > 1)$ and

$$\varphi(m) = 2^{a-1} \cdot q^{c-1} \cdot (p-1) \cdot (q-1).$$

If $b = 1$, then

$$\varphi(m) = 2^{a-1} \cdot q^{c-1} \cdot (p-1) \cdot (q-1).$$

Thus $p - 1 = 2^{x}, q - 1 = 2^{y} \cdot p^{t}$ for $y \geq 0, t \geq 1$ and

$$\varphi(m) = 2^{a-1+y} \cdot q^{c-1} \cdot p^{t}.$$

Thus $p = 2^{x} + 1$ is a Fermat prime and q is a prime of the form

$$q = 2^{y} \cdot (2^{x} + 1)^{t} + 1.$$

For example, if $x = 1, p = 3, q = 2^{y} \cdot 3^{t} + 1$ is prime if $y = t = 1$.

3 The case of Dedekind's arithmetical function

Let f be the Dedekind's function ψ. Therefore, again 2 must be a divisor of n, because in $\underline{\text{Set}}(n, \psi)$ for $n \geq 3$, all numbers must be even, i.e., n again has the form of (2).

Therefore,

$$\psi(m) = 2^{a-1} \cdot \prod_{i=1}^{k} p_i^{\beta_i - 1} \cdot (p_i + 1) \in \underline{\text{Set}}(n).$$

Hence, as above, $p_1 + 1 = 2^{b_1}$, because if $p_1 + 1$ has a divisor different from 2, it must be a divisor of n, while $p_1 \geq 3$ is the minimal one. By the same reason, for each i $(2 \leq i \leq k)$

$$p_i + 1 = 2^{b_i} \cdot \prod_{j=1}^{i-1} p_j^{\gamma_{i,j}},$$

where $\gamma_{i,j} \geq 0$, i.e., p_j cannot be a divisor of $p_i - 1$. Obviously, $p_k + 1$ does not have a divisor greater than p_k.

Therefore, as above

$$\psi(m) = 2^{a-1+\sum\limits_{i=1}^{k} b_i} \cdot \prod_{i=1}^{k} p_i^{\beta_i - 1 + \sum\limits_{j=1}^{i-1} \gamma_{i,j}}.$$

Hence, $\psi(m) \in \underline{\text{Set}}(n, \psi)$ only if (exactly as above)

$$\delta(n) \leq a - 1 + \sum_{i=1}^{k} b_i \leq \Delta(n),$$

i.e., $\Delta(n) \geq a + k - 1$ or $a \leq \Delta(n) - k + 1$, and for each i $(1 \leq i \leq k - 1)$

$$\delta(n) \leq \beta_i - 1 + \sum_{j=1}^{i-1} \gamma_{i,j} \leq \Delta(n),$$

and

$$\delta(n) \leq \beta_k - 1,$$

i.e., $\beta_k \geq 2$. Also, if for p_i there is no $p_s > p_i$ for which p_i is a divisor of $p_s - 1$, then β_i must be greater than 1.

For example, when $n = 24 = 2^3 \cdot 3$, then

$$\underline{\text{Set}}(24, \psi) = \{2 \cdot 3, 2^2 \cdot 3, 2 \cdot 3^2, 2^2 \cdot 3^2, 2 \cdot 3^3, 2^2 \cdot 3^3\},$$

because for $b \geq 3$ and $c \geq 1$

$$\psi(2^b \cdot 3^c) = 2^{b+1} \cdot 3^c,$$

i.e., it cannot be a member of $\underline{\text{Set}}(24, \psi)$ for $b \geq 3$. Obviously, $\underline{\text{Set}}(50, \psi) = \varnothing$.

4 A particular case for the sum of divisors function

Let $n = 2^\alpha \cdot p$, where p is a prime number. Then $m = 2^a \cdot p^b$ with $\alpha \geq a \geq 1, \alpha \geq b \geq 1$. Then

$$\sigma(m) = (2^{a+1} - 1) \cdot (p^b + p^{b-1} + \cdots + p + 1).$$

As the expression $p^b + p^{b-1} + \cdots + p + 1$ must be even, then b must be odd.

Let $p^b + p^{b-1} + \cdots + p + 1 = 2^s$ and $2^{a+1} - 1 = p^c$.

For example, if $c = 1$, then $p = 2^{a+1} - 1$ is a Mersenne prime. If $b = 1$, then $p = 2^s - 1 = 2^{a+1} - 1$, i.e., $s = a + 1$ and

$$m = 2^a \cdot p^1 = 2^a \cdot (2^{a+1} - 1)$$

for $\alpha - 1 \geq a \geq 1$. Thus $n = 2^\alpha \cdot (2^{\alpha+1} - 1), m = 2^a \cdot (2^{a+1} - 1)$ and

$$\sigma(m) = 2^{a+1} \cdot (2^{a+1} - 1).$$

If $a = 1$, then $n = 2^\alpha \cdot 3$ and $m = 2 \cdot 3, \sigma(m) = 2^2 \cdot 3$ for $\alpha \geq 2$. Therefore, we have here $\underline{\text{Set}}(n, \sigma) \neq \varnothing$.

5 Conclusion

In the paper, the object $\underline{\text{Set}}(n, f)$ was defined, where n is a natural number and f is an arithmetic function and we discussed the cases, when f is the functions φ, ψ and σ. At the moment, an **Open Problem** is to investigate the other cases for the last arithmetic function, as well as the case when f is another arithmetic function.

References

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