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On generalized hyperharmonic numbers of order r, $H^r_{n,m}(\sigma)$

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Abstract: In this paper, we define generalized hyperharmonic numbers of order r, $H^r_{n,m}(\sigma)$, for $m \in \mathbb{Z}^+$ and give some applications by using generating functions of these numbers. For example, for $n, r, s \in \mathbb{Z}^+$ such that $1 \le s \le r$,

$$\sum_{k=1}^{n} \binom{n-k+s-1}{s-1} H^{r-s}_{k,m}(\sigma) = H^r_{n,m}(\sigma),$$

and

$$\sum_{k=1}^{n} \sum_{i=1}^{k} \frac{H^{r+1}_{k-i,m}(\sigma)\, D_r(k-i+r)}{(n-k)!\,(k-i+r)!} = H^{2r+2}_{n,m}(\sigma),$$

where $D_r(n)$ is an r-derangement number.

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1 Introduction

The harmonic numbers, denoted by H_n, are defined by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n} \frac{1}{k} \text{ for } n \geq 1,$$

and their generating function is

$$\sum_{n=0}^{\infty} H_n x^n = \frac{-\ln(1-x)}{1-x}.$$

In [10], it is known that

$$\sum_{k=0}^{n} \frac{H_k}{n-k+1} = H_{n+1}^2 - H_{n+1,2},$$

where $H_{n,2} = \sum_{k=1}^{n} \frac{1}{k^2}$.

Harmonic numbers are interesting research objects. Recently, these numbers have been generalized by several authors. There are a lot of works involving harmonic numbers and their generalizations ([3, 5–9]).

Guo and Cha [5] defined the generalized harmonic numbers by

$$H_0(\sigma) = 0 \text{ and } H_n(\sigma) = \sum_{k=1}^{n} \frac{\sigma^k}{k} \text{ for } n \geq 1,$$

where σ is an appropriate parameter, and their generating function is

$$\sum_{n=0}^{\infty} H_n(\sigma) x^n = \frac{-\ln(1-\sigma x)}{1-x}.$$

When $\sigma = 1/\alpha$ for $\alpha \in \mathbb{R}^+$, $H_n(1/\alpha) := \sum_{k=1}^{n} \frac{1}{k\alpha^k}$ are called the generalized harmonic numbers by Genčev [4].

The exponential generating function is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{1.1}$$

The derangement numbers d_n are given by the closed form formula

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

These numbers satisfy the recursive formula given by

$$d_n = (n-1)\left(d_{n-1} + d_{n-2}\right) \text{ for } n \geq 2,$$

with $d_0 = 1, d_1 = 0$. The generating function of d_n is given by

$$\sum_{n=0}^{\infty} d_n \frac{x^n}{n!} = \frac{1}{1-x} e^{-x}. \tag{1.2}$$

In [11], for $0 \leq r \leq n$, $D_r(n)$ denotes the number of derangements on $n+r$ elements under the restriction that the first r elements are in disjoint cycles. A closed form formula for $D_r(n)$ is also given by

$$D_r(n)=\sum_{k=r}^{n}(-1)^{n-k}\binom{k}{r}\frac{n!}{(n-k)!}.$$

The r-derangement numbers $D_r(n)$ satisfy the recursive formula

$$D_r(n)=rD_{r-1}(n-1)+(n-1)D_r(n-2)+(n+r-1)D_r(n-1),\ n>2,r>0,$$

with initial conditions

$$D_1(n)=d_{n+1},\ D_r(r)=r!\ (r\geq 1)\text{ and }D_r(r+1)=r(r+1)!,\ r\geq 2.$$

The generating function of the r-derangement numbers $D_r(n)$ is given by

$$\sum_{n=0}^{\infty}D_r(n)\frac{x^n}{n!}=\frac{x^r}{(1-x)^{r+1}}e^{-x}. \tag{1.3}$$

Note that for $r=0$, we have $D_0(n)=d_n$. It is known that for $r\geq 1$,

$$\sum_{n=0}^{\infty}\binom{n+r-1}{n}x^n=\frac{1}{(1-x)^r}. \tag{1.4}$$

In [1,2], for $m\in\mathbb{Z}$, the polylogarithm is defined by

$$Li_m(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^m}. \tag{1.5}$$

Note that $Li_1(x)=\sum\limits_{n=1}^{\infty}\frac{x^n}{n}=-\log(1-x)$.

The Stirling numbers of the second kind $S_2(n,k)$ are defined by

$$x^n=\sum_{k=0}^{n}S_2(n,k)x^{\underline{k}},$$

where $x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{0}}=1$ and $x^{\underline{n}}=x(x-1)...(x-n+1)$.

The generating function of the Stirling numbers of the second kind $S_2(n,k)$ is given by

$$\sum_{n=k}^{\infty}S_2\left(n,k\right)\frac{x^n}{n!}=\frac{1}{k!}\left(e^x-1\right)^k\ \text{ for }k\geq 0. \tag{1.6}$$

2 Some results

In this section, we will define generalized harmonic numbers, $H_{n,m}\left(\sigma\right)$ and then give some applications of them.

Definition 2.1. *For $n,m\in\mathbb{Z}^+$, the generalized harmonic numbers, $H_{n,m}\left(\sigma\right)$, are defined by*

$$H_{0,m}\left(\sigma\right)=0\text{ and }H_{n,m}\left(\sigma\right)=\sum_{k=1}^{n}\frac{\sigma^k}{k^m}, \tag{2.1}$$

where σ is an appropriate parameter.

When $m=1$ in (2.1), we get $H_{n,1}\left(\sigma\right)=H_n\left(\sigma\right)$.

It is clearly seen that for $m > 0$, we have

$$\sum_{n=1}^{\infty} H_{n,m}(\sigma) x^n = \frac{Li_m(\sigma x)}{1-x}. \tag{2.2}$$

Definition 2.2. *For $r < 0$ or $n \leq 0$, $H_{0,m}^r(\sigma) = 0$ and for $n \geq 1$, the generalized hyperharmonic numbers of order r, $H_{n,m}^r(\sigma)$, are defined by*

$$H_{n,m}^r(\sigma) = \sum_{k=1}^{n} H_{k,m}^{r-1}(\sigma) \text{ for } r \geq 1, \tag{2.3}$$

where $H_{n,m}^0(\sigma) = \frac{\sigma^n}{n^m}$.

Note that for $r = 1$, $H_{n,m}^1(\sigma) = H_{n,m}(\sigma)$.

Theorem 2.1. *For $m \in \mathbb{Z}^+$ and $r \in \mathbb{Z}^+ \cup \{0\}$, we have*

$$\sum_{n=1}^{\infty} H_{n,m}^r(\sigma) x^n = \frac{Li_m(\sigma x)}{(1-x)^r}. \tag{2.4}$$

Proof. By (2.2) and (2.3), we have

$$\begin{aligned} \frac{Li_m(\sigma x)}{(1-x)^r} &= \frac{1}{(1-x)^{r-1}} \frac{Li_m(\sigma x)}{1-x} = \frac{1}{(1-x)^{r-1}} \sum_{n=1}^{\infty} H_{n,m}(\sigma) x^n \\ &= \frac{1}{(1-x)^{r-2}} \sum_{n=1}^{\infty} \sum_{k=1}^{n} H_{k,m}(\sigma) x^n = \frac{1}{(1-x)^{r-2}} \sum_{n=1}^{\infty} H_{n,m}^2(\sigma) x^n \\ &= \dots = \sum_{n=1}^{\infty} H_{n,m}^r(\sigma) x^n, \end{aligned}$$

as claimed. □

From Theorem 2.1, it is clearly seen that

$$H_{n,m}^r(\sigma) = H_{n,m}^{r-1}(\sigma) + H_{n-1,m}^r(\sigma).$$

Theorem 2.2. *For $n, m, r \in \mathbb{Z}^+$, we have*

$$H_{n,m}^r(\sigma) = \sum_{k=1}^{n} \binom{n-k+r-1}{r-1} \frac{\sigma^k}{k^m}.$$

Proof. By (1.4) and (2.4), we have

$$\begin{aligned} \sum_{n=1}^{\infty} H_{n,m}^r(\sigma) x^n &= \frac{Li_m(\sigma x)}{(1-x)^r} = \sum_{n=0}^{\infty} \binom{n+r-1}{n} x^n \sum_{n=1}^{\infty} \frac{\sigma^n}{n^m} x^n \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \binom{n-k+r-1}{r-1} \frac{\sigma^k}{k^m} x^n. \end{aligned}$$

Thus, by comparing the coefficients on both sides, the proof is complete. □

Theorem 2.3. *Let r, s be positive integers such that $1 \leq s \leq r$. For $n, m \in \mathbb{Z}^{+}$, we have*

$$\sum_{k=1}^{n} \binom{n-k+s-1}{s-1} H_{k,m}^{r-s}(\sigma) = H_{n,m}^{r}(\sigma), \tag{2.5}$$

and

$$\sum_{k=1}^{n} \binom{r}{n-k} (-1)^{k} H_{k,m}^{r}(\sigma) = \frac{(-\sigma)^{n}}{n^{m}}.$$

Proof. By (1.4) and (2.4), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} H_{n,m}^{r}(\sigma) x^{n} &= \frac{1}{(1-x)^{r-s}} Li_{m}(\sigma x) \frac{1}{(1-x)^{s}} \\
&= \sum_{n=1}^{\infty} H_{n,m}^{r-s}(\sigma) x^{n} \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} x^{n} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \binom{n-k+s-1}{s-1} H_{k,m}^{r-s}(\sigma) x^{n},
\end{aligned}$$

and by (2.4),

$$\begin{aligned}
Li_{m}(\sigma x) &= \sum_{n=1}^{\infty} \frac{\sigma^{n}}{n^{m}} x^{n} = \sum_{n=1}^{\infty} H_{n,m}^{r}(\sigma) x^{n} \sum_{n=0}^{\infty} \binom{r}{n} (-1)^{n} x^{n} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \binom{r}{n-k} (-1)^{n-k} H_{k,m}^{r}(\sigma) x^{n}.
\end{aligned} \tag{2.6}$$

Thus, by comparing the coefficients on both sides, we have the proof. □

For example, when $r = s$ in (2.5), we obtained Theorem 2.2.

Theorem 2.4. *For $n, m, r \in \mathbb{Z}^{+}$, we have*

$$\sum_{k=1}^{n} (-1)^{n+k} k! H_{k,m}^{r}(\sigma) S_{2}(n,k) = \sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^{i+k} \binom{n}{k} \frac{\sigma^{i}}{i^{m}} S_{2}(k,i)\, i! r^{n-k}.$$

Proof. Inserting $1 - e^{-x}$ in the place of x in (2.4), by (1.6), we have

$$\begin{aligned}
\sum_{k=1}^{\infty} H_{k,m}^{r}(\sigma) \left(1 - e^{-x}\right)^{k} &= \sum_{k=1}^{\infty} (-1)^{k} H_{k,m}^{r}(\sigma) k! \frac{(e^{-x}-1)^{k}}{k!} \\
&= \sum_{k=1}^{\infty} (-1)^{k} H_{k,m}^{r}(\sigma) k! \sum_{n=k}^{\infty} (-1)^{n} S_{2}(n,k) \frac{x^{n}}{n!} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^{n+k} H_{k,m}^{r}(\sigma) S_{2}(n,k) \frac{k!}{n!} x^{n},
\end{aligned} \tag{2.7}$$

and from (1.1) and (1.6),

$$\frac{Li_{m}\left(\sigma\left(1-e^{-x}\right)\right)}{e^{-rx}} = e^{rx} Li_{m}\left(\sigma\left(1-e^{-x}\right)\right)$$

$$= e^{rx}\sum_{i=1}^{\infty}(-1)^i\frac{\sigma^i}{i^m}i!\sum_{n=i}^{\infty}(-1)^n S_2(n,i)\frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{r^n}{n!}x^n\sum_{n=1}^{\infty}\sum_{i=1}^{n}(-1)^{n+i}\frac{\sigma^i}{i^m}S_2(n,i)\frac{i!}{n!}x^n$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{i=1}^{k}(-1)^{i+k}\frac{\sigma^i}{i^m}S_2(k,i)\frac{i!}{k!}\frac{r^{n-k}}{(n-k)!}x^n. \tag{2.8}$$

Thus, by comparing the coefficients on right sides of (2.7) and (2.8), we have the proof. □

Theorem 2.5. *For $r,s,m,t\in\mathbb{Z}^+$ and $n\in\mathbb{Z}^+\setminus\{1\}$, we have*

$$\sum_{k=1}^{n-1}H_{k,m}^{r}(\sigma)H_{n-k,t}^{s}(\sigma)=\sum_{k=1}^{n-1}\sum_{i=1}^{k}\binom{k-i+r+s-1}{k-i}\frac{\sigma^{i+n-k}}{i^m(n-k)^t}.$$

Proof. By (1.4), (1.5) and (2.4), we have

$$\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}H_{k,m}^{r}(\sigma)H_{n-k,t}^{s}(\sigma)x^n = \sum_{n=1}^{\infty}H_{n,m}^{r}(\sigma)x^n\sum_{n=1}^{\infty}H_{n,t}^{s}(\sigma)x^n=\frac{Li_m(\sigma x)Li_t(\sigma x)}{(1-x)^{r+s}}$$

$$= \sum_{n=0}^{\infty}\binom{n+r+s-1}{n}x^n\sum_{n=1}^{\infty}\frac{\sigma^n}{n^m}x^n\sum_{n=1}^{\infty}\frac{\sigma^n}{n^t}x^n$$

$$= \left(\sum_{n=1}^{\infty}\sum_{i=1}^{n}\binom{n-i+r+s-1}{n-i}\frac{\sigma^i}{i^m}x^n\right)\left(\sum_{n=1}^{\infty}\frac{\sigma^n}{n^t}x^n\right)$$

$$= \sum_{n=2}^{\infty}\sum_{k=1}^{n-1}\sum_{i=1}^{k}\binom{k-i+r+s-1}{k-i}\frac{\sigma^{i+n-k}}{i^m(n-k)^t}x^n.$$

Thus, by comparing the coefficients on both sides, we obtain the proof. □

Theorem 2.6. *For $n,m,r\in\mathbb{Z}^+$, we have*

$$\sum_{k=1}^{n}\sum_{i=1}^{k}H_{i,m}^{r+1}(\sigma)\frac{D_r(k-i+r)}{(n-k)!(k-i+r)!}=H_{n,m}^{2r+2}(\sigma),$$

and

$$\sum_{k=1}^{n}\frac{(-1)^{n-k}H_{k,m}^{r}(\sigma)-d_{n-k}H_{k,m}^{r-1}(\sigma)}{(n-k)!}=0.$$

Proof. From (1.1), (1.3) and (2.4), we have

$$\sum_{n=1}^{\infty}H_{n,m}^{2r+2}(\sigma)x^n = \frac{1}{(1-x)^{r+1}}Li_m(\sigma x)\frac{x^re^{-x}}{(1-x)^{r+1}}e^x\frac{1}{x^r}$$

$$= \sum_{n=1}^{\infty}H_{n,m}^{r+1}(\sigma)x^n\sum_{n=r}^{\infty}D_r(n)\frac{x^n}{n!}\sum_{n=0}^{\infty}\frac{x^n}{n!}\frac{1}{x^r}$$

$$= \sum_{n=1}^{\infty}\sum_{i=1}^{n}H_{i,m}^{r+1}(\sigma)\frac{D_r(n-i+r)}{(n-i+r)!}x^n\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{i=1}^{k}H_{i,m}^{r+1}(\sigma)\frac{D_r(k-i+r)}{(n-k)!(k-i+r)!}x^n.$$

By comparing the coefficients on both sides, the first equality is obtained.

For the second equality, by (1.1) and (2.4), we have

$$\begin{aligned} \frac{1}{(1-x)^r} Li_m(\sigma x) e^{-x} &= \sum_{n=1}^{\infty} H_{n,m}^r(\sigma) x^n \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^{n-k} \frac{H_{k,m}^r(\sigma)}{(n-k)!} x^n, \end{aligned} \tag{2.9}$$

and by (1.2) and (2.4)

$$\begin{aligned} \frac{1}{(1-x)^r} Li_m(\sigma x) e^{-x} &= \frac{1}{(1-x)^{r-1}} Li_m(\sigma x) \frac{e^{-x}}{1-x} \\ &= \sum_{n=1}^{\infty} H_{n,m}^{r-1}(\sigma) x^n \sum_{n=0}^{\infty} \frac{d_n}{n!} x^n \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} H_{k,m}^{r-1}(\sigma) \frac{d_{n-k}}{(n-k)!} x^n. \end{aligned} \tag{2.10}$$

Thus, by comparing the coefficients on right sides of (2.9) and (2.10), we have the second equality. □

Theorem 2.7. *For $n, m, r \in \mathbb{Z}^+$, we have*

$$\begin{aligned} \sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^i \frac{k!}{\sigma^k} \binom{r}{k-i} H_{i,m}^r(\sigma) S_2(n,k) &= \sum_{k=1}^{n} (-1)^k \frac{k!}{k^m} S_2(n,k) \\ &= \sum_{k=0}^{n-1} \sum_{i=0}^{k} (-1)^{i+1} \frac{i!}{(i+1)^{m-1}} \binom{n-1}{k} S_2(k,i). \end{aligned}$$

Proof. Inserting $\frac{1-e^{-x}}{\sigma}$ in the place of x in (2.6), by (1.6), then

$$\begin{aligned} Li_m(1-e^{-x}) &= \sum_{k=1}^{\infty} \sum_{i=1}^{k} (-1)^i \binom{r}{k-i} k! H_{i,m}^r(\sigma) \frac{(e^{-x}-1)^k}{\sigma^k k!} \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^{k} (-1)^i \binom{r}{k-i} \frac{k!}{\sigma^k} H_{i,m}^r(\sigma) \sum_{n=k}^{\infty} (-1)^n S_2(n,k) \frac{x^n}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^{n+i} \binom{r}{k-i} \frac{k!}{n! \sigma^k} H_{i,m}^r(\sigma) S_2(n,k) x^n, \end{aligned} \tag{2.11}$$

and inserting $1-e^{-x}$ in the place of x in (1.5), by (1.6), we have

$$\begin{aligned} Li_m(1-e^{-x}) &= \sum_{i=1}^{\infty} \frac{(1-e^{-x})^i}{i^m} = \sum_{i=1}^{\infty} (-1)^i \frac{i!}{i^m} \frac{(e^{-x}-1)^i}{i!} \\ &= \sum_{i=1}^{\infty} (-1)^i \frac{i!}{i^m} \sum_{n=i}^{\infty} (-1)^n S_2(n,i) \frac{x^n}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{n} (-1)^{n+i} \frac{i!}{n! i^m} S_2(n,i) x^n. \end{aligned} \tag{2.12}$$

By comparing the coefficients on right sides of (2.11) and (2.12), we obtain the first equality. For the second equality, we have

$$\begin{aligned}\frac{d}{dx}Li_m(1-e^{-x}) &= \frac{d}{dx}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{i=1}^{k}(-1)^{n+i}\binom{r}{k-i}\frac{k!}{n!\sigma^k}H_{i,m}^{r}(\sigma)\,S_2(n,k)x^n\right)\\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{i=1}^{k}(-1)^{n+i}\binom{r}{k-i}\frac{k!}{(n-1)!\sigma^k}H_{i,m}^{r}(\sigma)\,S_2(n,k)x^{n-1}\\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{n+1}\sum_{i=1}^{k}(-1)^{n+i+1}\binom{r}{k-i}\frac{k!}{n!\sigma^k}H_{i,m}^{r}(\alpha)\,S_2(n+1,k)x^n, \quad (2.13)\end{aligned}$$

and from (1.1) and (1.6)

$$\begin{aligned}\frac{d}{dx}Li_m(1-e^{-x}) &= \frac{Li_{m-1}(1-e^{-x})}{e^x-1} = \sum_{k=1}^{\infty}\frac{(1-e^{-x})^{k-1}}{k^{m-1}}e^{-x}\\ &= \sum_{k=0}^{\infty}(-1)^k\frac{k!}{(k+1)^{m-1}}\frac{(e^{-x}-1)^k}{k!}e^{-x}\\ &= \sum_{k=0}^{\infty}(-1)^k\frac{k!}{(k+1)^{m-1}}\sum_{n=k}^{\infty}(-1)^n\frac{S_2(n,k)}{n!}x^n\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k+n}k!}{(k+1)^{m-1}n!}S_2(n,k)x^n\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^{i+n}\binom{n}{k}\frac{i!}{(i+1)^{m-1}n!}S_2(k,i)x^n. \quad (2.14)\end{aligned}$$

By comparing the coefficients on right sides of (2.13) and (2.14), the second equality is given. Thus, we have the proof. □

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