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The Wiener, hyper-Wiener, Harary and *SK* indices of the $P(Z_{p^k \cdot q^r})$ power graph

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Abstract: The undirected $P(Z_n)$ power graph of a finite group of Z_n is a connected graph, the set of vertices of which is Z_n. Here $u, v \in P(Z_n)$ are two diverse adjacent vertices if and only if $u \neq v$ and $\langle v \rangle \subseteq \langle u \rangle$ or $\langle u \rangle \subseteq \langle v \rangle$. We will shortly name the undirected $P(Z_n)$ power graph as the power graph $P(Z_n)$. The Wiener, hyper-Wiener, Harary and *SK* indices of the $P(Z_n)$ power graph are in order as follows

$$\frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d(u,v), \quad \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d(u,v) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d^2(u,v),$$

$$\sum_{\{u,v\} \subseteq V(G)} \frac{1}{d(u,v)} \quad \text{and} \quad \frac{1}{2} \sum_{uv \in E(G)} (d_u + d_v).$$

In this article we focus more on the indices of $P(Z_n)$ power graph by Wiener, hyper-Wiener, Harary and *SK* the definition of the power graph is presented and the results and theorems which we need in our discussion are provided in the introduction. Finally, the main point of the article is that we calculate the Wiener, hyper-Wiener, Harary and *SK* indices of the power graph $P(Z_n)$ corresponding to the vertex $n = p^k \cdot q^r$. These are as follows: p, q are distinct primes and k, r are nonnegative integers.

Keywords: Wiener index, Hyper-Wiener index, Harary index, *SK* index, Undirected power graph.
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1 Introduction

The power graph was first introduced and studied by Kelarev and Quinn in the relevant literature as the directed power graph of finite semigroups [7]. We use the definition of undirected power graph introduced into the literature by Chakrabarty and others [3]. In this study our new theorems on the indices of power graph of finite cyclic groups are introduced, on which our article is based. Our article focuses on a finite and cyclic group G of order $n = p^k \cdot q^r$ (where p and q are distinct primes, $n \in Z^+$, and k and r are non-negative integers). This corresponds to a power graph $P(G)$ with m edges. Since any cyclic group of finite order $(Z_n, +)$ is isomorphic to the cyclic group (i.e., $G \cong Z_n$), we consider $G = (Z_n, +)$. In our research, original theorems are presented, which are not previously introduced in the literature regarding the calculations of the Wiener, hyper-Wiener, Harary, and SK indices of the $P(Z_{p^k \cdot q^r})$ power graph, using n, m, and the Euler ϕ function. This study may be deemed significantly for mathematical science because many mathematicians focus on power graphs and mathematical fields as in terms of power graphs using the Wiener, hyper-Wiener, Harary, and SK indices. As there is not any specific source of literature studying power graphs using the Wiener, hyper-Wiener, Harary, and SK indices, this kind of our research method can be also seen as an original approach.

2 Preliminaries

Let G be a group (or semigroup). The set of its points (vertices) is denoted by the undirected power graph, or simply the power graph $P(G)$. The elements of $P(G)$ are the elements of G. The relationships between points $u, v \in G$ and $n \in Z^+$ are:

$$u \sim v \iff u \neq v,\ u^n = v \text{ or } v^n = u \text{ (for multiplicative groups)}$$
$$u \sim v \iff u \neq v,\ nu = v \text{ or } nv = u \text{ (for additive groups)}$$
$$u \sim v \iff u \neq v,\ \langle v \rangle \subseteq \langle u \rangle \text{ or } \langle u \rangle \subseteq \langle v \rangle, \text{ [3].}$$

Let G be a graph with n points that is simply connected. The Wiener index of this graph is defined as follows [8]:

$$W(G) = \frac{1}{2} \cdot \sum_{\{u,v\} \subseteq V(G)} d(u,v).$$

This G graph hyper-Wiener index is defined as follows [4].

$$WW(G) = \frac{1}{2} \cdot \sum_{\{u,v\} \subseteq V(G)} d(u,v) + \frac{1}{2} \cdot \sum_{\{u,v\} \subseteq V(G)} d^2(u,v).$$

The Harary index of this graph is defined as [6, 8]:

$$H(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{d(u,v)}.$$

The *SK* index of this graph is defined as [9]:

$$SK(G) = \frac{1}{2} \cdot \sum_{uv \in E(G)} (d_u + d_v).$$

Theorem 2.1 [2]. *Let $P(Z_n)$ be a power graph of with n vertices and m edges. Then*

$$W(P(Z_n)) = \frac{1}{2} \cdot \left(\binom{2 \cdot n}{2} + \sum_{d|n} \phi(d) \cdot (\phi(d) - 2 \cdot d) \right). \tag{2.1.1}$$

Result 2.2 [3]. The number of edges of the undirected power graph $P(Z_n)$ is given by

$$m = \frac{1}{2} \cdot \sum_{d|n} \{2 \cdot d - \phi(d) - 1\} \cdot \phi(d). \tag{2.2.1}$$

Definition 2.3. Given G as a simply connected graph, the first and second Zagreb indices of this graph are:

$$M_1(G) = \sum_{u \in V(G)} {d_u}^2$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_u \cdot d_v.$$

Additionally, the Zagreb index of this graph is defined as:

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v).$$

Here, d_u and d_v represent the degrees of vertices u and v in G, respectively, [5].

3 Main results

In this study, the main aim is to present original results on the Wiener, hyper-Wiener, Harary, and SK indices of the $P\left(Z_{p^k \cdot q^r}\right)$ power graph, where p and q are distinct prime numbers and k, r are nonnegative integers. We use the definition of the relationship between power graphs as identified by Chakrabarty et al. (2009).

Theorem 3.1. L*et $P(Z_n)$ be a power graph with $n = p^k \cdot q^r$ vertices and m edges, where p and q are distinct prime numbers, $n \in Z^+$, k and r are nonnegative integers. Then,*

$$W(P(Z_n)) = \frac{1}{2} \cdot \left(\left(\frac{2 \cdot n}{2} \right) - 1 \right) + \frac{(1 - p^{2 \cdot k})}{2} + \frac{(1 - q^{2 \cdot r})}{2} + \frac{(\phi(p \cdot q) - 2 \cdot p \cdot q)}{2} \cdot \left(\frac{1 - p^{2 \cdot k}}{1 + p} \right) \cdot \left(\frac{1 - q^{2 \cdot r}}{1 + q} \right). \tag{3.1.1}$$

Proof. Let $P(Z_n)$ be a power graph with $n = p^k \cdot q^r$ vertices and m edges, where p and q are distinct prime numbers, $n \in Z^+$, k and r are nonnegative integers. To determine the Wiener index of $P(Z_n)$, we first identify all the positive divisors of $n = p^k \cdot q^r$. The positive divisors of $n = p^k \cdot q^r$ are as follows:

$$
\begin{aligned}
&\left\{1, p, p^2, p^3, \cdots, p^k\right\},\\
&\left\{q, q^2, q^3, \cdots, q^r\right\},\\
&\left\{p\cdot q, p\cdot q^2, p\cdot q^3, \ldots, p\cdot q^r\right\},\\
&\left\{p^2\cdot q, p^2\cdot q^2, p^2\cdot q^3, \ldots, p^2\cdot q^r\right\},\\
&\left\{p^3\cdot q, p^3\cdot q^2, p^3\cdot q^3, \ldots, p^3\cdot q^r\right\},\\
&\quad\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots\\
&\left\{p^k\cdot q, p^k\cdot q^2, p^k\cdot q^3, \ldots, p^k\cdot q^r\right\}.
\end{aligned}
$$

By using Equation (2.1.1) from Theorem 2.1 in the preliminary section, we obtain:

$$
\begin{aligned}
W\left(P\left(Z_n\right)\right) = \frac{1}{2}\cdot\binom{2\cdot n}{2} - \frac{1}{2} + \frac{1}{2}\cdot\phi(p)\cdot\left(\phi(p) - 2\cdot p\right)\cdot\left(p^0 + p^2 + p^4 + \cdots + p^{2\cdot(k-1)}\right)\\
+\frac{1}{2}\cdot\phi(q)\cdot\left(\phi(q) - 2\cdot q\right)\cdot\left(q^0 + q^2 + q^4 + \cdots + q^{2\cdot(r-1)}\right)\\
+\frac{1}{2}\cdot\phi(p)\cdot\phi(q)\cdot\left(\phi(p)\cdot\phi(q) - 2\cdot p\cdot q\right)\cdot\left(q^0 + q^2 + q^4 + \cdots + q^{2\cdot(r-1)}\right)\\
+\frac{1}{2}\cdot p^2\cdot\phi(p)\cdot\phi(q)\cdot\left(\phi(p)\cdot\phi(q) - 2\cdot p\cdot q\right)\cdot\left(q^0 + q^2 + q^4 + \cdots + q^{2\cdot(r-1)}\right)\\
+\frac{1}{2}\cdot p^4\cdot\phi(p)\cdot\phi(q)\cdot\left(\phi(p)\cdot\phi(q) - 2\cdot p\cdot q\right)\cdot\left(q^0 + q^2 + q^4 + \cdots + q^{2\cdot(r-1)}\right)\\
+\cdots+\frac{1}{2}\cdot p^{2\cdot(k-1)}\cdot\phi(p)\cdot\phi(q)\cdot\left(\phi(p)\cdot\phi(q) - 2\cdot p\cdot q\right)\cdot\left(q^0 + q^2 + q^4 + \cdots + q^{2\cdot(r-1)}\right).
\end{aligned}
$$

If we arrange the equation above, we obtain

$$
\begin{aligned}
W\left(P\left(Z_n\right)\right) = \frac{\left(\binom{2\cdot n}{2} - 1\right)}{2} + \frac{\phi(p)\cdot\left(\phi(p) - 2\cdot p\right)}{2}\cdot\left(\sum_{i=1}^{k}\left(p^2\right)^{i-1}\right)\\
+\frac{\phi(q)\cdot\left(\phi(q) - 2\cdot q\right)}{2}\cdot\left(\sum_{i=1}^{r}\left(q^2\right)^{i-1}\right)\\
+\frac{\phi(p\cdot q)\cdot\left(\phi(p\cdot q) - 2\cdot p\cdot q\right)}{2}\cdot\left(\sum_{i=1}^{k}\left(p^2\right)^{i-1}\right)\cdot\left(\sum_{i=1}^{r}\left(q^2\right)^{i-1}\right).
\end{aligned}
$$

On the other hand,

$$
\sum_{k=1}^{n} r^{k-1} = \frac{1-r^n}{1-r}, r \neq 1
$$

by using the provided equation and taking into account the terms, we deduce:

$$
W\left(P\left(Z_n\right)\right) = \frac{1}{2}\cdot\left(\binom{2\cdot n}{2} - 1\right) + \frac{\left(1-p^{2\cdot k}\right)}{2} + \frac{\left(1-q^{2\cdot r}\right)}{2} + \frac{\left(\phi(p\cdot q) - 2\cdot p\cdot q\right)}{2}\cdot\left(\frac{1-p^{2\cdot k}}{1+p}\right)\cdot\left(\frac{1-q^{2\cdot r}}{1+q}\right).
$$

This completes the proof. □

Example 1. By using the Theorem 3.1, we calculate the Wiener index of the power graph $P(Z_{18})$. $p=2$, $q=3$, $k=1$, $r=2$ and $n=2\cdot 3^2$.

$$W(P(Z_{18}))=\frac{1}{2}\cdot\left(\binom{2\cdot 18}{2}-1\right)+\frac{(1-2^2)}{2}+\frac{(1-3^4)}{2}$$
$$+\frac{1}{2}\cdot(\phi(2\cdot 3)-2\cdot 2\cdot 3)\cdot\left(\frac{1-2^2}{1+2}\right)\cdot\left(\frac{1-3^4}{1+3}\right)=173.$$

Theorem 3.2. L*et* $P(Z_n)$ *be a power graph of with* $n=p^k\cdot q^r$ *vertices and* m *edges, where* p *and* q *are distinct primes,* $n\in Z^+$, k *and* r *are nonnegative integers. Then,*

$$WW(P(Z_n))=2\cdot(1-p^{2\cdot k})+2\cdot(1-q^{2\cdot r})+2\cdot(\phi(p\cdot q)-2\cdot p\cdot q)\cdot\left(\frac{1-p^{2\cdot k}}{1+p}\right)\cdot\left(\frac{1-q^{2\cdot r}}{1+q}\right)+(3\cdot n+2)\cdot(n-1). \tag{3.2.1}$$

Proof. The hyper-Wiener of $P(Z_n)$ is determined as [1]:

$$WW(P(Z_n))=\frac{1}{2}\cdot\sum_{\{u,v\}\subseteq V(P(Z_n))}d(u,v)+\frac{1}{2}\cdot\sum_{\{u,v\}\subseteq V(P(Z_n))}d^2(u,v)$$
$$=W(P(Z_n))+\frac{1}{2}\cdot\sum_{\{u,v\}\subseteq V(P(Z_n))}d^2(u,v)$$
$$=\frac{1}{2}\cdot\left(\sum_{d(u,v)=1}1+\sum_{d(u,v)=2}2\right)+\frac{1}{2}\cdot\left(\sum_{d^2(u,v)=1}1+\sum_{d^2(u,v)=4}4\right)$$
$$=\frac{1}{2}\cdot\sum_{i=1}^{n}\left(d_{\bar{i}}+2\cdot\left(p^k\cdot q^r-1-d_{\bar{i}}\right)\right)+\frac{1}{2}\cdot\sum_{i=1}^{n}\left(d_{\bar{i}}+4\cdot\left(p^k\cdot q^r-1-d_{\bar{i}}\right)\right)$$
$$=\frac{6\cdot p^k\cdot q^r\cdot(p^k\cdot q^r-1)}{2}-2\cdot\sum_{i=1}^{n}d_{\bar{i}}$$
$$=6\cdot\binom{p^k\cdot q^r}{2}-4\cdot m\ .$$

As it is, if we write here the Equation (2.2.1) in the Result 2.2 instead of *m*,

$$WW\left(P\left(Z_{p^k\cdot q^r}\right)\right)=6\cdot\binom{p^k\cdot q^r}{2}-2\cdot\sum_{d\,|\,p^k\cdot q^r}(2\cdot d-\phi(d)-1)\cdot\phi(d)$$

$$=\left(6\cdot\binom{p^k\cdot q^r}{2}+2\cdot p^k\cdot q^r\right)+2\cdot\sum_{d\,|\,p^k\cdot q^r}\phi(d)\cdot(\phi(d)-2\cdot d). \tag{*}$$

is obtained. If we arrange the equation of (2.1.1) in Theorem 2.1 in the preliminary section,

$$2\cdot\sum_{d\,|\,p^k\cdot q^r}\phi(d)\cdot(\phi(d)-2\cdot d)=4\cdot W\left(P\left(Z_{p^k\cdot q^r}\right)\right)-2\cdot\binom{2\cdot p^k\cdot q^r}{2}, \tag{**}$$

we find the above equation. Here if we write the equation $(**)$ in its place within the equation (*) and

$$WW\left(P\left(Z_{p^k\cdot q^r}\right)\right)=4\cdot W\left(P\left(Z_{p^k\cdot q^r}\right)\right)+6\cdot\binom{p^k\cdot q^r}{2}-2\cdot\binom{2\cdot p^k\cdot q^r}{2}+2\cdot p^k\cdot q^r. \qquad (***)$$

is obtained. The equation in theorem in the preliminary section, specifically equation in that equation, when written in $(***)$, yields the following results,

$$WW\left(P\left(Z_n\right)\right)=2\cdot\left(1-p^{2\cdot k}\right)+2\cdot\left(1-q^{2\cdot r}\right)+2\cdot\left(\phi\left(p\cdot q\right)-2\cdot p\cdot q\right)\cdot\left(\frac{1-p^{2\cdot k}}{1+p}\right)\cdot\left(\frac{1-q^{2\cdot r}}{1+q}\right)+\left(3\cdot n+2\right)\cdot\left(n-1\right).$$

and proof is completed. □

Example 2. By using Theorem 3.2 we calculate the hyper-Wiener index of the power graph $P\left(Z_{18}\right)$. $p=2$, $q=3$, $k=1$, $r=2$ and $n=2\cdot 3^2$.

$$WW\left(P\left(Z_{18}\right)\right)=2\cdot\left(1-2^2\right)+2\cdot\left(1-3^4\right)+2\cdot\left(\phi\left(2\cdot 3\right)-2\cdot 2\cdot 3\right)\cdot\left(\frac{1-2^2}{1+2}\right)\cdot\left(\frac{1-3^4}{1+3}\right)+\left(3\cdot 18+2\right)\cdot\left(18-1\right)$$
$$=386.$$

Theorem 3.3. *Let $P\left(Z_n\right)$ be a power graph with $n=p^k\cdot q^r$ vertices and m edges, where p and q are distinct prime numbers, $n\in Z^+$, k and r are nonnegative integers. Then*

$$H\left(P\left(Z_n\right)\right)=\frac{\left(n-1\right)^2}{2}-\frac{\left(1-p^{2k}\right)}{2}-\frac{\left(1-q^{2r}\right)}{2}-\frac{1}{2}\cdot\left(\phi\left(p\cdot q\right)-2\cdot p\cdot q\right)\cdot\left(\frac{1-p^{2k}}{1+p}\right)\cdot\left(\frac{1-q^{2r}}{1+q}\right). \qquad (3.3.1)$$

Proof. Let $P\left(Z_n\right)$, be a power graph with $n=p^k\cdot q^r$ vertices and m edges, where p and q are distinct prime numbers, $n\in Z^+$, k and r are nonnegative integers. The Harary index of the power graph $P\left(Z_{p^k\cdot q^r}\right)$ is obtained [1].

$$H\left(P\left(Z_{p^k\cdot q^r}\right)\right)=\sum_{\{u,v\}\subseteq V\left(P\left(Z_{p^k\cdot q^r}\right)\right)}\frac{1}{d\left(u,v\right)}=\left(\sum_{d(u,v)^{-1}=1}1+\sum_{d(u,v)^{-1}=\frac{1}{2}}\frac{1}{2}\right)$$
$$=\sum_{i=1}^{p^k\cdot q^r}\left(d_{\bar{i}}+\frac{1}{2}\cdot\left(p^k\cdot q^r-1-d_{\bar{i}}\right)\right)=\binom{p^k\cdot q^r}{2}+m.$$

On the other hand, Equation (2.2.1) in Result 2.2 is substituted in m and is found the following result.

$$H\left(P\left(Z_{p^k\cdot q^r}\right)\right)=\binom{p^k\cdot q^r}{2}-\frac{1}{2}\cdot\sum_{d|n}\phi\left(d\right)\cdot\left(\phi\left(d\right)-2\cdot d\right)-\frac{1}{2}\cdot\sum_{d|n}\cdot\left(d\right).$$

$\sum_{d|n}\phi\left(d\right)=n$, if this equation use, is found the following result

$$H\left(P\left(Z_n\right)\right)=\binom{n}{2}-\frac{1}{2}\cdot\sum_{d|n}\phi\left(d\right)\cdot\left(\phi\left(d\right)-2\cdot d\right)-\frac{n}{2}$$
$$=\frac{n\cdot\left(n-2\right)}{2}-\frac{1}{2}\cdot\sum_{d|n}\phi\left(d\right)\cdot\left(\phi\left(d\right)-2\cdot d\right). \qquad (*)$$

According to the (**) equations in Theorem 3.2, the equation is found $(*)$,

$$-\frac{1}{2}\cdot\sum_{d|n}\phi(d)\cdot(\phi(d)-2\cdot d)=-W(P(Z_n))+\frac{1}{2}\cdot\binom{2\cdot n}{2}$$

$$H(P(Z_n))=3\cdot\binom{n}{2}-W(P(Z_n)).$$

If the equations in Theorem 3.1, specifically Equation (3.1.1), are used, the proof is completed as follows.

$$H(P(Z_n))=\frac{(n-1)^2}{2}-\frac{(1-p^{2k})}{2}-\frac{(1-q^{2r})}{2}-\frac{1}{2}\cdot(\phi(p\cdot q)-2\cdot p\cdot q)\cdot\left(\frac{1-p^{2k}}{1+p}\right)\cdot\left(\frac{1-q^{2r}}{1+q}\right). \qquad \square$$

Example 3. By using Theorem 3.3, we calculate the Harary index of the power graph $P(Z_{18})$.

$p=2$, $q=3$, $k=1$, $r=2$ and $n=2\cdot 3^2$.

$$H(P(Z_{18}))=\frac{(18-1)^2}{2}-\frac{(1-2^2)}{2}-\frac{(1-3^4)}{2}-\frac{1}{2}\cdot(\phi(2\cdot 3)-2\cdot 2\cdot 3)\cdot\left(\frac{1-2^2}{1+2}\right)\cdot\left(\frac{1-3^4}{1+3}\right)$$
$$=286.$$

Theorem 3.4. L*et* $P(Z_n)$, *be a power graph with* $n=p^k\cdot q^r$ *vertices and* m *edges, where* p *and* q *are distinct prime numbers,* $n\in Z^+$, k *and* r *are nonnegative integers.*

$$SK\left(P\left(Z_{p^k\cdot q^r}\right)\right)=\sum_{i=0}^{k}\frac{\left(\phi\left(p^{k-i}\cdot q^r\right)\cdot d_{\frac{2}{p^i}}\right)}{2}+\sum_{j=0}^{k}\sum_{i=1}^{r}\frac{\left(\phi\left(p^{k-j}\cdot q^{r-i}\right)\cdot d_{\frac{2}{p^j\cdot q^i}}\right)}{2}. \qquad (3.4.1)$$

Proof. Let $P(Z_n)$, be a power graph with $n=p^k\cdot q^r$ vertices and m edges, where p and q are distinct prime numbers, $n\in Z^+$, k and r are nonnegative integers. All the positive divisors of $n=p^k\cdot q^r$ is found as follows.

$$\{1, p, p^2, p^3,\ldots, p^k\},$$
$$\{q, q^2, q^3,\ldots, q^r\},$$
$$\{p\cdot q, p\cdot q^2, p\cdot q^3,\ldots, p\cdot q^r\},$$
$$\{p^2\cdot q, p^2\cdot q^2, p^2\cdot q^3,\ldots, p^2\cdot q^r\},$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$\{p^k\cdot q, p^k\cdot q^2, p^k\cdot q^3,\ldots, p^k\cdot q^r\}$$

Let $a\cdot\overline{1}\in Z_{p^k\cdot q^r}$ be. The number of generators of the cyclic group $a\cdot\overline{1}$ is $\phi\left(\frac{n}{(n,a)}\right)$. Moreover, $o(a\cdot\overline{1})=\frac{n}{(n,a)}$ is. The power graph SK index is found as $P\left(Z_{p^k\cdot q^r}\right)$

$$SK\left(P\left(Z_{p^k\cdot q^r}\right)\right)=\frac{1}{2}\cdot\sum_{uv\in E\left(P\left(Z_{p^k\cdot q^r}\right)\right)}\left(d_u+d_v\right). \tag{*}$$

Definition 2.3 corresponds to

$$M_1\left(P\left(Z_{p^k\cdot q^r}\right)\right)=\sum_{u\in V\left(P\left(Z_{p^k\cdot q^r}\right)\right)}d_u{}^2=\sum_{uv\in E\left(P\left(Z_{p^k\cdot q^r}\right)\right)}\left(d_u+d_v\right). \tag{**}$$

So, from (*) and (**),

$$SK\left(P\left(Z_{p^k\cdot q^r}\right)\right)=\frac{1}{2}\cdot\sum_{u\in V\left(P\left(Z_{p^k\cdot q^r}\right)\right)}d_u{}^2$$

$$=\frac{1}{2}\cdot\left(\phi\left(\frac{n}{(1,n)}\right)\cdot d_{\bar{1}}{}^2+\phi\left(\frac{n}{(p,n)}\right)\cdot d_{\bar{p}}{}^2+\phi\left(\frac{n}{(p^2,n)}\right)\cdot d_{\overline{p^2}}{}^2\right)$$

$$+\frac{1}{2}\cdot\left(\phi\left(\frac{n}{(p^3,n)}\right)\cdot d_{\overline{p^3}}{}^2+\cdots+\phi\left(\frac{n}{(p^k,n)}\right)\cdot d_{\overline{p^k}}{}^2+\phi\left(\frac{n}{(q,n)}\right)\cdot d_{\bar{q}}{}^2\right)$$

$$+\frac{1}{2}\cdot\left(\phi\left(\frac{n}{(q^2,n)}\right)\cdot d_{\overline{q^2}}{}^2+\phi\left(\frac{n}{(q^3,n)}\right)\cdot d_{\overline{q^3}}{}^2\right)+\cdots+\frac{1}{2}\cdot\phi\left(\frac{n}{(q^r,n)}\right)\cdot d_{\overline{q^r}}{}^2$$

$$+\frac{1}{2}\cdot\left(\phi\left(\frac{n}{(p\cdot q,n)}\right)\cdot d_{\overline{p\cdot q}}{}^2+\phi\left(\frac{n}{(p\cdot q^2,n)}\right)\cdot d_{\overline{p\cdot q^2}}{}^2\right)+\cdots$$

$$+\frac{1}{2}\cdot\left(\phi\left(\frac{n}{(p\cdot q^r,n)}\right)\cdot d_{\overline{p\cdot q^r}}{}^2+\phi\left(\frac{n}{(p^2\cdot q,n)}\right)\cdot d_{\overline{p^2\cdot q}}{}^2+\right)$$

$$+\frac{1}{2}\cdot\left(\phi\left(\frac{n}{(p^2\cdot q^2,n)}\right)\cdot d_{\overline{p^2\cdot q^2}}{}^2+\phi\left(\frac{n}{(p^2\cdot q^3,n)}\right)\cdot d_{\overline{p^2\cdot q^3}}{}^2\right)+\cdots+$$

$$+\frac{1}{2}\cdot\left(\phi\left(\frac{n}{(p^2\cdot q^r,n)}\right)\cdot d_{\overline{p^2\cdot q^r}}{}^2+\phi\left(\frac{n}{(p^3\cdot q,n)}\right)\cdot d_{\overline{p^3\cdot q}}{}^2+\right)$$

$$+\frac{1}{2}\cdot\left(\phi\left(\frac{n}{(p^3\cdot q^2,n)}\right)\cdot d_{\overline{p^3\cdot q^2}}{}^2+\phi\left(\frac{n}{(p^3\cdot q^3,n)}\right)\cdot d_{\overline{p^3\cdot q^3}}{}^2\right)+\cdots+$$

$$+\frac{1}{2}\cdot\phi\left(\frac{n}{(p^3\cdot q^r,n)}\right)\cdot d_{\overline{p^3\cdot q^r}}{}^2+\cdots+\frac{1}{2}\cdot\phi\left(\frac{n}{(p^k\cdot q,n)}\right)\cdot d_{\overline{p^k\cdot q}}{}^2$$

$$+\frac{1}{2}\cdot\left(\phi\left(\frac{n}{(p^k\cdot q^2,n)}\right)\cdot d_{\overline{p^k\cdot q^2}}{}^2+\phi\left(\frac{n}{(p^k\cdot q^3,n)}\right)\cdot d_{\overline{p^k\cdot q^3}}{}^2\right)+\cdots+$$

$$+\frac{1}{2}\cdot\phi\left(\frac{n}{(n,n)}\right)\cdot d_{\bar{n}=\bar{0}}{}^2.$$

This equation is obtained. If necessary arrangements are made by taking them in common brackets, the and proof is completed:

$$SK\left(P\left(Z_{p^k\cdot q^r}\right)\right)=\frac{1}{2}\cdot\sum_{i=0}^{k}\left(\phi\left(p^{k-i}\cdot q^r\right)\cdot d_{\overline{p^i}}{}^2\right)+\frac{1}{2}\cdot\sum_{j=0}^{k}\sum_{i=1}^{r}\left(\phi\left(p^{k-j}\cdot q^{r-i}\right)\cdot d_{\overline{p^j\cdot q^i}}{}^2\right).SK\left(P\left(Z_{2\cdot 3^2}\right)\right)$$

$$=\sum_{i=0}^{1}\frac{\left(\cdot\left(3^2\right)\cdot d_{\overline{2^i}}{}^2\cdot\phi\left(2^{1-i}\right)\right)}{2}+\sum_{j=0}^{1}\sum_{i=1}^{2}\frac{\left(\phi\left(2^{1-j}\right)\cdot d_{\overline{2^j\cdot 3^i}}{}^2\cdot\phi\left(3^{2-i}\right)\right)}{2}. \qquad \square$$

Example 4. By using Theorem 3.4, we calculate the SK index of the power graph $P\left(Z_{18}\right)$. $p=2$, $q=3$, $k=1$, $r=2$ and $n=2\cdot 3^2$.

$$SK\left(P\left(Z_{2\cdot 3^2}\right)\right)=\sum_{i=0}^{1}\frac{\left(\phi\left(3^2\right)\cdot d_{\overline{2^i}}{}^2\cdot\phi\left(2^{1-i}\right)\right)}{2}+\sum_{j=0}^{1}\sum_{i=1}^{2}\frac{\left(\phi\left(2^{1-j}\right)\cdot d_{\overline{2^j\cdot 3^i}}{}^2\cdot\phi\left(3^{2-i}\right)\right)}{2}$$

$$d_{\bar{0}}=d_{\bar{1}}=17,\ d_{\bar{2}}=14,\ d_{\bar{3}}=11,\ d_{\bar{6}}=16,\ d_{\bar{9}}=9.$$

$$SK\left(P\left(Z_{2\cdot 3^2}\right)\right)=\frac{1}{2}\cdot\left(6\cdot d_{\bar{1}}{}^2\cdot\phi(2)+6\cdot d_{\bar{2}}{}^2+d_{\bar{3}}{}^2\cdot\phi(3)+d_{\bar{9}}{}^2+2\cdot d_{\bar{6}}{}^2+d_{\bar{0}}{}^2\right)=2017.$$

4 Conclusions

The power graph of the $(Z_{p^k\cdot q^r},+)$ group, denoted as $P(Z_{p^k\cdot q^r})$, includes distinct prime numbers *p* and *q*, along with nonnegative integers *k* and *r*. Employing (utilizing) the Euler ϕ function, this power graph enables us to produce the derivation of Wiener, hyper-Wiener, Harary, and SK theorems. Results related to the calculations of these indices by Wiener, hyper-Wiener, Harary, and *SK* are obtained. The other indices of the $P(Z_{p^k\cdot q^r})$ power graph can also be studied and calculated by other researchers in the future. In addition it can be said that this kind of studies should be enhanced because it is possible that the index calculations of the power graph of $P\left(Z_n\right)$ are furthermore conducted in terms of any nonnegative integer *n*.

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