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# New Fibonacci-type pulsated sequences

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**Abstract:** A new Fibonacci-type sequence from pulsated type is introduced. The explicit form of its members is given.

**Keywords:** Fibonacci sequence, Pulsated sequence.
**2020 Mathematics Subject Classification:** 11B39.

## 1 Introduction

Approximately 40 years ago, the first extension of the Fibonacci sequence in the form of two or more sequences, was introduced in [5].

This idea was developed in different directions (see, e.g., [1–4, 6–8, 10–12]). One of these new ideas was introduced in [3, 4]. This direction was called *pulsated sequences.*

In the present paper, a new type of pulsated sequences is constructed.

## 2 Main results

Let $a, b, c, d$ be fixed real numbers. Let us define the following Fibonacci sequence(s) of pulsated type:

$$\begin{aligned}
\alpha_0 &= a,\\
\beta_1 &= b,\\
\alpha_1 &= c,\\
\gamma_1 &= d,\\
\beta_{3k+2} &= \beta_{3k+1} + \alpha_{3k},\\
\alpha_{3k+2} &= \alpha_{3k+1} + \alpha_{3k},\\
\gamma_{3k+2} &= \gamma_{3k+1} + \alpha_{3k},\\
\alpha_{3k+3} &= \alpha_{3k+2} + \beta_{3k+2} + \gamma_{3k+2},\\
\beta_{3k+4} &= \alpha_{3k+3} + \gamma_{3k+2},\\
\alpha_{3k+4} &= \alpha_{3k+3} + \alpha_{3k+2},\\
\gamma_{3k+4} &= \alpha_{3k+3} + \beta_{3k+2},
\end{aligned}$$

where $k \geq 0$ is an integer. The first members of this new sequence are the following:

| $n$ | $\beta_n$ | $\alpha_n$ | $\gamma_n$ |
|---|---|---|---|
| 0 | | $a$ | |
| 1 | $b$ | $c$ | $d$ |
| 2 | $a+b$ | $a+c$ | $a+d$ |
| 3 | | $3a+b+c+d$ | |
| 4 | $4a+b+c+2d$ | $4a+b+2c+d$ | $4a+2b+c+d$ |
| 5 | $7a+2b+2c+3d$ | $7a+2b+3c+2d$ | $7a+3b+2c+2d$ |
| 6 | | $7(3a+b+c+d)$ | |
| 7 | $28a+10b+9c+9d$ | $28a+9b+10c+9d$ | $28a+9b+9c+10d$ |
| 8 | $49a+17b+16c+16d$ | $49a+16b+17c+16d$ | $49a+16b+16c+17d$ |
| 9 | | $49(3a+b+c+d)$ | |
| 10 | $196a+65b+65c+66d$ | $196a+65b+66c+65d$ | $196a+66b+65c+65d$ |
| 11 | $343a+114b+114c+115d$ | $343a+114b+115c+114d$ | $343a+115b+114c+114d$ |
| 12 | | $343(3a+b+c+d)$ | |
| 13 | $1372a+458b+457c+457d$ | $1372a+456b+457c+456d$ | $1373a+457b+457c+458d$ |
| 14 | $2401a+801b+800c+800d$ | $2401a+800b+801c+800d$ | $2401a+800b+800c+801d$ |
| 15 | | $2401(3a+b+c+d)$ | |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

**Theorem.** For every four real numbers $a, b, c, d$ and for every integer $k \geq 1$:

$$\begin{aligned}
\beta_{6k+1} &= 4 \cdot 7^{2k-1}a + \left(\tfrac{4\cdot 7^{2k-1}-1}{3} + 1\right) b + \tfrac{4\cdot 7^{2k-1}-1}{3}c + \tfrac{4\cdot 7^{2k-1}-1}{3}d,\\
\alpha_{6k+1} &= 4 \cdot 7^{2k-1}a + \tfrac{4\cdot 7^{2k-1}-1}{3}b + \left(\tfrac{4\cdot 7^{2k-1}-1}{3} + 1\right) c + \tfrac{4\cdot 7^{2k-1}-1}{3}d,\\
\gamma_{6k+1} &= 4 \cdot 7^{2k-1}a + \tfrac{4\cdot 7^{2k-1}-1}{3}b + \tfrac{4\cdot 7^{2k-1}-1}{3}c + \left(\tfrac{4\cdot 7^{2k-1}-1}{3} + 1\right) d,\\
\beta_{6k+2} &= 7^{2k}a + \left(\tfrac{7^{2k}-1}{3} + 1\right) b + \tfrac{7^{2k}-1}{3}c + \tfrac{7^{2k}-1}{3}d,\\
\alpha_{6k+2} &= 7^{2k}a + \tfrac{7^{2k}-1}{3}b + \left(\tfrac{7^{2k}-1}{3} + 1\right) c + \tfrac{7^{2k}-1}{3}d,\\
\gamma_{6k+2} &= 7^{2k}a + \tfrac{7^{2k}-1}{3}b + \tfrac{7^{2k}-1}{3}c + \left(\tfrac{7^{2k}-1}{3} + 1\right) d,\\
\alpha_{6k+3} &= 7^{2k}(3a + b + c + d),\\
\beta_{6k+4} &= 4 \cdot 7^{2k}a + \tfrac{4\cdot 7^{2k}-1}{3}b + \tfrac{4\cdot 7^{2k}-1}{3}c + \left(\tfrac{4\cdot 7^{2k}-1}{3} + 1\right) d,\\
\alpha_{6k+4} &= 4 \cdot 7^{2k}a + \tfrac{4\cdot 7^{2k}-1}{3}b + \left(\tfrac{4\cdot 7^{2k}-1}{3} + 1\right) c + \tfrac{4\cdot 7^{2k}-1}{3}d,\\
\gamma_{6k+4} &= 4 \cdot 7^{2k}a + \left(\tfrac{4\cdot 7^{2k}-1}{3}\right) b + \tfrac{4\cdot 7^{2k}-1}{3}c + \tfrac{4\cdot 7^{2k}-1}{3}d,\\
\beta_{6k+5} &= 7^{2k+1}a + \tfrac{7^{2k+1}+1}{3}b + \tfrac{7^{2k+1}-1}{3}c + \left(\tfrac{7^{2k+1}-1}{3} + 1\right) d,\\
\alpha_{6k+5} &= 7^{2k+1}a + \tfrac{7^{2k+1}-1}{3}b + \left(\tfrac{7^{2k+1}-1}{3} + 1\right) c + \tfrac{7^{2k+1}-1}{3}d,\\
\gamma_{6k+5} &= 7^{2k+1}a + \left(\tfrac{7^{2k+1}-1}{3} + 1\right) b + \tfrac{7^{2k+1}-1}{3}c + \tfrac{7^{2k+1}-1}{3}d,\\
\alpha_{6k+6} &= 7^{2k+1}(3a + b + c + d).
\end{aligned}$$

**Proof.** For $k = 1$ the assertion is valid (see the above table). Let us assume that it is valid for some $k$. Then

$$\begin{aligned}
\beta_{6(k+1)+1} &= \beta_{6k+7}\\
&= \alpha_{6k+6} + \gamma_{6k+5}\\
&= 7^{2k+1}(3a + b + c + d) + 7^{2k+1}a + \left(\tfrac{7^{2k+1}-1}{3} + 1\right) b + \tfrac{7^{2k+1}-1}{3}c + \tfrac{7^{2k+1}-1}{3}d\\
&= 4 \cdot 7^{2k+1}a + \left(\tfrac{4\cdot 7^{2k+1}-1}{3} + 1\right) b + \tfrac{4\cdot 7^{2k+1}-1}{3}c + \tfrac{4\cdot 7^{2k+1}-1}{3}d.\\
\alpha_{6(k+1)+1} &= \alpha_{6k+7}\\
&= \alpha_{6k+6} + \alpha_{6k+5}\\
&= 7^{2k+1}(3a + b + c + d) + 7^{2k+1}a + \tfrac{7^{2k+1}-1}{3}b + \left(\tfrac{7^{2k+1}-1}{3} + 1\right) c + \tfrac{7^{2k+1}-1}{3}d,\\
&= 4 \cdot 7^{2k+1}a + \tfrac{4\cdot 7^{2k+1}-1}{3}b + \left(\tfrac{4\cdot 7^{2k+1}-1}{3} + 1\right) c + \tfrac{4\cdot 7^{2k+1}-1}{3}d.
\end{aligned}$$

$$
\begin{aligned}
\gamma_{6(k+1)+1} &= \gamma_{6k+7} \\
&= \alpha_{6k+6} + \beta_{6k+5} \\
&= 7^{2k+1}(3a+b+c+d) + 7^{2k+1}a + \tfrac{7^{2k+1}+1}{3}b + \tfrac{7^{2k+1}-1}{3}c + \left(\tfrac{7^{2k+1}-1}{3}+1\right)d \\
&= 4\cdot 7^{2k+1}a + \tfrac{4\cdot 7^{2k+1}-1}{3}b + \tfrac{4\cdot 7^{2k+1}-1}{3}c + \left(\tfrac{4\cdot 7^{2k+1}-1}{3}+1\right)d. \\
\beta_{6(k+1)+2} &= \beta_{6k+8} \\
&= \beta_{6k+7} + \alpha_{6k+6} \\
&= 4\cdot 7^{2k+1}a + \left(\tfrac{4\cdot 7^{2k+1}-1}{3}+1\right)b + \tfrac{4\cdot 7^{2k+1}-1}{3}c + \tfrac{4\cdot 7^{2k+1}-1}{3}d + 7^{2k+1}(3a+b+c+d) \\
&= 7^{2k+2}a + \left(\tfrac{7^{2k+2}-1}{3}+1\right)b + \tfrac{7^{2k+2}-1}{3}c + \tfrac{7^{2k+2}-1}{3}d. \\
\alpha_{6(k+1)+2} &= \alpha_{6k+8} \\
&= \alpha_{6k+7} + \alpha_{6k+6} \\
&= 4\cdot 7^{2k+1}a + \tfrac{4\cdot 7^{2k+1}-1}{3}b + \left(\tfrac{4\cdot 7^{2k+1}-1}{3}+1\right)c + \tfrac{4\cdot 7^{2k+1}-1}{3}d + 7^{2k+1}(3a+b+c+d) \\
&= 7^{2k+2}a + \tfrac{7^{2k+2}-1}{3}b + \left(\tfrac{7^{2k+2}-1}{3}+1\right)c + \tfrac{7^{2k+2}-1}{3}d. \\
\gamma_{6(k+1)+2} &= \gamma_{6k+8} \\
&= \gamma_{6k+7} + \alpha_{6k+6} \\
&= 4\cdot 7^{2k+1}a + \tfrac{4\cdot 7^{2k+1}-1}{3}b + \tfrac{4\cdot 7^{2k+1}-1}{3}c + \left(\tfrac{4\cdot 7^{2k+1}-1}{3}+1\right)d + 7^{2k+1}(3a+b+c+d) \\
&= 7^{2k+2}a + \tfrac{7^{2k+2}-1}{3}b + \tfrac{7^{2k+2}-1}{3}c + \left(\tfrac{7^{2k+2}-1}{3}+1\right)d. \\
\alpha_{6(k+1)+3} &= \alpha_{6k+9} \\
&= \beta_{6k+8} + \alpha_{6k+8} + \gamma_{6k+8} \\
&= 7^{2k+2}a + \left(\tfrac{7^{2k+2}-1}{3}+1\right)b + \tfrac{7^{2k+2}-1}{3}c + \tfrac{7^{2k+2}-1}{3}d \\
&+ 7^{2k+2}a + \tfrac{7^{2k+2}-1}{3}b + \left(\tfrac{7^{2k+2}-1}{3}+1\right)c + \tfrac{7^{2k+2}-1}{3}d \\
&+ 7^{2k+2}a + \tfrac{7^{2k+2}-1}{3}b + \tfrac{7^{2k+2}-1}{3}c + \left(\tfrac{7^{2k+2}-1}{3}+1\right)d \\
&= 7^{2k+2}(3a+b+c+d).
\end{aligned}
$$

The remaining equalities are checked in the same manner. □

# 3 Conclusion

The Finonacci-type sequence discussed in the paper has a new form. In future, we will define other sequences of the same type and their properties will be studied.

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