Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 4, 774–788
DOI: 10.7546/nntdm.2023.29.4.774-788

n-Rooks and n-queens problem on planar and modular chessboards with hexagonal cells

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Received: 17 June 2022 **Revised:** 27 July 2023
Accepted: 25 November 2023 **Online First:** 27 November 2023

Abstract: We show the existence of solutions to the n-rooks problem and n-queens problem on chessboards with hexagonal cells, problems equivalent to certain three and six direction riders on ordinary chessboards. Translating the problems into graph theory problems, we determine the independence number (maximum size of independent set) of rooks graph and queens graph. We consider the $n \times n$ planar diamond-shaped H_n with hexagonal cells, and the board H_n as a flat torus T_n. Here, a rook can execute moves on lines perpendicular to the six sides of the cell it is placed, and a queen can execute moves on those lines together with lines through the six corners of the cell it is placed.

Keywords: n-Queens problem, n-Rooks problem, Non-attacking fairy chess riders, Queens graph, Rooks graph, Independent sets.
2020 Mathematics Subject Classification: 05C69, 05C99.

1 Introduction

The problem to place n queens on the $n \times n$ chessboard so that no two queens attack each other is referred to as the *n-queens problem*. This problem dates back to 1848, when Max Bezzel [2] posed the problem of putting eight non-attacking queens on a standard chessboard. The n-queens problems have been widely studied (see [3, 9, 20] and references therein). Several extensions of the n-queens problem, that is considering other board topologies and dimensions [4–7, 12, 14], have also been explored. As stated in [1], Polya [15] is the first to consider the n-queens problem on an $n \times n$ *modular chessboard* $-$ a torus formed from an $n \times n$ chessboard with opposite sides identified. He proved that a solution to the n-queens problem exists on an $n \times n$ modular board if and only if $\gcd(n, 6) = 1$. Monsky [13] determined the maximum number of non-attacking queens for other values of n. A very comprehensive survey of past results on n-queens problem was done by Bell and Stevens [1].

The n-queens problem on a board with hexagonal cells called a square beehive where a 'queen' can execute moves on lines perpendicular to any of the six sides of a cell on which it is placed, has been investigated by Theron and Geldenhuys in [18]. In [8], Joe DeMaio and Hong Lien Tran considered the problem of placing the maximum number of non-attacking chess pieces on a triangular honeycomb chessboard. Wagon [19] presented properties of graphs related to chess pieces on a triangular chessboard.

Analogous to the n-queens problem is the n-rooks problem which focuses on placing n rooks on the $n \times n$ chessboard so that no two rooks attack each other.

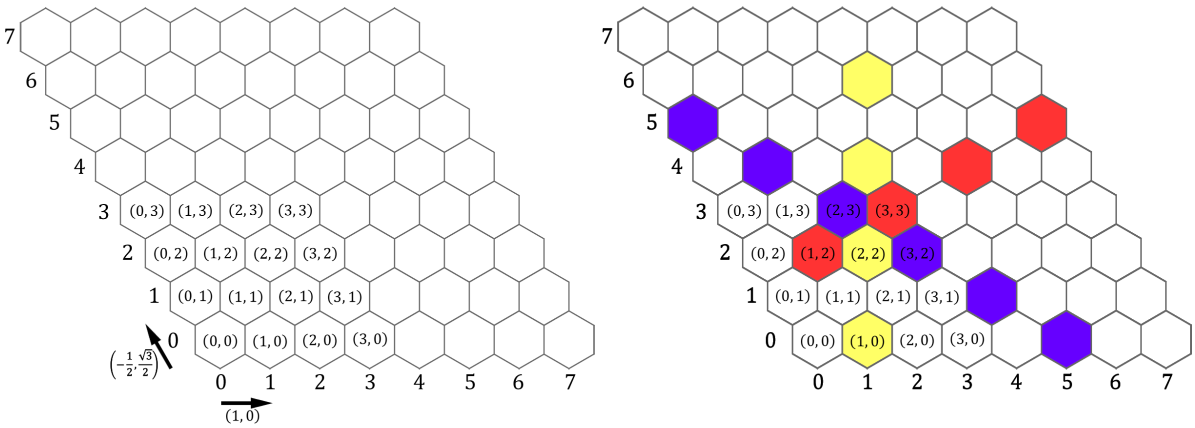
In this work, we show the existence of solutions to the n-queens problem on a planar diamond-shaped chessboard H_n with regular hexagonal cells of H. E. de Vasa [16] where there are n cells on each side of the board. Here, the queen can execute moves on lines perpendicular the six sides or through the six corners of the cell it is placed. We reformulate the problems into graph theory problems dealing with the rooks graph and queens graph. Determining an independent set (with size n) of a rooks graph or queens graph respectively corresponds to determining a solution to n-rooks or n-queens problem. In Section 2, we introduce a coordinate system of the chessboard and the preliminaries on chess-piece graphs. Sections 3 and 4 focus on the independent sets of rooks graph and queens graph on H_n respectively. In Section 5, results are presented when we consider the board H_n as a flat torus T_n.

When our skewed coordinates below are reinterpreted as ordinary plane Cartesian coordinates, our problems are transformed into problems of placing non-attacking "fairy chess pieces" called "compound rides" on a square chessboard [11]. A rook becomes a rider with three move directions: horizontal $(1, 0)$, vertical $(0, 1)$, and $45°$ $(1, 1)$. It is called a "semi-queen" in [12]. A queen has all our rook's moves and three more: $(1, -1), (2, 1)$ and $(1, 2)$. However, the squareboard model obscures the symmetries we will investigate. Neither our latter piece nor our analysis of solutions by symmetries appear in Kotesovec's compendium [12].

2 The chessboard and chess-piece graphs

Consider a planar diamond-shaped chessboard H_n where there are n regular hexagonal cells on each side of the board (Figure 1). Let the distance between the centers of two cells with a common edge to be 1 unit. We set up a coordinate system for the board as follows. Assign the bottom left-hand corner cell with coordinates $(0, 0)$. Consider the basis vectors $(1, 0)$ and $(\frac{-1}{2}, \frac{\sqrt{3}}{2})$ in

Figure 1(a) and assign each cell with coordinates (a, b) if its center is the point $a(1,0) + b(\frac{-1}{2}, \frac{\sqrt{3}}{2})$ where $a, b \in \{0, 1, \ldots, n-1\}$.



(a) (b)

Figure 1. (a) An 8×8 diamond-shaped chessboard H_8 showing vectors $(1, 0)$ and $(\frac{-1}{2}, \frac{\sqrt{3}}{2})$ and some coordinates of the hexagonal cells.
(b) Cells belonging to one fers-line are given the same color.

On H_n, the lines through the edges of the hexagonal cells are referred to as the rows, c-diagonals ("columns") and difference diagonals (abbreviated as d-diagonals). Two cells (x_1, z_1) and (x_2, z_2) belong to a row when $z_1 = z_2$ but $x_1 \neq x_2$. They belong to a c-diagonal when $x_1 = x_2$ but $z_1 \neq z_2$. The chessboard H_n has n rows and n c-diagonals which we can number from 0 to $n-1$, beginning from the bottom left-hand corner.

Two cells (x_1, z_1) and (x_2, z_2) of H_n belong to a d-diagonal when $z_1 - x_1 = z_2 - x_2$. This d-diagonal is referred to as d-diagonal $z_1 - x_1$. H_n has $2n-1$ d-diagonals.

The lines through the corners of the hexagonal cells are refered to as *fers-lines* l_1, l_2, l_3. Two cells (x_1, z_1) and (x_2, z_2) of H_n belong to a l_1 fers-line when $z_1 + x_1 = z_2 + x_2$. They belong to a l_2 fers-line when $2x_1 - z_1 = 2x_2 - z_2$. They belong to a l_3 fers-line when $2z_1 - x_1 = 2z_2 - x_2$. The l_1, l_2, l_3 fers-lines containing cell (x_1, z_1) is referred to as $z_1 + x_1$, $2x_1 - z_1$, $2z_1 - x_1$ respectively. There are $2n-1, 3n-2$ and $3n-2$ $l_1, l_2,$ and l_3 fers-lines respectively. In Figure 1(b), cells in l_1 fers-line, l_2 fers-line, and l_3 fers-line are colored blue, yellow, and red, respectively.

In this variant chess [16], a rook moves along rows, c-diagonals and d-diagonals, while a bishop moves along fers-lines. A queen is a combination of a rook and a bishop, that is, it moves along rows, c-diagonals, d-diagonals and fers-lines.

The n-rooks and n-queens problems are commonly translated into graph theory problems known as domination and independence. Vertices of the graphs represent the cells of the board and are considered *adjacent* if and only if the chess-piece in question can legally move between the corresponding cells. In particular, the vertices of the *rooks graph* R_n^h and the *queens graph* Q_n^h obtained from H_n are the n^2 hexagonal cells of H_n. In R_n^h, two cells are adjacent if they belong to a row, a c-diagonal or a d-diagonal, while in Q_n^h two cells are adjacent if they belong to a row, a c-diagonal, a d-diagonal or a fers-line.

A rook on a cell (x, y) of R^h_n is said to *dominate* (x, y) and any cell adjacent to (x, y). A set S of cells is a *dominating set* of R^h_n if every cell of R^h_n is either in S or adjacent to a cell in S. If no two cells in S are adjacent, then S is an *independent set*. The *independence number* of R^h_n, denoted by $\beta(R^h_n)$ is the maximum size of an independent set of R^h_n. An independent set with this maximum size is called a *maximum independent set*. Any maximum independent set of R^h_n must also be a dominating set of R^h_n, but not conversely. Similar definitions also hold for other chess-piece graphs, or for any graph in general.

In Figure 2(a), a rook at cell $(1, 2)$ dominates the red cells in an 8×8 chess board H_8. The set $S = \{(0, 0), (1, 2), (2, 5), (3, 7), (4, 6), (5, 4), (6, 1), (7, 3)\}$ is a dominating set of R^h_8. S is also an independent set of R^h_8. S is a maximum independent set of R^h_8 since $|S| = n$. S is also a dominating set of Q^h_8 but it is not an independent set of Q^h_8 (Figure 2(b)). In Q^h_8, the cell $(1, 2)$ is adjacent to cells $(0, 0)$ and $(5, 4)$. The cells dominated by a queen at $(1, 2)$ are colored yellow.

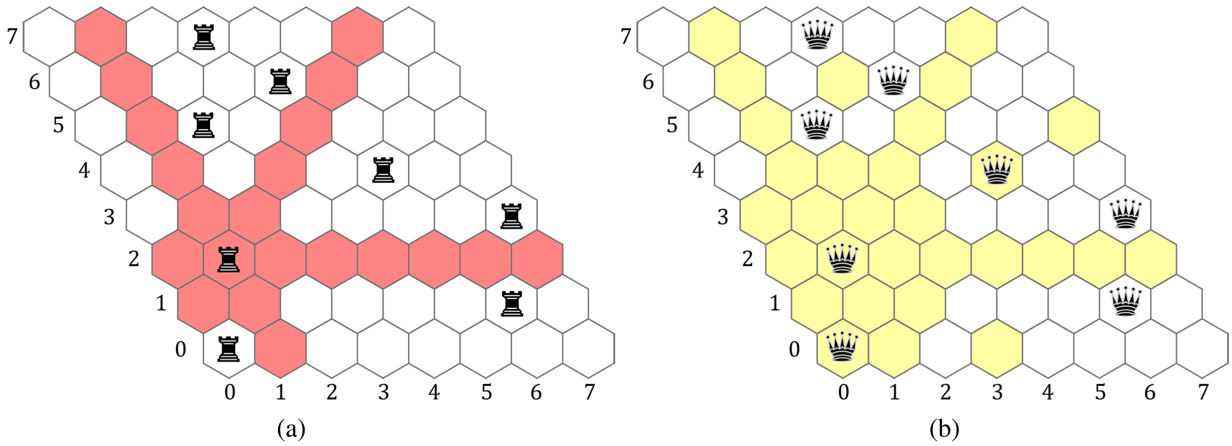


Figure 2. A dominating set of (a) R^h_8 and (b) Q^h_8. Cells dominated by a rook or a queen at cell $(1, 2)$ are colored red or yellow, respectively.

An independent set S with size $|S| = n$ is a solution to the n-rooks or n-queens problem on H_n. A solution to the n-queens problem is also a solution to the n-rooks problem, but the converse is not always true. We first show results on the n-rooks problem on H_n. Then we consider the n-queens problem on H_n.

3 The n-rooks problem on H_n

We show the existence of an independent set S of R^h_n by showing the existence of a certain permutation of the finite set $N = \{0, 1, \ldots, n-1\}$ and an injective map from N to $N' = \{-(n-1), -(n-2), \ldots, (n-2), (n-1)\}$. In the n-rooks problem, each row of H_n must contain at most one rook, hence $|S|$ should be at most n.

Theorem 3.1. *The set S with $|S| = n$ is an independent set of R^h_n if and only if $S = \{(x, f(x)) : x \in N\}$, where f is a permutation of N such that the function $h : N \to N'$ defined by $h(x) = f(x) - x$ is an injective map.*

Proof. Suppose $S = \{(x : f(x)) : x \in N\}$, where f is a permutation of N, and $h(x) = f(x) - x$ is injective. Since f is a permutation then each row and each c-diagonal of H_n contains an element of S, and no two elements of S are in the same row nor in the same c-diagonal. No two elements of S are in the same d-diagonal because h is injective. Thus S is an independent set of R_n^h with $|S| = n$.

Conversely, if S is an independent set of R_n^h with $|S| = n$, then each row and c-diagonal of R_n^h contains exactly one element of S. So for each pair $x, y \in N$ with $x \neq y$, then $f(x) \neq f(y)$; that is f is an injective map. Since N is finite, then f is also a surjective map. Thus f is a permutation of N.

Now, since S is an independent set then any two cells $(x, f(x))$ and $(x', f(x'))$ in S do not belong to the same d-diagonal, that is, $f(x) - x \neq f(x') - x'$. Thus for any $x, x' \in N$, $h(x) \neq h(x')$. So $h : N \to N'$ is an injective map. □

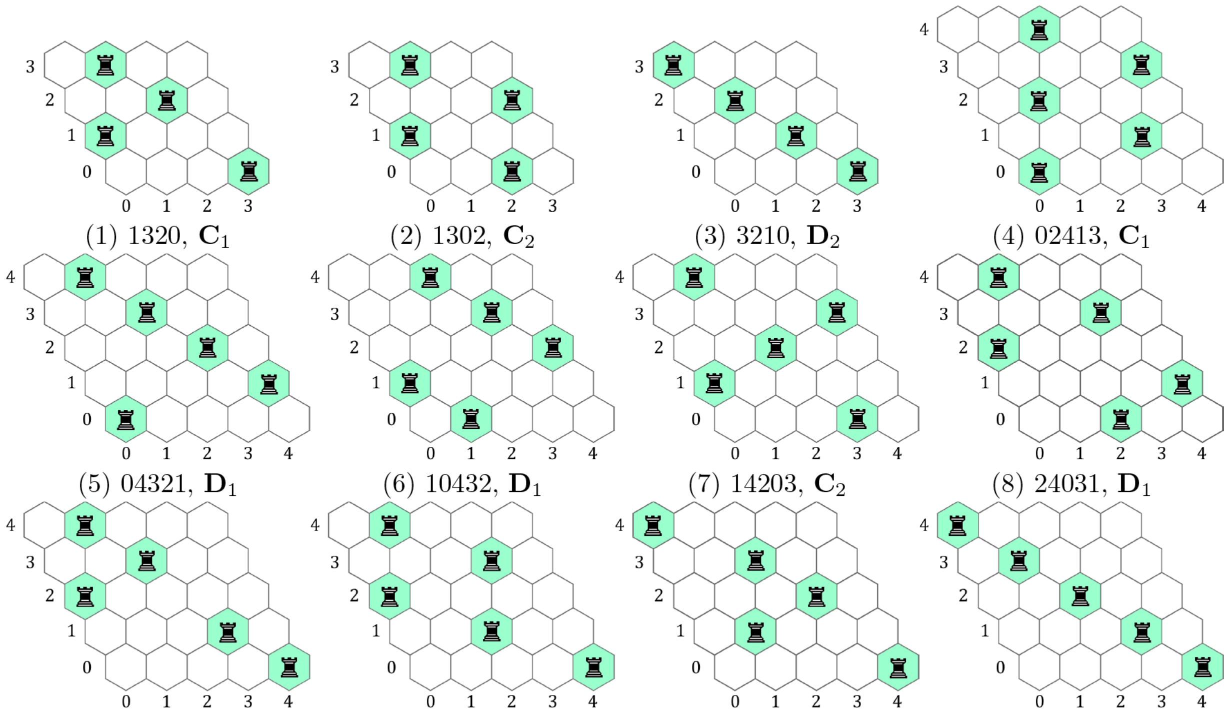
It should be noted that by Theorem 3.1, checking a permutation to be a solution by testing injectivity of certain functions gives better computational efficiency than checking pairs of chess pieces for attack. As an example, for any positive integer n, the set $S = \{(x, n-1-x) : x \in N\}$ is an independent set of R_n^h of cardinality n because $f(x) = n - 1 - x$ is a permutation of N, and $h : N \to N'$ with $h(x) = f(x) - x = n - 1 - 2x$ is injective. Since $\beta(R_n^h) \leq n$, then $\beta(R_n^h) = n$.

The chessboard H_n has symmetry group $\mathrm{Sym}(\mathrm{H}_n) \cong \mathbf{D}_2$, dihedral group of order 4. $\mathrm{Sym}(\mathrm{H}_n)$ consists of the identity i, the $180°$-rotation symmetry r at the center of H_n, and two reflection symmetries m_1, m_2 with axes bisecting H_n. If solutions to the n-rooks or n-queens problems that differ only by the symmetry operations of rotation and reflections of H_n are identified as one, we obtain *fundamental solutions*. We can classify a fundamental solution according to its symmetry, which is a subgroup of $\mathrm{Sym}(\mathrm{H}_n)$. These subgroups are $\{i\} \cong \mathbf{C}_1$, $\{i, r\} \cong \mathbf{C}_2$, $\{i, m_1\} \cong \mathbf{D}_1$, $\{i, m_2\} \cong \mathbf{D}_1$, and $\{i, r, m_1, m_2\} \cong \mathbf{D}_2$. It should be noted that if a solution to the n-rooks problem has a reflection symmetry m_2 with axis passing through the centers of cells $(0, n-1)$ and $(n-1, 0)$, then all non-attacking rooks should be placed on cells belonging to l_1 fers-line $n-1$; otherwise two rooks would belong to the same d-diagonal. In fact such a solution has $\mathbf{D}_2$ symmetry; meaning no solution exists with symmetry group $\{i, m_2\} \cong \mathbf{D}_1$. The solution with $\mathbf{D}_2$ symmetry corresponds to the independent set $S = \{(x, n-1-x) : x \in N\}$.

A solution with $\mathbf{D}_2$ or $\mathbf{C}_1$ symmetry gives rise to 1 or 4 solutions, respectively. A solution with either $\mathbf{D}_1$ or $\mathbf{C}_2$ symmetry generates 2 solutions. The fundamental solutions for $n = 4$ and $n = 5$ are presented in Figure 3. Each digit in the codes below each solution represents the rows when a rook is in c-diagonal $0, 1, 2, 3$ and so on. The symmetry of a solution is also presented at the end of the codes. When $n = 6$ there are 83 solutions, 25 are fundamental solutions which are presented in Figure 4. In Figure 4, #25 has $\mathbf{D}_2$ symmetry, while #s 13 and 18 have $\mathbf{D}_1$ symmetry. Those with $\mathbf{C}_2$ symmetry are #s $3, 16, 20, 21, 24$, while the rest have $\mathbf{C}_1$ symmetry.

Table 1 summarizes the number of solutions to the n-rooks problem on H_n for small values of n. In obtaining these solutions, a program for listing all permutations of n was modified by making sure that the test for injectivity was satisfied. The Total column of Table 1 reproduces numbers for n up to 21 given in [12] (p. 672) for rook + semi-rider $[1, 1]$ or semi-queen. This set of numbers belongs to the sequence of numbers called A099152 (see [17]), the number of $n \times n$

permutation matrices in which the sums of the entries of each NorthEast-SouthWest diagonal are 0 or 1.



(1) 1320, $\mathbf{C}_1$ (2) 1302, $\mathbf{C}_2$ (3) 3210, $\mathbf{D}_2$ (4) 02413, $\mathbf{C}_1$

(5) 04321, $\mathbf{D}_1$ (6) 10432, $\mathbf{D}_1$ (7) 14203, $\mathbf{C}_2$ (8) 24031, $\mathbf{D}_1$

(9) 24310, $\mathbf{C}_1$ (10) 24130, $\mathbf{C}_1$ (11) 41320, $\mathbf{D}_1$ (12) 43210, $\mathbf{D}_2$

Figure 3. Fundamental solutions to the n-rooks problem on H_4 (#1–#3) and on H_5(#4–#12).

Table 1. Number of solutions to the n-rooks problem on H_n for $3 \leq n \leq 7$.

n	$\mathbf{C}_1$ symmetry	$\mathbf{C}_2$ symmetry	$\mathbf{D}_1$ symmetry	$\mathbf{D}_2$ symmetry	Total
3	0	0	1	1	3
4	1	1	0	1	7
5	3	1	4	1	23
6	17	5	2	1	83
7	86	9	21	1	405

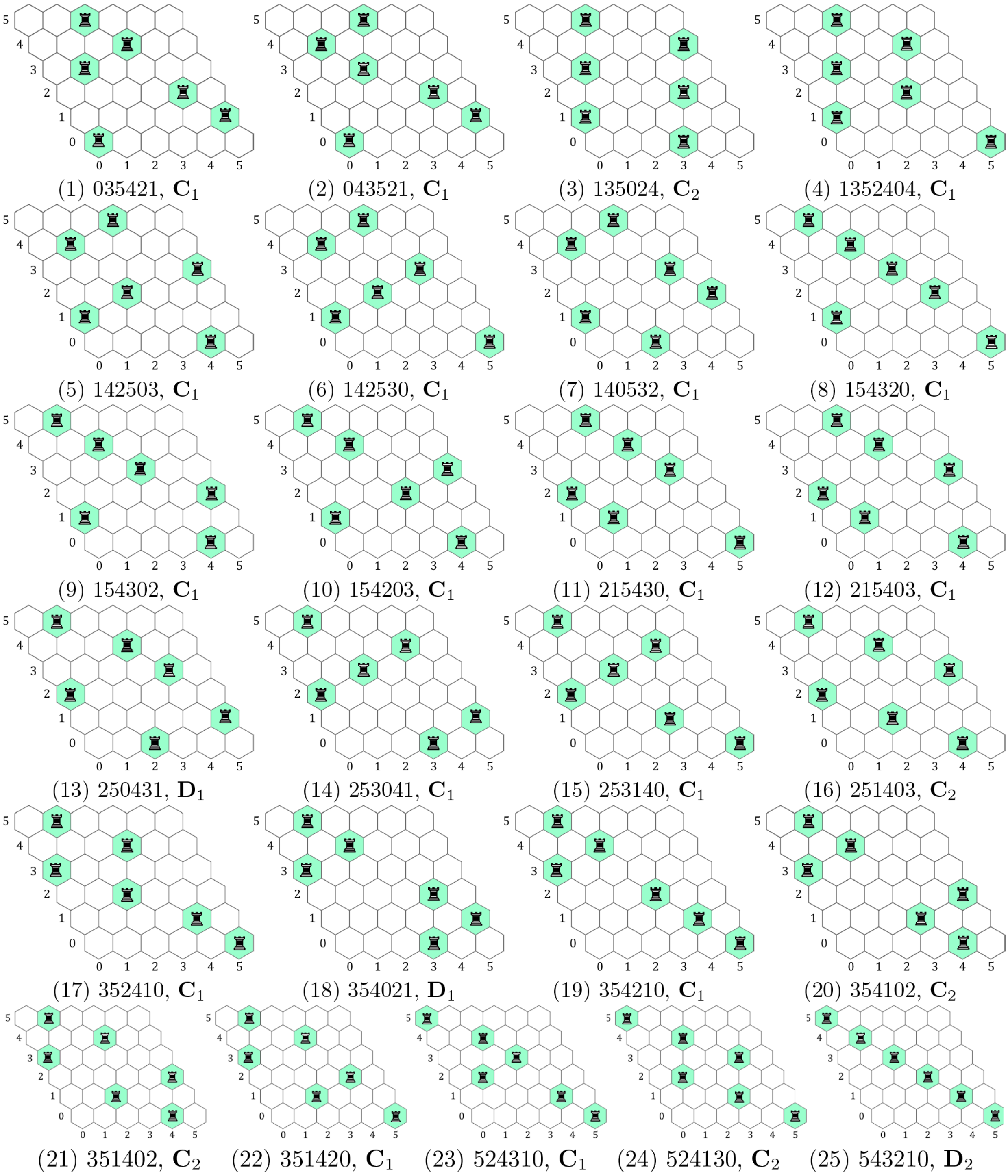


Figure 4. Fundamental solutions to the n-rooks problem on H_6.

4 The n-queens problem on H_n

Let us now consider the n-queens problem on H_n. We place n queens on H_n so that no two queens lie on the same row, c-diagonal, d-diagonal or fers-line. We show the existence of an independent set S of Q_n^h by showing the existence of permutation f of the finite set N and injective maps $h : N \to N'$, $g_1 : N \to N_1'$, and $g_2, g_3 : N \to N_2'$ where $N_1' = \{0, 1, \ldots, 2(n-1)\}$ and $N_2' = \{-(n-1), -(n-2), \ldots, 2(n-1)\}$. Similar to the n-rooks problem on H_n, an independent set S with size $|S| = n$ is a solution to the n-queens problem on H_n. Here, each row of H_n must contain at most one queen, so $|S|$ is at most n.

Theorem 4.1. *The set S with $|S| = n$ is an independent set of Q_n^h if and only if $S = \{(x, f(x)) : x \in N\}$ where f is a permutation of N such that the following functions are injective maps: $h : N \to N'$ defined by $h(x) = f(x) - x$, $g_1 : N \to N_1'$ defined by $g_1(x) = f(x) + x$, and $g_2, g_3 : N \to N_2'$ defined by $g_2(x) = 2x - f(x)$ and $g_3(x) = 2f(x) - x$.*

Proof. Suppose $S = \{(x, f(x)) : x \in N\}$, where f is a permutation of N. Since f is a permutation, then each row and each c-diagonal of H_n contains an element of S, and no two elements of S are in the same row nor in the same c-diagonal. Let $h : N \to N'$ be an injective map defined by $h(x) = f(x) - x$. Then no two elements of S are in the same d-diagonal. Let $g_1 : N \to N_1'$ and $g_2, g_3 : N \to N_2'$ be injective maps defined by $g_1(x) = f(x) + x$, $g_2(x) = 2x - f(x)$, and $g_3(x) = 2f(x) - x$. Then no two elements of S are in the same fers-lines. Thus S is an independent set of Q_n^h with $|S| = n$.

Conversely, if S is an independent set of Q_n^h with $|S| = n$, then each row and c-diagonal of Q_n^h contains exactly one element of S. So for each pair $x, y \in N$ with $x \neq y$, $f(x) \neq f(y)$, that is, f is an injective map. Thus f is a permutation of finite set N.

Now, since S is an independent set, then any two cells $(x, f(x))$ and $(x', f(x'))$ in S do not belong to the same d-diagonal, that is, $f(x) - x \neq f(x') - x$. Thus for any $x, x' \in N$, $h(x) \neq h(x')$, that is, $h : N \to N'$ is an injective map.

S being an independent set also implies that any two cells $(x, f(x))$ and $(x', f(x'))$ in S do not belong to the same fers-lines. Meaning $f(x) + x \neq f(x') + x'$, $2x - f(x) \neq 2x' - f(x')$, and $2f(x) - x \neq 2f(x') - x'$. Thus for any $x, x' \in N$, $g_1(x) \neq g_1(x')$, $g_2(x) \neq g_2(x')$, and $g_3(x) \neq g_3(x')$. Therefore, g_1, g_2, g_3 are injective maps. □

It can observed that an independent set of Q_n^h is also an independent set of R_n^h. For $n \in \{3, 4, 5\}$, examining the independent sets of R_n^h with size n, we found that there exists no independent sets of Q_n^h with size n. When $n \in \{1, 2, 3\}$, $\beta(Q_n^h) = 1$. When n is 4 or 5, $\beta(Q_n^h)$ is 3 or 4, respectively. When $n = 6, \beta(Q_6^h) = 6$. There are maximum two independent sets of Q_6^h where one corresponds to the solution in Figure 4 #16 while the other is the image of this solution under a reflection symmetry of H_n.

Theorem 4.2. *Let $n \geq 6$ be an integer, then the independence number $\beta(Q_n^h)$ of Q_n^h is n.*

Proof. Since no row in H_n can contain more than one queen, then $\beta(Q_n^h) \leq n$. We show by cases that for a given $n \geq 6$, the set $S = \{(x, f(x)) : x \in N, f : N \to N\}$ is an independent set of Q_n^h with cardinality n.

Suppose n is an even integer. Define $f(x)$ as

$$f(x) = \begin{cases} n - 2 - 2x & \text{if } x \in \{0, 1, \ldots, \frac{n-2}{2}\} \\ 2n - 1 - 2x & \text{if } x \in \{\frac{n}{2}, \frac{n+2}{2}, \ldots, n - 1\} \end{cases}.$$

When n is an odd integer such that $n \equiv 1$ or $2 \pmod 3$. Let $f(x) = [3x]_n$, where $[3x]_n$ is to be interpreted as the remainder of $3x$ on division by n.

Let us settle the case where n is an odd integer such that $n \equiv 0 \pmod 3$. Note that such n satisfies $n \equiv 3 \pmod 6$. First consider $n \equiv 4 \pmod 6$. Define $f(x)$ as

$$f(x) = \begin{cases} 3x & \text{if } x \in \{0, 1, \dots, \frac{n-1}{3}\} \\ n-2 & \text{if } x = \frac{n+5}{3} \\ \frac{3x-n}{2} & \text{if } x \in \{\frac{n+2}{3}, \frac{n+8}{3}, \dots, n-2\} \\ \frac{3x-n-7}{2} & \text{if } x \in \{\frac{n+11}{3}, \frac{n+17}{3}, \dots, n-1\} \end{cases}. \tag{1}$$

Let $S' = \{(y, f(y)) : y \in N\}$ where $f(y)$ is defined as in (1). Then S' is an independent set of Q_n^h with cardinality n, and $\beta(Q_n^h) = n$. The set $S = \{(y-1, f(y)-1) : y > 0 \text{ and } (y, f(y)) \in S'\}$ is an independent set of Q_n^h where n is an odd integer such that $n \equiv 3 \pmod 6 \equiv 0 \pmod 3$. In other words, we take the non-attacking queens solution corresponding to the independent set S' and remove the leftmost c-diagonal and bottom row of the board (also removing the queen at $(0, 0)$ position), then relabel cells of the board such that the lowest leftmost cell is $(0, 0)$. The result is a set of n non-attacking queens on H_n where $n \equiv 0 \pmod 3$, n an odd integer. It then follows that $\beta(Q_n^h) = n$. □

Table 2 summarizes the number of solutions to the n-queens problem on H_n for small values of n. There are no solutions with reflection symmetry; otherwise two queens would belong to the same d-diagonal or l_1 fers-line. Thus there are no solutions with $\mathbf{D}_1$ or $\mathbf{D}_2$ symmetry.

Table 2. Number of solutions to the n-queens problem on H_n for $6 \le n \le 10$.

n	$\mathbf{C}_1$ symmetry	$\mathbf{C}_2$ symmetry	Total
6	0	1	2
7	3	1	14
8	1	0	4
9	1	1	6
10	2	1	10

The fundamental solutions for $n \in \{7, 8, 9, 10\}$ are presented in Figure 5. Interestingly, if we let $S = \{(x, f(x)) : x \in \{0, 1, \dots, 7\}\}$ be the independent set corresponding to the non-attacking queens in solution #1, the independent sets S' corresponding to other 3 fundamental solutions are, for certain $i \in \mathbb{Z}$, $S' = \{(x, F(x)) : F(x) \equiv f(x) + i \pmod n\}$. When $n = 8$, there is only one fundamental solution with $\mathbf{C}_1$ symmetry (Figure 5 #5); hence a total of 4 solutions. When $n = 9$, there are 6 solutions and 2 are fundamental where #6 and #7 have $\mathbf{C}_1$ and $\mathbf{C}_2$ symmetry, respectively. When $n = 10$, there are 10 solutions and 3 are fundamental (#8–#10). A solution to the n-queens problem on H_{11} is shown in #11.

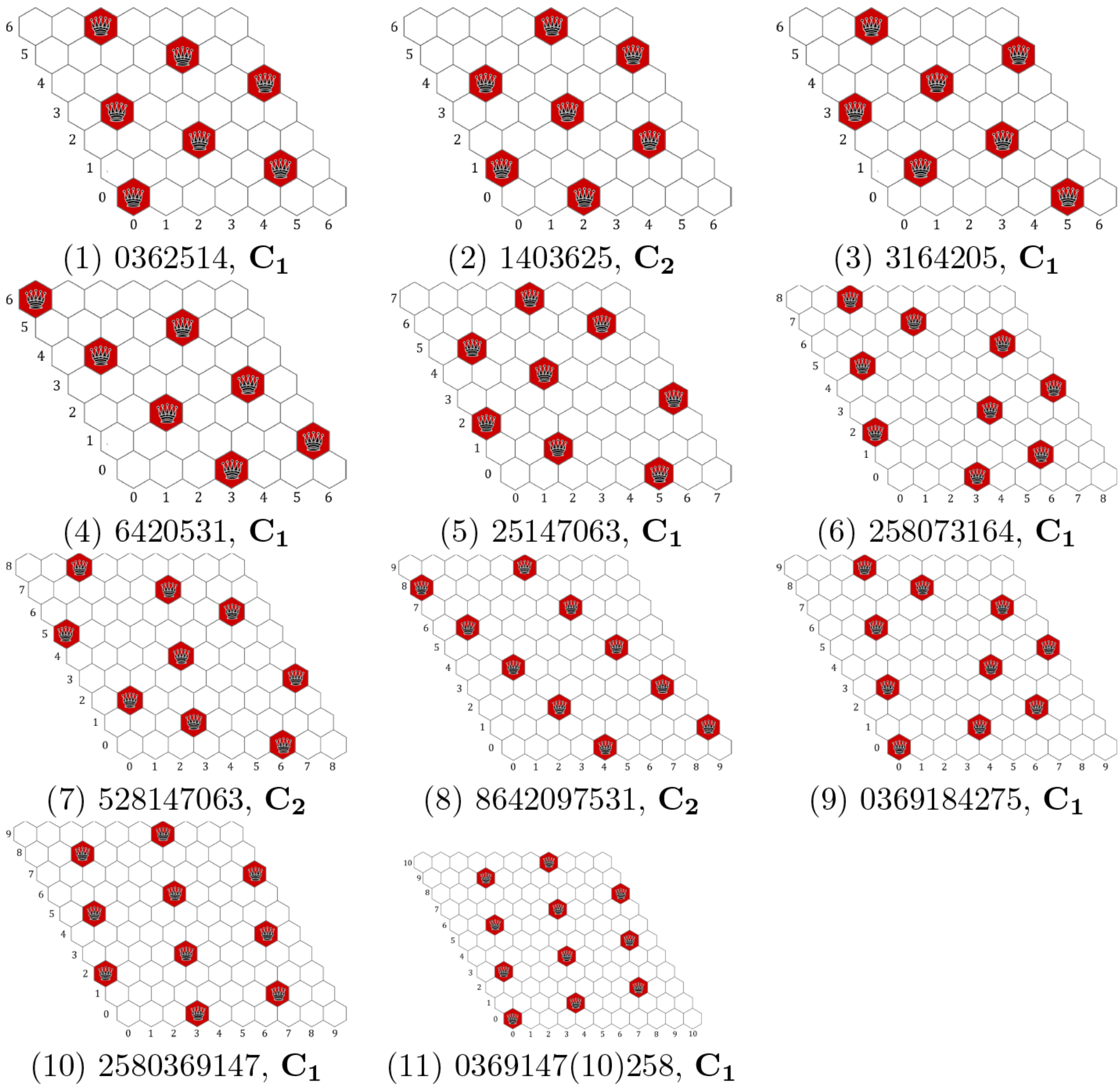
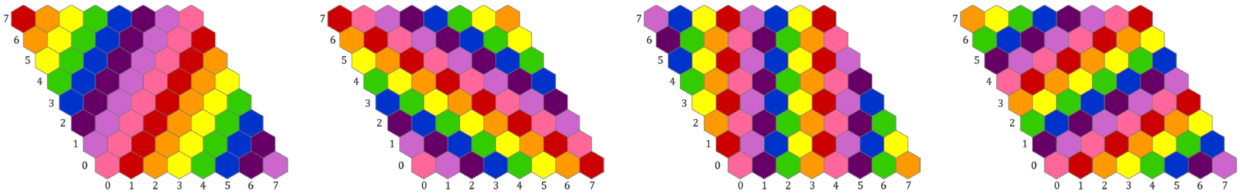


Figure 5. Fundamental solutions to the n-queens problem on H_7 (#1–4), on H_8 (#5), on H_9 (#6, 7), on H_{10} (#8–10). #11 is a solutions to the n-queens problem on H_{11}.

5 The n-rooks and n-queens problem on $\mathbf{T}_n$

We now determine the independence number of rooks graph R_n^t and queens graph Q_n^t on a torus chessboard, also called *modular chessboard*, with hexagonal cells. A *torus* or, more precisely, a *flat torus* T_n is the board H_n with the opposite edges (top and bottom, left and right) identified. We can think of a torus by starting with the board H_n, and first gluing the top and bottom edges of this board together to form a cylindrical tube, and then bending and stretching this tube so that the left and right ends of the tube also come together. We consider the torus where the edges of cells in row 0 are glued with the edges of cells in row $n-1$, and the edges of cells in c-diagonal 0 are glued with the edges of cells in c-diagonal $n-1$.

The torus chessboard T_n has n rows and n c-diagonals as in H_n. It has also n d-diagonals and n of each fers-lines l_1, l_2 and l_3. Two cells (x_1, z_1) and (x_2, z_2) belong to a d-diagonal when $z_1 - x_1 \equiv z_2 - x_2 \pmod n$. They belong to a l_1 fers-line when $z_1 + x_1 \equiv z_2 + x_2 \pmod n$. They belong to a l_2 fers-line when $2x_1 - z_1 \equiv 2x_2 - z_2 \pmod n$; and to a l_3 fers-line when $2z_1 - x_1 \equiv 2z_2 - x_2 \pmod n$. The d-diagonal containing cell (x_1, z_1) is referred to as d-diagonal k where $k \equiv (z_1 - x_1) \pmod n$. The l_1, l_2, l_3 fers-lines containing cell (x_1, z_1) are referred to as l_1 fers-line k_1 where $k_1 \equiv (z_1 + x_1) \pmod n$, l_2 fers-line k_2 where $k_2 \equiv (2x_1 - z_1) \pmod n$, and l_3 fers-line k_3 where $k_3 \equiv (2z_1 - x_1) \pmod n$. In Figure 6(a), cells in a d-diagonal are given in the same color. In Figures 6 (b), (c) and (d), cells in a l_1 fers-line, a l_2 fers-line and a l_3 fers-line are also given the same color.



(a) (b) (c) (d)

Figure 6. Cells of a modular chessboard T_8 in a (a) d-diagonal, (b) l_1 fers-line, (c) l_2 fers-line, and (d) l_3 fers-line are given the same color.

Similar to the n-rooks problem and n-queens problem on H_n, a maximum independent set S with size $|S| = n$ is a solution to the n-rooks and n-queens problem on T_n. Similarly, each row of T_n must contain at most one rook or one queen, hence $|S| \leq n$. The result in Theorem 5.1 on rooks graph R_n^t on T_n is analogous to our result in Theorem 3.1 in Section 3. Their proofs are also analogous. In Theorem 3.1, one concludes that $h : N \to N'$ is an injective map. So in Theorem 5.1, $h : N \to N$ being an injective map implies that h is a permutation of a finite set N. Theorem 5.1 also follows immediately from Proposition 1 in [4].

Theorem 5.1. *The set S with $|S| = n$ is an independent set of R_n^t if and only if $S = \{(x, f(x)) : x \in N\}$ where f is a permutation of N such that the function $h : N \to N$ defined by $h(x) \equiv (f(x) - x) \pmod n$ is a permutation of N.*

Theorem 5.2. *If n is an even integer, then there does not exist a permutation f of N such that the function $h : N \to N$ defined by $h(x) \equiv (f(x) - x) \pmod n$ is a permutation.*

Proof. Let n be even. Suppose on the contrary that there exists a permutation f of N for which $h(x) \equiv (f(x) - x) \pmod n$ is a permutation. Then

$$\sum_{x=0}^{n-1} f(x) \equiv \sum_{x=0}^{n-1} h(x) \pmod n$$

$$\sum_{x=0}^{n-1} f(x) \equiv \sum_{x=0}^{n-1} (f(x) - x) \pmod n$$

$$\sum_{x=0}^{n-1} x \equiv \left(\sum_{x=0}^{n-1} x - \sum_{x=0}^{n-1} x \right) \pmod n \text{ since } f \text{ is a permutation of } N$$

$$\frac{n(n-1)}{2} \equiv 0 \pmod{n}$$
$$\frac{n(n-1)}{2} = kn \text{ where } k \text{ is an integer}$$
$$k = \frac{n-1}{2}$$, a contradiction since $n-1$ is an odd integer and k is an integer. □

Let n be an even integer. For each $x \in N_1 = N - \{\frac{n}{2}\}$, define $f, h : N_1 \to N$ by

$$f(x) \equiv \begin{cases} (n-1-x) \pmod{n} & \text{if } x \in \{0, 1, \ldots, \frac{n-2}{2}\} \\ (n-x) \pmod{n} & \text{if } x \in \{\frac{n+2}{2}, \frac{n+4}{2}, \ldots, n-1\} \end{cases}. \tag{2}$$

Theorem 5.3. *Let n be an even integer, then the independence number $\beta(R_n^t)$ of R_n^t is $n-1$.*

Proof. By Theorem 5.2, $\beta(R_n^t) \leq n-1$. Consider the set $S = \{(x, f(x)) : x \in N_1\}$ where $f(x)$ is defined as in (2) and $N_1 = N - \{\frac{n}{2}\}$. Analogous to the note on Theorem 3.1, it can be shown that S is an independent set of R_n^t of cardinality $n-1$. Thus $\beta(R_n^t) = n-1$. □

Figure 7 #6 presents 12 non-attacking rooks on T_{12} corresponding to the independent set described in the proof Theorem 5.3.

In T_n, given an independent set S_1 of R_n^t another independent set S_2 of R_n^t is obtained by translating every element of S_1 with a vector (a, b) in T_n. Figure 7 #4 shows a solution to the n-rooks problem on T_9. Another a solution to the n-rooks problem on T_9 is presented in Figure 7 #5, which can be obtained by translating every element of S with vector $(4, 0)$ in T_n. Figure 7 #s 1–3 show 3 of the 15 solutions to the n-rooks problem on T_5. The rest of the solutions are translates of those listed.

Table 3. Independence numbers of R_n^t and Q_n^t on T_n.

n	Graph G	$\beta(G)$
n is an odd integer	R_n^t	$\beta(R_n^t) = n$
n is an even integer	R_n^t	$\beta(R_n^t) = n-1$
$n > 6$ is an odd integer such that $n \equiv 1$ or $2 \pmod 3$	Q_n^t	$\beta(Q_n^t) = n$
n is an even integer or $n \equiv 0 \pmod 3$	Q_n^t	$\beta(Q_n^t) \leq n-1$

We now indulge ourselves to the queens graph Q_n^t on T_n. If $n = 1, 2, 3, 4, 5$, a queen alone dominates T_n, so $\beta(Q_n^t) = 1$. When $n = 6$, $\beta(Q_n^t) = 3$. We have now the following result on Q_n^t. Theorem 5.4 is analogous to Theorem 4.1 in Section 4. Their proofs are also analogous. As in the case of rooks graph in Theorem 5.1, the conclusion that h, g_1, g_2, g_3 in Theorem 4.1 are injective maps implies that they are permutations in Theorem 5.4.

Theorem 5.4. *The set S with $|S| = n$ is an independent set of Q_n^t if and only if $S = \{(x, f(x)) : x \in N\}$ where f is a permutation of N such that the following functions $h, g_1, g_2, g_3 : N \to N$ defined by $h(x) \equiv (f(x) - x) \pmod{n}$, $g_1(x) \equiv (f(x) + x) \pmod{n}$, $g_2(x) \equiv (2x - f(x)) \pmod{n}$, and $g_3(x) \equiv (2f(x) - x) \pmod{n}$ are permutations of N.*

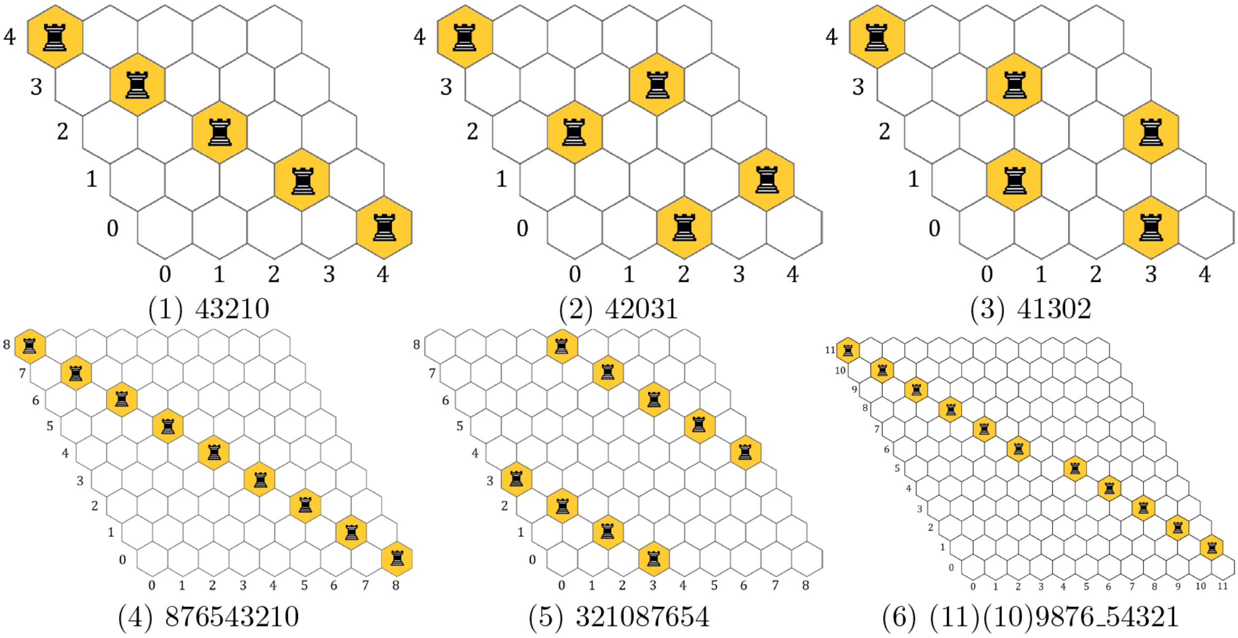


Figure 7. Solutions to the n-rooks problem on T_5 (#s 1–3), on T_9 (#s 4, 5), on T_{12} (#6).

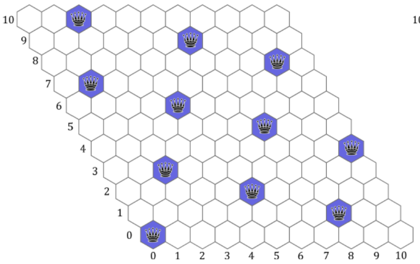
Theorem 5.5. *If n is an even integer or $n \equiv 0\,(mod\ 3)$, then there does not exist a permutation f of N such that the functions $g_1, h : N \to N$ defined by $g_1(x) \equiv (f(x) + x) \pmod n$ or $h(x) \equiv (f(x) - x) \pmod n$ are permutations.*

Proof. Now let $n \equiv 0 \pmod n$. Suppose on the contrary that the permutations f, g_1 and h of N exist. Since g_1, h are permutations of N, then

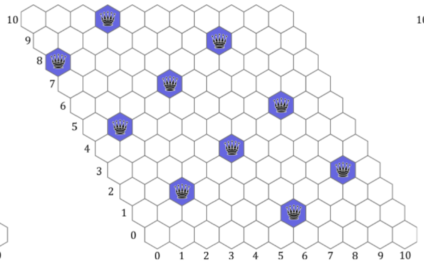
$$\begin{aligned}
\sum_{x=0}^{n-1}(g_1(x))^2 &\equiv \sum_{x=0}^{n-1}(h(x))^2 \pmod n\\
\sum_{x=0}^{n-1}(f(x)+x)^2 &\equiv \sum_{x=0}^{n-1}(f(x)-x)^2 \pmod n\\
\sum_{x=0}^{n-1}((f(x))^2+2xf(x)+x^2) &\equiv \sum_{x=0}^{n-1}((f(x))^2-2xf(x)+x^2) \pmod n\\
\sum_{x=0}^{n-1}4xf(x) &\equiv 0 \pmod n\\
4\sum_{x=0}^{n-1}x^2 &\equiv 0 \pmod n \text{ since } f \text{ is a permutation of } N\\
\frac{2n(n-1)(2n-1)}{3} &\equiv 0 \pmod n\\
\frac{2n(n-1)(2n-1)}{3} &= kn \text{ where } k \text{ is an integer}\\
k &= \frac{2(n-1)(2n-1)}{3}
\end{aligned}$$

We have a contradiction since neither $n-1$ nor $2n-1$ are multiples of 3 and k is an integer. □

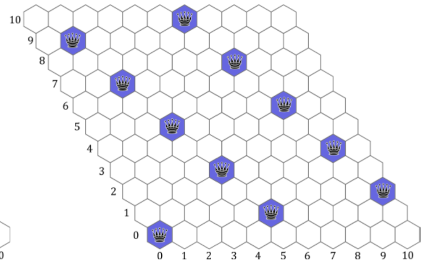
The independent sets of Q_7^h corresponding to the non-attacking queens on H_7 shown in Figure 5 #s 1–4, are also independent sets of Q_7^t on T_7. When $n = 7$, there are 7 solutions to the n-queens problem on T_7 which are translates of those in Figure 5. On T_{11}, we found 42 solutions which are translates of the 6 solutions shown in Figure 8.



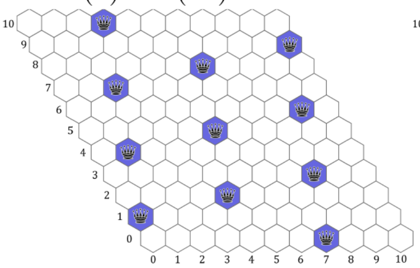
(1) 073(10)6295184



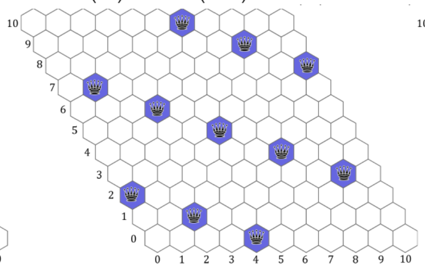
(2) 0852(10)741963



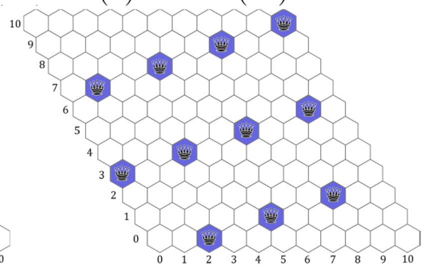
(3) 097531(10)8642



(4) 147(10)2580369



(5) 271605(10)4938



(6) 3704815926(10)

Figure 8. Solutions to the n-queens problem on T_{11}.

To end, one may consider other domination problems and properties of other chessboard graphs on the setting of hexagonal chessboards, including those surfaces with hexagonal cells.

Acknowledgements

The authors would like to thank the anonymous referees for their comments and suggestions that helped improve the quality of the article. The first author would like to express his gratitute to the CLSU-Academic Research Council for the support.

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