Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 4, 752–773
DOI: 10.7546/nntdm.2023.29.4.752-773

Notes on generalized and extended Leonardo numbers

Anthony G. Shannon[1], Peter J.-S. Shiue[2] and Shen C. Huang[3]

[1] Warrane College, University of New South Wales
Kensington, NSW 2033, Australia
e-mail: tshannon38@gmail.com

[2] Department of Mathematical Sciences, University of Nevada, Las Vegas
4505 S Maryland Pkwy, Las Vegas, NV 89154, USA
e-mail: shiue@unlv.nevada.edu

[3] Department of Mathematical Sciences, University of Nevada, Las Vegas
4505 S Maryland Pkwy, Las Vegas, NV 89154, USA
e-mail: huangs5@unlv.nevada.edu

Received: 19 September 2023 **Accepted:** 30 October 2023 **Online First:** 27 November 2023

Abstract: This paper both extends and generalizes recently published properties which have been developed by many authors for elements of the Leonardo sequence in the context of second-order recursive sequences. It does this by considering the difference equation properties of the homogeneous Fibonacci sequence and the non-homogeneous properties of their Leonardo sequence counterparts. This produces a number of new identities associated with a generalized Leonardo sequence and its associated algorithm, as well as some combinatorial results which lead into elegant properties of hyper-Fibonacci numbers in contrast to their ordinary Fibonacci number analogues, and as a convolution of Fibonacci and Leonardo numbers.

Keywords: Binet formulas, Leonardo sequences, Generalized Leonardo sequence, Extended Leonardo sequence, Fibonacci sequences, Hyper-Fibonacci sequences, Recurrence relations, Undetermined coefficients.
2020 Mathematics Subject Classification: 05A19, 11B37, 11B39.

1 Introduction

A revival of interest in these Leonard Fibonacci sequences occurred after the paper from Paula Catarino and Anabela Borges [4]. There was also some passing attention in the early days of the Fibonacci Association [3] in order to emphasize the genius of Leonard Fibonacci, but for the most part it was a case of converting non-homogeneous second order forms into higher order homogeneous forms. This possibly accounts for the relative dearth of number theory specifically about Leonardo sequences per se. We too consider some non-homogeneous properties to extend the work of Alwyn Horadam [9] to the Leonardo canvas. This results in a number of tables which, in themselves, suggest further work for the interested reader. Some applications follow with a number of well-known sequences from Koshy [12]. This culminates in a number of identities associated with a generalized Leonardo sequence and an associated algorithm, as well as some combinatorial results which lead into hyper-Fibonacci numbers $\{2, 5, 11, 21, 38, 66, 112, 187, \dots\}$ as a convolution of Fibonacci and Leonardo numbers.

2 Preliminaries

Consider the Fibonacci sequence $\{F_n\}$

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2, \tag{1}$$

with $F_0 = 0$ and $F_1 = 1$, and the Lucas sequence $\{L_n\}$

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2,, \tag{2}$$

with $L_0 = 2$ and $L_1 = 1$. The closed formulas for the Fibonacci sequence and Lucas sequence are

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \psi^n\right) \tag{3}$$

$$L_n = \phi^n + \psi^n, \tag{4}$$

respectively, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. These formulas are also known as Binet's formula.

We first consider the generalized Leonardo sequence.

Definition 2.1 (Kuhapatanakul *et al.* [13])**.** *The generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}$, with a fixed positive integer k, is defined by*

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k,\ n \geq 2, \tag{5}$$

with the initial conditions $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$.

A version of the Leonardo-like sequence $\{C_n(a, b, k)\}$, defined by

$$C_n(a, b, k) = C_{n-1}(a, b, k) + C_{n-2}(a, b, k) + k, \tag{6}$$

with $C_0(a, b, k) = b - a - k$, $C_1(a, b, k) = a$, and k is a constant, has been studied by Bicknell-Johnson and Bergum [3]. The generalized Leonardo sequence arises as a special case of C_n:

$$\mathcal{L}_{k,n} = C_n(1, 2 + k, k).$$

Theorem 2.1 (Kuhapatanakul *et al.* [13]). *The closed formula for the generalized Leonardo sequence* $\{\mathcal{L}_{k,n}\}$ *is*

$$\mathcal{L}_{k,n} = (1+k)F_{n+1} - k, \tag{7}$$

Corollary 2.1 (Catarino and Borges [4]). *Let* $\{Le_n\}$ *be the classical Leonardo sequence be defined by* $Le_n = Le_{n-1} + Le_{n-2} + 1,\ n \geq 2$ *with initial conditions* $Le_0 = Le_1 = 1$. *Then*

$$Le_n = 2F_{n+1} - 1.$$

Proof. Let $k = 1$ in the previous theorem. □

Corollary 2.2. *We have*

(i) $\mathcal{L}_{k,n} = (k+1)\frac{\mathcal{L}_{1,n}-1}{2} + 1$,

(ii) $\mathcal{L}_{k+1,n} - \mathcal{L}_{k,n} = \frac{\mathcal{L}_{1,n}-1}{2}$.

Proof. From Corollary 2.1, we have $\mathcal{L}_{1,n} = Le_n = 2F_{n+1} - 1$. Then $F_{n+1} = \frac{\mathcal{L}_{1,n}+1}{2}$. By Theorem 2.1, we have

$$\mathcal{L}_{k,n} = (1+k)F_{n+1} - k = (1+k)\frac{\mathcal{L}_{1,n}+1}{2} - k = (k+1)\frac{\mathcal{L}_{1,n}-1}{2} + 1,$$

which proves *(i)*.

Next, again by Theorem 2.1,

$$\begin{aligned} \mathcal{L}_{k+1,n} &= (k+2)F_{n+1} - (k+1), \\ \mathcal{L}_{k,n} &= (k+1)F_{n+1} - k. \end{aligned}$$

Subtract these two equations yields

$$\begin{aligned} \mathcal{L}_{k+1,n} - \mathcal{L}_{k,n} &= (k+2)F_{n+1} - k - 1 - (k+1)F_{n+1} + k = F_{n+1} - 1 \\ &= \frac{\mathcal{L}_{1,n}-1}{2}, \end{aligned}$$

which proves *(ii)*. □

3 Main results

Let $\{a_n\}$ be a sequence of order 2 satisfying the following homogeneous linear recurrence relation:

$$a_n = pa_{n-1} + qa_{n-2}, \quad n \geq 2, \tag{8}$$

where $a_0, a_1, p, q \neq 0$ are given constants. Let α and β be two roots of the characteristic equation of (8):

$$x^2 - px - q = 0. \tag{9}$$

He and Shiue [7] proved the following theorem that gives the general formula of $\{a_n\}$.

Theorem 3.1 (He and Shiue [7]). *Let $\{a_n\}$ be a sequence of order 2 satisfying the linear recurrence relation* (8). *Then*

$$a_n = \begin{cases} \left(\frac{a_1-\beta a_0}{\alpha-\beta}\right)\alpha^n - \left(\frac{a_1-\alpha a_0}{\alpha-\beta}\right)\beta^n, & \text{if } \alpha \neq \beta; \\ na_1\alpha^{n-1} - (n-1)a_0\alpha^n, & \text{if } \alpha = \beta, \end{cases} \tag{10}$$

where α and β are the two roots of (9).

Corollary 3.1. *If $a_0 = 0$ and $a_1 = 1$, then the general formula is given by*

$$a_n = \begin{cases} \left(\frac{1}{\alpha-\beta}\right)\alpha^n - \left(\frac{1}{\alpha-\beta}\right)\beta^n, & \text{if } \alpha \neq \beta; \\ n\alpha^{n-1}, & \text{if } \alpha = \beta. \end{cases} \tag{11}$$

Corollary 3.2. *If $a_0 = 1$ and $a_1 = 1$, then the general formula is given by*

$$a_n = \begin{cases} \left(\frac{1-\beta}{\alpha-\beta}\right)\alpha^n - \left(\frac{1-\alpha}{\alpha-\beta}\right)\beta^n, & \text{if } \alpha \neq \beta; \\ n\alpha^{n-1} - (n-1)\alpha^n, & \text{if } \alpha = \beta. \end{cases} \tag{12}$$

Theorem 3.1, Corollary 3.1, and Corollary 3.2 will be used in the main results.

In this paper, we will consider the sequence $\{a_n(t,j)\}$ satisfying the second order non-homogeneous linear recurrence relation:

$$a_n(t,j) = pa_{n-1}(t,j) + qa_{n-2}(t,j) + (p+q-1)(tn+j), \quad n \geq 2,\ t,j \in \mathbb{Z}, \tag{13}$$

where $a_0(t,j)$, $a_1(t,j)$, p, and q, with $p+q \neq 1$, are given constants.

We will write $a_n(t,j)$ as w_n, to follow Horadam's [9] notation:

$$w_n \equiv w_n(w_0, w_1, p, q, t, j) = a_n(t,j), \tag{14}$$

with $w_0 = a_0(t,j)$, $w_1 = a_1(t,j)$, $w_n(w_0, w_1, p, q, 0, 0) = a_n$, $n \geq 2$.

We now give the general formula of w_n:

Theorem 3.2. *Let $\{w_n(w_0, w_1, p, q, t, j)\}$ be a sequence of order 2 satisfying the non-homogeneous linear relation in the following form:*

$$w_n = pw_{n-1} + qw_{n-2} + (p+q-1)(tn+j),\ n \geq 2,\ t,j \in \mathbb{Z}, \tag{15}$$

where $w_0,\ w_1,\ p,\ q$, with $p+q \neq 1$, are given constants. Then

$$w_n = w_n(w_0, w_1, p, q, 0, 0) + \left(j - \frac{t(p+2q)}{1-p-q}\right)(w_n(1,1,p,q,0,0)-1) + t\left(w_n(0,1,p,q,0,0) - n\right). \tag{16}$$

Proof. First we consider the homogeneous part

$$w_n(w_0, w_1, p, q, 0, 0) = pw_{n-1}(w_0, w_1, p, q, 0, 0) + qw_{n-2}(w_0, w_1, p, q, 0, 0).$$

Then the characteristic equation

$$x^2 = px + q$$

gives

$$x = \frac{p \pm \sqrt{p^2+4q}}{2}.$$

Let $\alpha = \frac{p+\sqrt{p^2+4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2+4q}}{2}$. Then the homogeneous solution of (15) is

$$w_n(w_0, w_1, p, q, 0, 0) = c_1\alpha^n + c_2\beta^n.$$

Suppose $\alpha \neq \beta$. Assume the particular solution is of the form

$$w_n^\rho = An + B,$$

where $A = A(t, j)$ and $B = B(t, j)$. Then we have

$$An + B = p(A(n-1) + B) + q(A(n-2) + B) + (p + q - 1)(tn + j).$$

Solving for A and B, we have

$$\begin{aligned} A &= -t, \\ B &= \frac{t(p+2q)}{1-p-q} - j. \end{aligned}$$

Then

$$w_n = c_1\alpha^n + c_2\beta^n - tn - j + \frac{t(p+2q)}{1-p-q}.$$

Using the initial conditions w_0 and w_1, we have

$$\begin{cases} w_0 &= c_1 + c_2 - j + \frac{t(p+2q)}{1-p-q} \\ w_1 &= c_1\alpha + c_2\beta - t - j + \frac{t(p+2q)}{1-p-q}. \end{cases}$$

Multiplying the first equation by α and subtract with the second, we have

$$\begin{aligned} w_0\alpha - w_1 &= c_2(\alpha - \beta) + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\alpha + t + j + \frac{t(p+2q)}{1-p-q} \\ \implies c_2 &= \frac{w_0\alpha - w_1}{\alpha - \beta} - \frac{t}{\alpha - \beta} + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\left(\frac{1-\alpha}{\alpha-\beta}\right). \end{aligned}$$

Then

$$c_1 = \frac{w_1 - w_0\beta}{\alpha - \beta} + \frac{t}{\alpha - \beta} + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\left(\frac{\beta - 1}{\alpha - \beta}\right).$$

Thus, the general solution is

$$\begin{aligned} w_n = &\left[\frac{w_1 - w_0\beta}{\alpha - \beta} + \frac{t}{\alpha - \beta} + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\left(\frac{\beta - 1}{\alpha - \beta}\right)\right]\alpha^n \\ &+ \left[\frac{w_0\alpha - w_1}{\alpha - \beta} - \frac{t}{\alpha - \beta} + \left(-j + \frac{t(p+2q)}{1-p-q}\right)\left(\frac{1-\alpha}{\alpha - \beta}\right)\right]\beta^n \\ &- tn - j + \frac{t(p+2q)}{1-p-q}. \end{aligned} \tag{17}$$

We can rewrite it as

$$\begin{aligned} w_n = &\left(\frac{w_1 - w_0\beta}{\alpha - \beta}\right)\alpha^n - \left(\frac{w_1 - w_0\alpha}{\alpha - \beta}\right)\beta^n - \left(j - \frac{t(p+2q)}{1-p-q}\right)\left(\frac{(\beta-1)\alpha^n + (1-\alpha)\beta^n}{\alpha - \beta}\right) \\ &+ t\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) - tn - j + \frac{t(p+2q)}{1-p-q}. \end{aligned}$$

Using (10), (11), and (12), we have

$$\begin{aligned} w_n &= w_n(w_0, w_1, p, q, 0, 0) + \left(j - \frac{t(p+2q)}{1-p-q}\right)(w_n(1,1,p,q,0,0) - 1) \\ &\quad + t\left(w_n(0,1,p,q,0,0) - n\right). \end{aligned} \tag{18}$$

for $\alpha \neq \beta$.

Now, if $\alpha = \beta$, we have

$$w_n = (c_1 + c_2 n)\,\alpha^n.$$

The solution is

$$w_n = (c_1 + c_2 n)\,\alpha^n - tn - j + \frac{t(p+2q)}{1-p-q}.$$

Using the initial conditions,

$$\begin{aligned} w_0 &= c_1 - j + \frac{t(p+2q)}{1-p-q} \\ w_1 &= c_1\alpha + c_2\alpha - t - j + \frac{t(p+2q)}{1-p-q}. \end{aligned}$$

Then

$$\begin{aligned} c_1 &= w_0 + j - \frac{t(p+2q)}{1-p-q} \\ c_2 &= \frac{w_1}{\alpha} - w_0 - j + \frac{t(p+2q)}{1-p-q} + \frac{1}{\alpha}\left(t + j - \frac{t(p+2q)}{1-p-q}\right) \end{aligned}$$

$$\begin{aligned} w_n &= \left(w_0 + j - \frac{t(p+2q)}{1-p-q}\right)\alpha^n + \left[\frac{w_1}{\alpha} - w_0 - j + \frac{t(p+2q)}{1-p-q} + \frac{1}{\alpha}\left(t + j - \frac{t(p+2q)}{1-p-q}\right)\right] n\alpha^n \\ &\quad - tn - j + \frac{t(p+2q)}{1-p-q} \\ &= \left(w_0 + j - \frac{t(p+2q)}{1-p-q}\right)(\alpha^n - n\alpha^n) + \left(w_1 + t + j - \frac{t(p+2q)}{1-p-q}\right) n\alpha^{n-1} \\ &\quad - tn - j + \frac{t(p+2q)}{1-p-q} \\ &= w_1 n\alpha^{n-1} - w_0(n-1)\alpha^n + \left(j - \frac{t(p+2q)}{1-p-q}\right)\left(n\alpha^{n-1} - (n-1)\alpha^n - 1\right) + tn\alpha^{n-1} - tn. \end{aligned}$$

Thus, if $\alpha = \beta$, the solution is

$$w_n = w_1 n\alpha^{n-1} - w_0(n-1)\alpha^n + \left(j - \frac{t(p+2q)}{1-p-q}\right)\left(n\alpha^{n-1} - (n-1)\alpha^n - 1\right) + tn\alpha^{n-1} - tn.$$

Using (10), (11), and (12), then

$$\begin{aligned} w_n &= w_n(w_0, w_1, p, q, 0, 0) + \left(j - \frac{t(p+2q)}{1-p-q}\right)(w_n(1,1,p,q,0,0) - 1) \\ &\quad + t(w_n(0,1,p,q,0,0) - n). \end{aligned}$$

The results for both cases are the same. □

Corollary 3.3 (Bicknell-Johnson *et al.* [3]). *Consider the Leonardo-like sequence* $C_n(a,b,k)$ *defined in* (6). *Using Horadam's notation, we have*

$$w_n = w_n(b-a-k, a, 1, 1, 0, k).$$

Then

$$w_n(b-a-k, a, 1, 1, 0, k) = aF_{n-2} + bF_{n-1} + k(F_n - 1). \tag{19}$$

Proof. By Theorem 3.2,

$$\begin{aligned} w_n(b-a-k, a, 1, 1, 0, k) &= w_n(b-a-k, a, 1, 1, 0, 0) + k(w_n(1,1,1,1,0,0) - 1) \\ &= w_n(b-a-k, a, 1, 1, 0, 0) + k(F_{n+1} - 1). \end{aligned}$$

Since $p = q = 1$ and $t = j = 0$, we can use Theorem 3.1, with $\alpha = \phi$ and $\beta = \psi$:

$$\begin{aligned} w_n(b-a-k, a, 1, 1, 0, 0) &= \frac{a - \psi(b-a-k)}{\phi - \psi}\phi^n - \frac{a - \phi(b-a-k)}{\phi - \psi}\psi^n \\ &= a\left(\frac{\phi^n - \psi^n}{\phi - \psi}\right) + (b-a-k)\left(\frac{\phi^{n-1} - \psi^{n-1}}{\phi - \psi}\right) \\ &= aF_n + (b-a-k)F_{n-1} = aF_{n-2} + (b-k)F_{n-1}. \end{aligned}$$

Hence, by Theorem 3.2,

$$\begin{aligned} w_n(b-a-k, a, 1, 1, 0, k) &= aF_{n-2} + (b-k)F_{n-1} + k(F_{n+1} - 1) \\ &= aF_{n-2} + bF_{n-1} + k(F_n - 1). \end{aligned}$$

□

Corollary 3.4. *Consider the general Leonardo sequence* $\{w_n(1,1,1,1,t,j)\}$. *Then*

$$w_n(1,1,1,1,t,j) = (1+3t+j)F_{n+1} + t(F_n - n - 3) - j. \tag{20}$$

Proof. Let $p = q = 1$ and $w_0 = w_1 = 1$ in (16) of Theorem 3.2. Recall that the Fibonacci sequence $\{F_n\}$ satisfies the second order linear recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \tag{21}$$

where $F_0 = 0$ and $F_1 = 1$. By (11), we have

$$w_n(0,1,1,1,0,0) = F_n,\ w_n(1,1,1,1,0,0) = F_{n+1}$$

where $\alpha = \phi$ and $\beta = \psi$. Then

$$\begin{aligned} w_n(1,1,1,1,t,j) &= w_n(1,1,1,1,0,0) + \left(j - \frac{t(1+2)}{1-1-1}\right)(w_n(1,1,1,1,0,0) - 1) \\ &\quad + t\,(w_n(0,1,1,1,0,0) - n) \\ &= F_{n+1} + (j+3t)\,(F_{n+1} - 1) + t(F_n - n) \\ &= (1+j+3t)F_{n+1} + tF_n - tn - j - 3t \\ &= (1+3t+j)F_{n+1} + t(F_n - n - 3) - j. \end{aligned}$$

□

Corollary 3.5 (Shannon *et al.* [18]). *Consider the sequence* $\{w_n(1,1,1,1,1,j)\}$ *of order* 2 *satisfying the non-homogeneous linear recurrence relation* (15). *Then*

$$w_n(1,1,1,1,1,j) = (4+j)F_{n+1} + F_n - n - 3 - j. \tag{22}$$

Proof. Let $t = 1$ in Corollary 3.4 yield the result. □

Corollary 3.6. *The closed formula for the generalized Leonardo sequence* $\{\mathcal{L}_{k,n}\}$ *defined in Definition 2.1 is*

$$\mathcal{L}_{k,n} = (1+k)F_{n+1} - k,$$

as given in Theorem 2.1.

Proof. Let $t = 0$ and $j = k$ in Corollary 3.4 yield the result. ☐

Corollary 3.7 (Shannon *et al.* [18])**.** *Consider the sequence* $\{w_n(1,1,1,1,1,0)\}$ *of order* 2 *satisfying the non-homogeneous linear recurrence relation* (15)*. Then*

$$w_n(1,1,1,1,1,0) = 4F_{n+1} + F_n - n - 3 \tag{23}$$

Proof. Let $t = 1$ and $j = 0$ in Corollary 3.4 yield the result. ☐

Theorem 3.3. *Let* $\{w_n(w_0, w_1, p, q, t, j)\}$ *be a sequence of order* 2 *satisfying the non-homogeneous linear relation in the following form:*

$$w_n = pw_{n-1} + qw_{n-2} + (p+q-1)(tn+j),\ n \geq 2,\ t, j \in \mathbb{Z}, \tag{24}$$

where $w_0,\ w_1,\ p,\ q$*, with* $p + q \neq 1$*, are given constants. Then*

$$w_n(w_0, w_1, p, q, t, j+1) - w_n(w_0, w_1, p, q, t, j) = w_n(1,1,p,q,0,0) - 1 \text{ and} \tag{25}$$

$$w_n(w_0, w_1, p, q, t, j+k) - w_n(w_0, w_1, p, q, t, j) = k\left(w_n(1,1,p,q,0,0) - 1\right). \tag{26}$$

Proof. Using the result from Theorem 3.2, we have

$$\begin{aligned} w_n(w_0, w_1, p, q, t, j+1) = w_n(w_0, w_1, p, q, 0, 0) &+ \left(j+1-\frac{t(p+2q)}{1-p-q}\right)(w_n(1,1,p,q,0,0)-1) \\ &+ t\left(w_n(0,1,p,q,0,0) - n\right) \end{aligned}$$

and

$$\begin{aligned} w_n(w_0, w_1, p, q, t, j) = w_n(w_0, w_1, p, q, 0, 0) &+ \left(j-\frac{t(p+2q)}{1-p-q}\right)(w_n(1,1,p,q,0,0)-1) \\ &+ t\left(w_n(0,1,p,q,0,0) - n\right). \end{aligned}$$

Subtracting the two equations yields

$$w_n(w_0, w_1, p, q, t, j+1) - w_n(w_0, w_1, p, q, t, j) = w_n(1,1,p,q,0,0) - 1.$$

The second result can be obtained by repeating the same process and replacing $j+1$ by $j+k$. ☐

Corollary 3.8. *Consider the Leonardo-like sequence* $\{w_n(b-a-k, a, 1, 1, 0, k)\}$*. Then for* $n \geq 2$*,*

$$w_n(b-a-k, a, 1, 1, 0, k+1) - w_n(b-a-k, a, 1, 1, 0, k) = F_{n+1} - 1.$$

Proof. By Theorem 3.3,

$$\begin{aligned} w_n(b-a-k, a, 1, 1, 0, k+1) - w_n(b-a-k, a, 1, 1, 0, k) &= w_n(1,1,1,1,0,0) - 1 \\ &= F_{n+1} - 1. \end{aligned}$$ ☐

Corollary 3.9 (Shannon *et al.* [17]). *Consider the general Leonardo sequence* $\{w_n(1, 1, 1, 1, t, j)\}$. *Then for* $n \geq 2$,

$$w_n(1, 1, 1, 1, t, j + 1) - w_n(1, 1, 1, 1, t, j) = F_{n+1} - 1. \tag{27}$$

Proof. Using Theorem 3.3. We have

$$w_n(1, 1, 1, 1, t, j + 1) - w_n(1, 1, 1, 1, t, j) = w_n(1, 1, 1, 1, 0, 0) - 1 = F_{n+1} - 1. \qquad \square$$

Note that $w_n(1, 1, 1, 1, 0, k) = \mathcal{L}_{k,n}$. Hence when $t = 0$, we have the same result as Corollary 2.2 (ii).

Next, note that this difference is independent of t. A table by Shannon and Deveci [18] for $t = 1$ is given here:

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
-3	1	1	1	2	4	8	15	27	47
-2	1	1	2	4	8	15	27	47	80
-1	1	1	3	6	12	22	39	67	113
0	1	1	4	8	16	29	51	87	146
1	1	1	5	10	20	36	63	107	179
2	1	1	6	12	24	43	75	127	212
3	1	1	7	14	28	50	87	147	245
Differences	0	0	1	2	4	7	12	20	33

Table 1. ”Extended Leonardo sequence”, [18].

We now give two more tables with $t = 2$ and $t = 3$ to show the Independence of t:

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
-3	1	1	3	7	15	29	53	93	159
-2	1	1	4	9	19	36	65	113	192
-1	1	1	5	11	23	43	77	133	225
0	1	1	6	13	27	50	89	153	258
1	1	1	7	15	31	57	101	173	291
2	1	1	8	17	35	64	113	193	324
3	1	1	9	19	39	71	125	213	357
Differences	0	0	1	2	4	7	12	20	33

Table 2. Extended Leonardo sequence with $t = 2$.

n / j	0	1	2	3	4	5	6	7	8
-3	1	1	5	12	26	50	91	159	271
-2	1	1	6	14	30	57	103	179	304
-1	1	1	7	16	34	64	115	199	337
0	1	1	8	18	38	71	127	219	370
1	1	1	9	20	42	78	139	239	403
2	1	1	10	22	46	85	151	259	436
3	1	1	11	24	50	92	163	279	469
Differences	0	0	1	2	4	7	12	20	33

Table 3. Extended Leonardo sequence with $t = 3$.

Theorem 3.4. *Let $\{a_n\}$ be a sequence of order 2 satisfying the non-homogeneous linear relation:*

$$a_n = a_{n-1} + a_{n-2} + Ck^n,\ n \geq 2, \tag{28}$$

where $a_0 = 0, a_1 = 1$, $C \neq 0$, $k \neq 0$, and $k^2 - k - 1 \neq 0$. Then

$$a_n = \left(1 - \frac{Ck^3}{k^2 - k - 1}\right) F_n + \left(1 - \frac{Ck^2}{k^2 - k - 1}\right) F_{n-1} + \frac{Ck^{n+2}}{k^2 - k - 1}. \tag{29}$$

Proof. The homogeneous solution is

$$a_n = c_1\phi^n + c_2\psi^n,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

The particular solution can be found using the method of undetermined coefficients. Assume the particular solution is of the form $a_n^* = Ak^n$, where A is a constant. Then

$$Ak^n = Ak^{n-1} + Ak^{n-2} + Ck^n$$
$$\implies A = \frac{Ck^2}{k^2 - k - 1}.$$

Hence, the general solution to (28) is

$$a_n = c_1\phi^n + c_2\psi^n + \frac{Ck^{n+2}}{k^2 - k - 1}.$$

With $a_0 = a_1 = 1$, we have the following system

$$\begin{cases} 1 &= c_1 + c_2 + \frac{Ck^2}{k^2-k-1} \\ 1 &= c_1\phi + c_2\psi + \frac{Ck^3}{k^2-k-1} \end{cases} \implies \begin{cases} c_1 + c_2 &= 1 - \frac{Ck^2}{k^2-k-1} \\ c_1\phi + c_2\psi &= 1 - \frac{Ck^3}{k^2-k-1} \end{cases}.$$

Then

$$\begin{aligned} c_2(\phi - \psi) &= \phi - \frac{Ck^2\phi}{k^2 - k - 1} - 1 + \frac{Ck^3}{k^2 - k - 1} \\ \implies c_2 &= \frac{\phi - 1}{\sqrt{5}} + \frac{Ck^3 - Ck^2\phi}{\sqrt{5}(k^2 - k - 1)} \\ &= -\frac{\psi}{\sqrt{5}} + \frac{Ck^3 - Ck^2\phi}{\sqrt{5}(k^2 - k - 1)}. \end{aligned}$$

Moreover,

$$\begin{aligned}
c_1 &= 1 - \frac{Ck^2}{k^2-k-1} + \frac{\psi}{\sqrt{5}} - \frac{Ck^3 - Ck^2\phi}{\sqrt{5}(k^2-k-1)} \\
&= \frac{(k^2-k-1-Ck^2)(\phi-\psi) + \psi(k^2-k-1) - Ck^3 + Ck^2\phi}{\sqrt{5}(k^2-k-1)} \\
&= \frac{-(k+1)(\phi-\psi) + (k^2-Ck^2)(\phi-\psi) + k^2\psi - (k+1)\psi - Ck^3 + Ck^2\phi}{\sqrt{5}(k^2-k-1)} \\
&= \frac{-(k+1)\phi + k^2\left((1-C)(\phi-\psi) - \alpha k + C\phi + \psi\right)}{\sqrt{5}(k^2-k-1)} \\
&= \frac{-(k+1)\phi + k^2(\phi + C\psi - Ck)}{\sqrt{5}(k^2-k-1)} = \frac{\phi}{\sqrt{5}} - \frac{Ck^3 - Ck^2\psi}{\sqrt{5}(k^2-k-1)}.
\end{aligned}$$

Hence, the general solution to (28) is

$$\begin{aligned}
a_n &= \left(\frac{\phi}{\sqrt{5}} - \frac{Ck^3 - Ck^2\psi}{\sqrt{5}(k^2-k-1)}\right)\phi^n - \left(\frac{\psi}{\sqrt{5}} - \frac{Ck^3 - Ck^2\phi}{\sqrt{5}(k^2-k-1)}\right)\psi^n + \frac{Ck^{n+2}}{k^2-k-1} \\
&= \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}} - \frac{Ck^3(\phi^n - \psi^n)}{\sqrt{5}(k^2-k-1)} + \frac{Ck^2\phi\psi(\phi^{n-1} - \psi^{n-1})}{\sqrt{5}(k^2-k-1)} + \frac{Ck^{n+2}}{k^2-k-1} \\
&= F_{n+1} - \frac{Ck^3}{k^2-k-1}F_n - \frac{Ck^2}{k^2-k-1}F_{n-1} + \frac{Ck^{n+2}}{k^2-k-1} \\
&= \left(1 - \frac{Ck^3}{k^2-k-1}\right)F_n + \left(1 - \frac{Ck^2}{k^2-k-1}\right)F_{n-1} + \frac{Ck^{n+2}}{k^2-k-1}.
\end{aligned}$$ □

Corollary 3.10 (Shannon *et al.* [18]). *Consider a sequence $\{a_{j,n}\}$ of order 2 satisfying the following non-homogeneous linear recurrence relation:*

$$a_{j,n} = a_{j,n-1} + a_{j,n-2} + (-1)^n j,\ n \geq 2,\ j \geq 0, \tag{30}$$

where $a_0 = 0,\ = a_1 = 1$. Then

$$a_{j,n} = F_{n+1} + jF_{n-2} + (-1)^n j,\ n \geq 2. \tag{31}$$

Proof. Let $C = j$ and $k = -1$. Then

$$\begin{aligned}
a_n &= \left(1 - \frac{-j}{1}\right)F_n + \left(1 - \frac{j}{1}\right)F_{n-1} + \frac{(-1)^{n+2}j}{1} \\
&= (1+j)F_n + (1-j)F_{n-1} + (-1)^n j \\
&= F_{n+1} + jF_{n-2} + (-1)^n j.
\end{aligned}$$ □

Corollary 3.11 (Shannon *et al.* [18]). *Consider a sequence $\{a_n\}$ of order 2 satisfying the following non-homogeneous linear recurrence relation:*

$$a_n = a_{n-1} + a_{n-2} + (-1)^n,\ n \geq 2, \tag{32}$$

where $a_0 = 0, a_1 = 1$. Then

$$a_n = 2F_n + (-1)^n. \tag{33}$$

Proof. Let $j = 1$ in the previous corollary. Then

$$a_n = F_{n+1} + F_{n-2} + (-1)^n = F_n + F_{n-1} + F_n - F_{n-1} + (-1)^n = 2F_n + (-1)^n. \qquad \square$$

Corollary 3.12 (Shannon *et al.* [18])**.** *Consider a sequence* $\{a_{j,n}\}$ *of order* 2 *satisfying the following non-homogeneous linear recurrence relation: Let*

$$a_{j,n} = a_{j,n-1} + a_{j,n-2} + (-1)^n j,\ n \geq 2,\ j \geq 0,$$

where $a_0 = 0,\ = a_1 = 1$*. Then*

$$a_{j+1,n} - a_{j,n} = F_{n-2} + (-1)^n,\ n \geq 2. \tag{34}$$

Proof. By Corollary 3.10, we have

$$a_{j,n} = F_{n+1} + jF_{n-2} + (-1)^n j$$

and

$$a_{j+1,n} = F_{n+1} + (j+1)F_{n-2} + (-1)^n(j+1).$$

Then

$$a_{j+1,n} - a_{j,n} = F_{n-2} + (-1)^n. \qquad \square$$

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	1	2	3	5	8	13	21	34
1	0	1	2	2	5	6	12	17	30	46
2	0	1	3	2	7	7	16	21	39	58
3	0	1	4	2	9	8	20	25	48	70
4	0	1	5	2	11	9	24	29	57	82
5	0	1	6	2	13	10	28	33	66	94
6	0	1	7	2	15	11	32	37	75	106
7	0	1	8	2	17	12	36	41	84	118
8	0	1	9	2	19	13	40	45	93	130
9	0	1	10	2	21	14	44	49	102	142
10	0	1	11	2	23	15	48	53	111	154
Differences	0	0	1	0	2	1	4	4	9	12

Table 4. Table of values for Corollary 3.14.

4 Examples

Consider

$$w_n(w_0, w_1, p, q, 0, 0) = pw_{n-1}(w_0, w_1, p, q, 0, 0) + qw_{n-2}(w_0, w_1, p, q, 0, 0),\ n \geq 2,$$

where w_0, w_1, p, and $q \neq 0$ are given constants. The following table by Koshy [12] lists some well-known sequences:

Sequence of numbers	w_0	w_1	p	q
Fibonacci F_n	0	1	1	1
Lucas L_n	2	1	1	1
Pell P_n	0	1	2	1
Pell–Lucas Q_n	2	2	2	1
Mersenne M_n	0	1	3	-2
Jacobsthal J_n	0	1	1	2
Jacobsthal–Lucas $\mathcal{J}_n$	2	1	1	2
Balancing B_n	0	1	6	-1
Lucas-balancing C_n	1	3	6	-1
M. Ward W_n	1	1	4	-1
Fermat of the first kind T_n	1	3	3	-2
Fermat of the second kind S_n	2	3	3	-2

Table 5. Some well-known sequences, [12].

Example 4.1. *Let* $w_0 = 2,\ w_1 = 1,\ p = q = 1$ *in* (15)*, i.e.,*

$$w_n(2,1,1,1,t,j) = w_{n-1}(2,1,1,1,t,j) + w_{n-2}(2,1,1,1,t,j) + tn + j,\ n \geq 2,\ t \in \mathbb{Z},$$

Then

(1). $w_n(2,1,1,1,t,j) = L_n + (j+3t)(F_{n+1}-1) + t(F_n - n)$*;*

(2). $w_n(2,1,1,1,t,j+1) - w_n(2,1,1,1,t,j) = F_{n+1} - 1.$

Proof. Since $p = q = 1$, $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $w_n(w_0,w_1,p,q,0,0) = w_n(2,1,1,1,0,0) = L_n$, $w_n(0,1,p,q,0,0) = w_n(0,1,1,1,0,0) = F_n$, and $w_n(1,1,p,q,0,0) = w_n(1,1,1,1,0,0) = F_{n+1}$. Then by Theorem 3.2,

$$\begin{aligned} w_n(2,1,1,1,t,j) &= w_n(2,1,1,1,0,0) + \left(j - \frac{t(1+2)}{1-1-1}\right)(w_n(1,1,1,1,0,0) - 1) \\ &\quad + t\,(w_n(0,1,1,1,0,0) - n) \\ &= L_n + (j+3t)(F_{n+1}-1) + t(F_n - n)\,. \end{aligned}$$

We use Theorem 3.3 to obtain the second result. We have

$$\begin{aligned} w_n(2,1,1,1,t,j+1) - w_n(2,1,1,1,t,j) &= w_n(1,1,1,1,0,0) - 1 \\ &= F_{n+1} - 1. \end{aligned}$$ □

We give three tables to show this difference.

For $t = 1$:

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
-3	2	1	2	3	6	11	20	35	60
-2	2	1	3	5	10	18	32	55	93
-1	2	1	4	7	14	25	44	75	126
0	2	1	5	9	18	32	56	95	159
1	2	1	6	11	22	39	68	115	192
2	2	1	7	13	26	46	80	135	225
3	2	1	8	15	30	53	92	155	258
Differences	0	0	1	2	4	7	12	20	33

Table 6. Values of $w_n(2, 1, 1, 1, 1, j)$.

For $t = 2$:

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
-3	2	1	4	8	17	32	58	101	172
-2	2	1	5	10	21	39	70	121	205
-1	2	1	6	12	25	46	82	141	238
0	2	1	7	14	29	53	94	161	271
1	2	1	8	16	33	60	106	181	304
2	2	1	9	18	37	67	118	201	337
3	2	1	10	20	41	74	130	221	370
Differences	0	0	1	2	4	7	12	20	33

Table 7. Values of $w_n(2, 1, 1, 1, 2, j)$.

For $t = 3$:

j \ n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
-3	2	1	6	13	28	53	96	167	284
-2	2	1	7	15	32	60	108	187	317
-1	2	1	8	17	36	67	120	207	350
0	2	1	9	19	40	74	132	227	383
1	2	1	10	21	44	81	144	247	416
2	2	1	11	23	48	88	156	267	449
3	2	1	12	25	52	95	168	287	482
Differences	0	0	1	2	4	7	12	20	33

Table 8. Values of $w_n(2, 1, 1, 1, 3, j)$.

We can see that the difference resembles the sequence $\{F_{n+1}-1\}$.

Example 4.2. *Let* $w_0=0$, $w_1=1$, $p=2$, *and* $q=1$ *in* (15)*, i.e.,*

$$w_n(0,1,2,1,t,j)=2w_{n-1}(0,1,2,1,t,j)+w_{n-2}(0,1,2,1,t,j)+2(tn+j),\ n\geq 2,\ t\in\mathbb{Z}.$$

Then

(1). $w_n(0,1,2,1,t,j)=(1+t)P_n+(j+2t)(P_{n+1}-P_n-1)-tn$*;*

(2). $w_n(0,1,2,1,t,j+1)-w_n(0,1,2,1,t,j)=P_{n+1}-P_n-1.$

Proof. Since $p=2$ and $q=1,\alpha=1+\sqrt{2},\beta=1-\sqrt{2}$, $w_n(w_0,w_1,p,q,0,0)=w_n(0,1,2,1,0,0)=P_n$. Then by Theorem 3.2,

$$\begin{aligned}w_n(0,1,2,1,t,j)&=w_n(0,1,2,1,0,0)+\left(j-\frac{t(2+2)}{1-2-1}\right)(w_n(1,1,2,1,0,0)-1)\\&\quad+t\left(w_n(0,1,2,1,0,0)-n\right)\\&=P_n+(j+2t)\left(P_{n+1}-P_n-1\right)+tP_n-tn\\&=(1+t)P_n+(j+2t)(P_{n+1}-P_n-1)-tn.\end{aligned}$$

We use Theorem 3.3 to obtain the second result. Then

$$w_n(0,1,2,1,t,j+1)-w_n(0,1,2,1,t,j)=w_n(1,1,2,1,0,0)-1=P_{n+1}-P_n-1.\qquad\square$$

Remark 4.1. *In Example 4.2, the following identity is used:*

$$w_n(1,1,2,1,0,0)=w_{n+1}(0,1,2,1,0,0)-w_n(0,1,2,1,0,0).$$

Proof.

$$\begin{aligned}w_n(0,1,2,1,0,0)&=\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}\\w_{n+1}(0,1,2,1,0,0)&=\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}\\w_{n+1}(0,1,2,1,0,0)-w_n(0,1,2,1,0,0)&=\frac{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}}{2\sqrt{2}}\\&\quad-\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}\\&=\frac{\sqrt{2}(1+\sqrt{2})^n+\sqrt{2}(1-\sqrt{2})^n}{2\sqrt{2}}\\&=\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{2}\\&=w_n(1,1,2,1,0,0).\end{aligned}\qquad\square$$

Example 4.3. *Let* $w_0=2$, $w_1=2$, $p=2$, *and* $q=1$ *in* (15)*, i.e.,*

$$w_n(2,2,2,1,t,j)=2w_{n-1}(2,2,2,1,t,j)+w_{n-2}(2,2,2,1,t,j)+2(tn+j),\ n\geq 2,\ t\in\mathbb{Z}.$$

Then

(1). $w_n(2,2,2,1,t,j)=Q_n+(j+2t)(P_{n+1}-P_n-1)+t(P_n-n)$*;*

(2). $w_n(2,2,2,1,t,j+1)-w_n(2,2,2,1,t,j)=P_{n+1}-P_n-1.$

Proof. Similar to the last example, $\alpha = 1+\sqrt{2}$, $\beta = 1-\sqrt{2}$, $w_n(w_0, w_1, p, q, 0, 0) = w_n(2, 2, 2, 1, 0, 0) = Q_n$ and $w_n(0, 1, 2, 1, 0, 0) = P_n$. Then by Theorem 3.2,

$$\begin{aligned} w_n(2,2,2,1,t,j) &= w_n(2,2,2,1,0,0) + \left(j - \frac{t(2+2)}{1-2-1}\right)(w_n(1,1,2,1,0,0)-1) \\ &\quad + t\,(w_n(0,1,2,1,0,0)-n) \\ &= Q_n + (j+2t)\,(P_{n+1}-P_n-1) + t(P_n - n). \end{aligned}$$

The second result is the same as the last example. □

Example 4.4. *Let* $w_0 = 0$*,* $w_1 = 1$*,* $p = 1$*, and* $q = 2$ *in* (15)*, i.e.,*

$$w_n(0,1,1,2,t,j) = w_{n-1}(0,1,1,2,t,j) + 2w_{n-2}(0,1,1,2,t,j) + 2(tn+j),\ n \geq 2,\ t \in \mathbb{Z}.$$

Then

(1). $w_n(0,1,1,2,t,j) = (1+t)J_n + \left(j + \frac{5t}{2}\right)(\mathcal{J}_n - 1) - tn$*;*

(2). $w_n(0,1,1,2,t,j+1) - w_n(0,1,1,2,t,j) = \mathcal{J}_n - 1$.

Proof. Since $p = 1$ and $q = 2$, we have $\alpha = -1$, $\beta = 2$, and $w_n(w_0, w_1, p, q, 0, 0) = w_n(0, 1, 1, 2, 0, 0) = J_n$. Then by Theorem 3.2,

$$\begin{aligned} w_n(0,1,1,2,t,j) &= w_n(0,1,1,2,0,0) + \left(j - \frac{t(1+2\cdot 2)}{1-p-q}\right)(w_n(1,1,1,2,0,0)-1) \\ &\quad + t\,(w_n(0,1,1,2,0,0)-n) \\ &= J_n + \left(j+\frac{5t}{2}\right)(\mathcal{J}_n - 1) + tJ_n - tn \\ &= (1+t)J_n + \left(j+\frac{5t}{2}\right)(\mathcal{J}_n-1) - tn. \end{aligned}$$

We use Theorem 3.3 to obtain the second result. Then

$$w_n(0,1,1,2,t,j+1) - w_n(0,1,1,2,t,j) = w_n(1,1,2,1,0,0) - 1 = \mathcal{J}_n - 1.$$ □

Remark 4.2. *In Example 4.4, the following identity is used:*

$$w_{n+1}(0,1,1,2,0,0) = w_n(1,1,1,2,0,0).$$

Proof.

$$\begin{aligned} w_{n+1}(0,1,1,2,0,0) &= \frac{1}{3}\left((-1)^{n+2} + 2^{n+1}\right) = \frac{1}{3}\left((-1)^n + 2^{n+1}\right) \\ &= w_n(1,1,1,2,0,0) = J_{n+1} = \mathcal{J}_n. \end{aligned}$$ □

Example 4.5. *Let* $w_0 = 1$*,* $w_1 = 1$*,* $p = 1$*, and* $q = 2$ *in* (15)*, i.e.,*

$$w_n(1,1,1,2,t,j) = w_{n-1}(1,1,1,2,t,j) + 2w_{n-2}(1,1,1,2,t,j) + tn + j,\ n \geq 2,\ t \in \mathbb{Z}.$$

Then

(1). $w_n(1,1,1,2,t,j) = \mathcal{J}_n + \left(j + \frac{5t}{2}\right)(\mathcal{J}_n - 1) + t(J_n - n)$*;*

(2). $w_n(1,1,1,2,t,j+1) - w_n(1,1,1,2,t,j) = \mathcal{J}_n - 1$.

Proof. Similar to the last example, we have $\alpha = -1$, $\beta = 2$, $w_n(w_0, w_1, p, q, 0, 0) = w_n(1, 1, 1, 2, 0, 0) = \mathcal{J}_n$. Then by Theorem 3.2,

$$\begin{aligned} w_n(1,1,1,2,t,j) &= w_n(1,1,1,2,0,0) + \left(j - \frac{t(1+2\cdot 2)}{1-1-2}\right)(w_n(1,1,1,2,0,0)-1) \\ &\quad + t\,(w_n(0,1,1,2,0,0)-n) \\ &= \mathcal{J}_n + \left(j + \frac{5t}{2}\right)(\mathcal{J}_n - 1) + t(J_n - n). \end{aligned}$$

The second result is the same as the previous example. □

Remark 4.3. *In Examples 4.2, 4.3, 4.4, 4.5, the homogeneous parts are Pell sequence* $\{P_n\}$*, Pell–Lucas sequence* $\{Q_n\}$*, Jacobsthal sequence* $\{J_n\}$*, and Jacobsthal–Lucas sequence* $\{\mathcal{J}_n\}$*, respectively.*

5 Some identities involving the generalized Leonardo sequence

Theorem 5.1. *Let* $\{\mathcal{L}_{k,n}\}$ *denote the generalized Leonardo sequence. Then*

1. *(*Shattuck [19]*)* $\mathcal{L}_{k,n}^2 - \mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} = (-1)^n(k+1)^2 + k(k+1)F_{n-2}$*;*
2. *(*Kuhapatanakul [13]*)* $\mathcal{L}_{k,m}\mathcal{L}_{k,n-1} + \mathcal{L}_{k,m-1}\mathcal{L}_{k,n} = \mathcal{L}_{k,m+1}\mathcal{L}_{k,n+1} - (k+1)\mathcal{L}_{k,m+n} - k$.

Proof. By Theorem 2.1, we can write the generalized Leonardo sequence as

$$\mathcal{L}_{k,n} = (1+k)F_{n+1} - k. \tag{35}$$

Then

$$\begin{aligned} \mathcal{L}_{k,n}^2 - \mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} &= (1+k)^2F_{n+1}^2 - 2k(1+k)F_{n+1} + k^2 - ((1+k)F_n - k)\left((1+k)F_{n+2} - k\right) \\ &= (1+k)^2\left(F_{n+1}^2 - F_nF_{n+2}\right) - k(k+1)\left(2F_{n+1} - F_n - F_{n+2}\right) \\ &= (1+k)^2(-1)^n - k(1+k)\left(2F_{n+1} - F_n - F_{n+1} - F_n\right) \\ &= (1+k)^2(-1)^n - k(1+k)F_{n-2}, \end{aligned}$$

by Cassini's identity.

For the second result, we first note Honsberger's identity

$$F_{n-1}F_m + F_nF_{m+1} = F_{m+n}.$$

Then

$$\begin{aligned} \mathcal{L}_{k,m}\mathcal{L}_{k,n-1} &= (1+k)^2F_{m+1}F_n - k(1+k)(F_{m+1} + F_n) + k^2, \\ \mathcal{L}_{k,m-1}\mathcal{L}_{k,n} &= (1+k)^2F_mF_{n+1} - k(1+k)(F_m + F_{n+1}) + k^2, \\ \mathcal{L}_{k,m+1}\mathcal{L}_{k,n+1} &= (1+k)^2F_{m+2}F_{n+2} - k(1+k)(F_{m+2} + F_{n+2}) + k^2 \\ &= (1+k)^2(F_{m+1}F_{n+1} + F_{m+1}F_n + F_{n+1}F_m + F_mF_n) \\ &\quad - k(1+k)(F_{m+1} + F_m + F_{n+1} + F_n) + k^2, \\ \mathcal{L}_{k,m+n} &= (1+k)F_{m+n+1} - k \end{aligned}$$

Then

$$\mathcal{L}_{k,m}\mathcal{L}_{k,n-1}\mathcal{L}_{k,m-1}\mathcal{L}_{k,n} - \mathcal{L}_{k,m+1}\mathcal{L}_{k,n+1} = k^2 - (1+k)^2(F_{m+1}F_{n+1} + F_mF_n)$$
$$= k^2 - (1+k)^2F_{m+n+1}.$$

Finally,

$$\mathcal{L}_{k,m}\mathcal{L}_{k,n-1}\mathcal{L}_{k,m-1}\mathcal{L}_{k,n} - \mathcal{L}_{k,m+1}\mathcal{L}_{k,n+1} + (1+k)\mathcal{L}_{k,m+n} + k = 0. \qquad \square$$

Theorem 5.2. *Let*

$$a_0F_{n+t} + a_1F_{n+t-1} + \cdots + a_tF_n = 0, \tag{36}$$

where $a_0 + a_1 + \cdots + a_t = 0$*,* $a_i \in \mathbb{Z}$*,* $(i = 0, 1, 2, \ldots, t)$*,* t *is a fixed positive integer. Then*

$$a_0\mathcal{L}_{k,n+t-1} + a_1\mathcal{L}_{k,n+t-2} + \cdots + a_t\mathcal{L}_{k,n-1} = 0. \tag{37}$$

Proof. Since $\mathcal{L}_{k,n} = (1+k)F_{n+1} - k$, we have

$$\begin{aligned} &a_0\mathcal{L}_{k,n+t-1} + a_1\mathcal{L}_{k,n+t-2} + \cdots + a_t\mathcal{L}_{k,n-1} \\ &\quad= a_0[(1+k)F_{n+t} - k] + a_1[(1+k)F_{n+t-1} - k] + \cdots + a_t[(1+k)F_n - k] \\ &\quad= (1+k)[a_0F_{n+t} + a_1F_{n+t-1} + \cdots + a_tF_n] - k[a_0 + a_1 + \cdots + a_t] \\ &\quad= (1+k)\cdot 0 - k \cdot 0 = 0. \end{aligned} \qquad \square$$

Remark 5.1. (36) *can be obtained by computing* $(x^2 - x - 1)x^n(x-1)p(x)$*, where* $p(x)$ *is a polynomial over* $\mathbb{Z}$ *first, then replace each* x^{n+i} *by* F_{n+i}*.*

Algorithm 1 Obtaining this identity

Input: A polynomial $p(x)$ over $\mathbb{Z}$
Output: An identity with generalized Leonard sequence
1: $g(x) \leftarrow (x^2 - x - 1) \cdot x^n \cdot (x-1) \cdot p(x)$
2: Replace each x^{n+i} by F_{n+i}
3: Verify the coefficients of F_{n+i} sums to zero
4: Replace each F_{n+i} by $\mathcal{L}_{n+i-1}$
5: Ouput the identity

Example 5.1. *It is known that*

$$F_n + F_{n+1} + F_{n+6} - 3F_{n+4} = 0.$$

Hence $a_0 = 1$*,* $a_1 = 0$*,* $a_2 = -3$*,* $a_3 = a_4 = 0$*,* $a_5 = 1$*,* $a_6 = 1$*, i.e.* $\sum a_i = 0$*. Then*

$$\mathcal{L}_{k,n+5} - 3\mathcal{L}_{k,n+3} + \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} = 0,$$

or

$$\mathcal{L}_{k,n+5} + \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} = 3\mathcal{L}_{k,n+3}. \tag{38}$$

Example 5.2. *Let* $f(x) = (x^2 - x - 1)x^n$ *and* $p(x) = (x-1)(2x^3 + 3x - 1)$*. Then* $g(x) = f(x) \cdot p(x) = 2x^{n+6} - 4x^{n+5} + 3x^{n+4} - 5x^{n+3} + 2x^{n+2} + 3x^{n+1} - x^n$*. Replacing each* x^{n+i} *by* F_{n+i}*, we have*

$$2F_{n+6} - 4F_{n+5} + 3F_{n+4} - 5F_{n+3} + 2F_{n+2} + 3F_{n+1} - F_n = 0. \tag{39}$$

The coefficients are

$$a_0 = 2,\ a_1 = -4,\ a_2 = 3,\ a_3 = -5,\ a_4 = 2,\ a_5 = 3,\ a_6 = -1,$$

which gives

$$\sum a_i = 0.$$

Then we have

$$2\mathcal{L}_{k,n+5} - 4\mathcal{L}_{k,n+4} + 3\mathcal{L}_{k,n+3} - 5\mathcal{L}_{k,n+2} + 2\mathcal{L}_{k,n+1} + 3\mathcal{L}_{k,n} - \mathcal{L}_{k,n-1} = 0,\ n \geq 1.$$

Example 5.3. *Let* $f(x) = (x^2 - x - 1)x^n$ *and let* $p(x) = (x-1)(2x^2 + x + 1)$. *Then* $g(x) = f(x) \cdot p(x) = 2x^{n+5} - 3x^{n+4} - x^{n+3} + x^{n+1} + x^n$. *Replacing each* x^{n+i} *by* F_{n+i}, *we have*

$$2F_{n+5} - 3F_{n+4} - F_{n+3} + F_{n+1} + F_n = 0. \tag{40}$$

The coefficients are

$$a_0 = 2,\ a_1 = -3,\ a_2 = -1,\ a_3 = 0,\ a_4 = 1,\ a_5 = 1,$$

which gives

$$\sum a_i = 0.$$

Then we have

$$2\mathcal{L}_{k,n+4} - 3\mathcal{L}_{k,n+3} - \mathcal{L}_{k,n+2} + \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} = 0,\ n \geq 1.$$

6 Combinatorial conclusion

Jarden [10] has also considered Leonardo sequences from the point of view of the following variation of the Leonardo equation related to equation (5):

$$a_n = a_{n-1} + a_{n-2} \mp 1,\ n \geq 2, \tag{41}$$

and the associated 3^{rd} order linear recurrence

$$b_n = 2b_{n-1} - b_{n-3},\ n \geq 3, \tag{42}$$

to which the Leonardo sequences conform as in equation (5) with $k = \mp 1$. In fact, Jarden considers the sequences in Tables 1, 2, and 3, which can bring out the corresponding relations with the Fibonacci and Lucas sequences. $\{u_n\}$ is the sequence of differences, and is related to the generalized Fibonacci numbers of Jarden in Table 9 [10] and the hyper-Fibonacci and hyper-Lucas numbers in Table 10 [6] with further generalized and extended Leonardo numbers.

(-1)	0	1	2	3	4	5	6	7	8
U_n	1	2	2	3	4	6	9	14	22
V_n	3	2	4	5	8	12	19	30	48
$(+1)$	0	1	2	3	4	5	6	7	8
U_n	-1	0	0	1	2	4	7	12	20
V_n	1	0	2	3	6	10	17	28	46

Table 9. Jarden's example of equation (5) with $k = \mp 1$.

Table 10 below is copied from Table 1 [1]. It shows the interested reader the salient features of these sequences, both horizontally and vertically, as well as diagonally. Further properties to be investigated include intersections between sequences [8] and step functions within sequences [5]. The last of these leads to s-Pascal triangles, as in Table 11.

$\boldsymbol{n}$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	$\cdots$
$\boldsymbol{F_n^{(0)}}$	0	1	1	2	3	5	8	13	21	34	$\cdots$
$\boldsymbol{L_n^{(0)}}$	2	1	3	4	7	11	18	29	47	76	$\cdots$
$\boldsymbol{F_n^{(1)}}$	0	1	2	4	7	12	20	33	54	88	$\cdots$
$\boldsymbol{L_n^{(1)}}$	2	3	6	10	17	28	46	75	122	198	$\cdots$
$\boldsymbol{F_n^{(2)}}$	0	1	3	7	14	26	46	79	133	221	$\cdots$
$\boldsymbol{L_n^{(2)}}$	2	5	11	21	38	66	112	187	309	507	$\cdots$
$\boldsymbol{F_n^{(3)}}$	0	1	4	11	25	51	97	176	309	530	$\cdots$
$\boldsymbol{L_n^{(3)}}$	2	7	18	39	77	143	225	442	751	1258	$\cdots$

Table 10. Hyper-Fibonacci and hyper-Lucas numbers.

1													**1**
1	1	1											**3**
1	2	3	2	1									**9**
1	3	6	7	6	3	1							**27**
1	4	10	16	19	16	10	4	1					**81**
1	5	15	30	45	51	45	30	15	5	1			**243**
1	6	21	50	90	126	141	126	90	50	21	6	1	**729**

Table 11. A simple s-Pascal triangle.

If we then add along the leading diagonals in Table 11, we seem to arrive at the Tribonacci numbers, which can generate third-order Leonardo numbers.

In a different, but somewhat similar manner, Lind [14] defined $L(n,r)$ the r-th order nonlinear binomial sum as the sum of the first r terms of the $(n-1)$-th row of the ordinary Pascal's triangle plus the terms of the rising stair-step (or rising) diagonal originating at the r-th term, which can be applied to any of these tables. For example, in Table 11, we can have

$$L(1,3)=1,\ L(2,3)=3,\ L(3,3)=6,\ L(4,3)=12,\ L(4,4)=18.$$

All of these can provide a nexus between the numerical results in this paper and the recent combinatorial work of Shattuck [19], who provided a framework for these and other identities satisfied by the Leonardo numbers in the notation of section 3 and other generalized and extended Fibonacci numbers. The initial step in extending Corollary 3.12 is

$$w_n = w_{n-1} + w_{n-2} + tn + j,\ n \geq 2,\ j > -4,$$

and

$$w_n = w_{n-1} + F_{n+1} - 1. \tag{43}$$

One can then extend the process to other second order sequences [15] or to other orders and other dimensions [16] for further related combinatorial properties. In this way, one can relate

$$w_n = w_{n-1} + w_{n-2} + tn + j,\ n \geq 2,\ t \geq 1,$$

and

$$w_n = w_{n-1} + F_n^{[k]}, \tag{44}$$

in which $F_n^{[k]}$ is hyper-Fibonacci sequence, as in Table 10, the rows of which as k increases can be seen as staked on top of one another for a third dimension. These can be developed further [2]. We note the neat recurrence relation

$$F_n^{[k]} = F_{n-1}^{[k]} + F_n^{[k-1]},\ k, n > 0, \tag{45}$$

with boundary conditions $F_n^{[0]} = F_n$ and $F_0^{[k]} = 0$; and with an elegant characteristic polynomial

$$(x^2 - x - 1)(x - 1)^k,$$

so that

$$F_n^{[k]} = \sum_{j=1}^{n} \binom{k+n-j-1}{k-1} F_j; \tag{46}$$

see [11] for details, including their relation to the infinite matrix in which $F_n^{[k]}$ is the entry in the n-th row and k-th column, and from there to Stirling numbers of the first kind.

References

[1] Alp, Y. (2023). Hybrid hyper-Fibonacci and hyper-Lucas numbers. *Notes on Number Theory and Discrete Mathematics*, 29(1), 154–170.

[2] Bahşi, M. & Solak, S. (2016). A symmetric algorithm for golden ratio in hyper-Horadam numbers. *Chinese Journal of Mathematics*, Article ID 4361582.

[3] Bicknell-Johnson, M.& Bergum, G. E. (1988). The generalized Fibonacci numbers $\{C_n\}$, $C_n = C_{n-1} + C_{n-2} + k$. *Applications of Fibonacci Numbers: Proceedings of The Second International Conference on Fibonacci Numbers and Their Applications*, San Jose State University, California, USA, Springer, 193–205.

[4] Catarino, P. M., & Borges, A. (2019). On Leonardo Numbers. *Acta Mathematica Universitatis Comenianae*, 89(1), 75–86.

[5] Chu, H. V., Irmak, N., Miller, S. J., Szalay, L., & Zhang, S. X. Schreier multisets and the s-step Fibonacci sequences (*in press*)

[6] Dil, A., & Mező, I. (2008). A symmetric algorithm for hyperharmonic and Fibonacci numbers. *Applied Mathematics and Computation*, 206(2), 942–951.

[7] He, T.-X., & Shiue, P. J.-S. (2009). On sequences of numbers and polynomials defined by linear recurrence relations of order 2. *International Journal of Mathematics and Mathematical Sciences*, 21, Article ID 709386.

[8] Horadam, A. F. (1966). Generalization of two theorems of K. Subba Rao. *Bulletin of Calcutta Mathematical Society*, 58(1), 23–29.

[9] Horadam, A. F. (1965). Basic properties of a certain generalized sequence of numbers. *The Fibonacci Quarterly*, 3(3), 161–176.

[10] Jarden, D. (1996). Recurring Sequences. *Jerusalem: Riveon Lematematika*, 95–102.

[11] Komatsu, T.,& Szalay, L. (2018). A new formula for hyper-Fibonacci numbers, and the number of occurrences. *Turkish Journal of Mathematics*, 42(3), 993–1004.

[12] Koshy, T. (2019). *Fibonacci and Lucas Numbers with Applications* (Volume 2), John Wiley & Sons.

[13] Kuhapatanakul, K., & Chobsorn, J. (2022). On the generalized Leonardo numbers. *Integers*, 22, A48.

[14] Lind, D. A. (1965). On a class of non-linear binomial sums. *The Fibonacci Quarterly*, 3(4), 292–298.

[15] Ollerton, R. L., & Shannon, A. G. (1998). Some Properties of Generalized Pascal Squares and Triangles. *The Fibonacci Quarterly*, 36(2), 98–109.

[16] Shannon, A. G. (1977). Tribonacci numbers and Pascal's pyramid. *The Fibonacci Quarterly*, 15(3), 268–275.

[17] Shannon, A. G. (2019). A note on generalized Leonardo numbers. *Notes on Number Theory and Discrete Mathematics*, 25(3), 97–101.

[18] Shannon, A. G., & Deveci, Ö. (2022). A note on generalized and extended Leonardo sequences. *Notes on Number Theory and Discrete Mathematics*, 28(1), 109–114.

[19] Shattuck, M. (2022). Combinatorial proofs of identities for the generalized Leonardo numbers. *Notes on Number Theory and Discrete Mathematics*, 28(4), 778–790.