Notes on generalized and extended Leonardo numbers

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Abstract: This paper both extends and generalizes recently published properties which have been developed by many authors for elements of the Leonardo sequence in the context of second-order recursive sequences. It does this by considering the difference equation properties of the homogeneous Fibonacci sequence and the non-homogeneous properties of their Leonardo sequence counterparts. This produces a number of new identities associated with a generalized Leonardo sequence and its associated algorithm, as well as some combinatorial results which lead into elegant properties of hyper-Fibonacci numbers in contrast to their ordinary Fibonacci number analogues, and as a convolution of Fibonacci and Leonardo numbers.

Keywords: Binet formulas, Leonardo sequences, Generalized Leonardo sequence, Extended Leonardo sequence, Fibonacci sequences, Hyper-Fibonacci sequences, Recurrence relations, Undetermined coefficients.

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1 Introduction

A revival of interest in these Leonard Fibonacci sequences occurred after the paper from Paula Catarino and Anabela Borges [4]. There was also some passing attention in the early days of the Fibonacci Association [3] in order to emphasize the genius of Leonard Fibonacci, but for the most part it was a case of converting non-homogeneous second order forms into higher order homogeneous forms. This possibly accounts for the relative dearth of number theory specifically about Leonardo sequences per se. We too consider some non-homogeneous properties to extend the work of Alwyn Horadam [9] to the Leonardo canvas. This results in a number of tables which, in themselves, suggest further work for the interested reader. Some applications follow with a number of well-known sequences from Koshy [12]. This culminates in a number of identities associated with a generalized Leonardo sequence and an associated algorithm, as well as some combinatorial results which lead into hyper-Fibonacci numbers \{2, 5, 11, 21, 38, 66, 112, 187, \ldots\} as a convolution of Fibonacci and Leonardo numbers.

2 Preliminaries

Consider the Fibonacci sequence \(\{F_n\}\)

\[ F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad (1) \]

with \(F_0 = 0\) and \(F_1 = 1\), and the Lucas sequence \(\{L_n\}\)

\[ L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \quad , \quad (2) \]

with \(L_0 = 2\) and \(L_1 = 1\). The closed formulas for the Fibonacci sequence and Lucas sequence are

\[ F_n = \frac{1}{\sqrt{5}} (\phi^n - \psi^n) \quad (3) \]

\[ L_n = \phi^n + \psi^n, \quad (4) \]

respectively, where \(\phi = \frac{1+\sqrt{5}}{2}\) and \(\psi = \frac{1-\sqrt{5}}{2}\). These formulas are also known as Binet’s formula.

We first consider the generalized Leonardo sequence.

**Definition 2.1** (Kuhapatanakul et al. [13]). The generalized Leonardo sequence \(\{L_{k,n}\}\), with a fixed positive integer \(k\), is defined by

\[ L_{k,n} = L_{k,n-1} + L_{k,n-2} + k, \quad n \geq 2, \quad (5) \]

with the initial conditions \(L_{k,0} = L_{k,1} = 1\).

A version of the Leonardo-like sequence \(\{C_n(a, b, k)\}\), defined by

\[ C_n(a, b, k) = C_{n-1}(a, b, k) + C_{n-2}(a, b, k) + k, \quad (6) \]

with \(C_0(a, b, k) = b - a - k, \quad C_1(a, b, k) = a,\) and \(k\) is a constant, has been studied by Bicknell-Johnson and Bergum [3]. The generalized Leonardo sequence arises as a special case of \(C_n\):

\[ L_{k,n} = C_n(1, 2 + k, k). \]
**Theorem 2.1** (Kuhapatanakul et al. [13]). The closed formula for the generalized Leonardo sequence \( \{ L_{k,n} \} \) is
\[
L_{k,n} = (1 + k)F_{n+1} - k, \tag{7}
\]

**Corollary 2.1** (Catarino and Borges [4]). Let \( \{ L_n \} \) be the classical Leonardo sequence be defined by \( L_n = L_{n-1} + L_{n-2} + 1, \ n \geq 2 \) with initial conditions \( L_0 = L_1 = 1 \). Then
\[
L_n = 2F_{n+1} - 1.
\]

**Proof.** Let \( k = 1 \) in the previous theorem. \( \square \)

**Corollary 2.2.** We have

(i) \( L_{k,n} = (k + 1)\frac{L_{1,n-1}}{2} + 1, \)
(ii) \( L_{k+1,n} - L_{k,n} = \frac{L_{1,n-1}}{2}. \)

**Proof.** From Corollary 2.1, we have \( L_{1,n} = L_n = 2F_{n+1} - 1 \). Then \( F_{n+1} = \frac{L_{1,n+1}}{2} \). By Theorem 2.1, we have
\[
L_{k,n} = (1 + k)F_{n+1} - k = (1 + k)\frac{L_{1,n+1}}{2} - k = (k + 1)\frac{L_{1,n} - 1}{2} + 1,
\]
which proves (i).

Next, again by Theorem 2.1,
\[
L_{k+1,n} = (k + 2)F_{n+1} - (k + 1),
L_{k,n} = (k + 1)F_{n+1} - k.
\]

Subtract these two equations yields
\[
L_{k+1,n} - L_{k,n} = (k + 2)F_{n+1} - k - 1 - (k + 1)F_{n+1} + k = F_{n+1} - 1 = \frac{L_{1,n} - 1}{2},
\]
which proves (ii). \( \square \)

3 \quad Main results

Let \( \{ a_n \} \) be a sequence of order 2 satisfying the following homogeneous linear recurrence relation:
\[
a_n = pa_{n-1} + qa_{n-2}, \quad n \geq 2, \tag{8}
\]
where \( a_0, a_1, p, q \neq 0 \) are given constants. Let \( \alpha \) and \( \beta \) be two roots of the characteristic equation of (8):
\[
x^2 - px - q = 0. \tag{9}
\]
He and Shiue [7] proved the following theorem that gives the general formula of \( \{ a_n \} \).
Theorem 3.1 (He and Shiue [7]). Let \( \{a_n\} \) be a sequence of order 2 satisfying the linear recurrence relation (8). Then

\[
a_n = \begin{cases} \frac{a_1 - \beta a_0}{\alpha - \beta} \alpha^n - \frac{a_1 - \alpha a_0}{\alpha - \beta} \beta^n, & \text{if } \alpha \neq \beta; \\ \frac{n}{n\alpha^n - (n - 1)\alpha^n}, & \text{if } \alpha = \beta, \end{cases}
\]

where \( \alpha \) and \( \beta \) are the two roots of (9).

Corollary 3.1. If \( a_0 = 0 \) and \( a_1 = 1 \), then the general formula is given by

\[
a_n = \begin{cases} \frac{1}{\alpha - \beta} \alpha^n - \frac{1}{\alpha - \beta} \beta^n, & \text{if } \alpha \neq \beta; \\ \frac{n}{n\alpha^n - (n - 1)\alpha^n}, & \text{if } \alpha = \beta. \end{cases}
\]

Corollary 3.2. If \( a_0 = 1 \) and \( a_1 = 1 \), then the general formula is given by

\[
a_n = \begin{cases} \frac{1 - \beta}{\alpha - \beta} \alpha^n - \frac{1 - \alpha}{\alpha - \beta} \beta^n, & \text{if } \alpha \neq \beta; \\ \frac{n}{n\alpha^n - (n - 1)\alpha^n}, & \text{if } \alpha = \beta. \end{cases}
\]

Theorem 3.1, Corollary 3.1, and Corollary 3.2 will be used in the main results.

In this paper, we will consider the sequence \( \{a_n(t, j)\} \) satisfying the second order non-homogeneous linear recurrence relation:

\[
a_n(t, j) = pa_{n-1}(t, j) + qa_{n-2}(t, j) + (p + q - 1)(tn + j), \quad n \geq 2, \ t, j \in \mathbb{Z},
\]

where \( a_0(t, j) \), \( a_1(t, j) \), \( p \), and \( q \), with \( p + q \neq 1 \), are given constants.

We will write \( a_n(t, j) \) as \( w_n \), to follow Horadam’s [9] notation:

\[
w_n \equiv w_n(w_0, w_1, p, q, t, j) = a_n(t, j),
\]

with \( w_0 = a_0(t, j), w_1 = a_1(t, j), w_n(w_0, w_1, p, q, 0, 0) = a_n, n \geq 2. \)

We now give the general formula of \( w_n \):

Theorem 3.2. Let \( \{w_n(w_0, w_1, p, q, t, j)\} \) be a sequence of order 2 satisfying the non-homogeneous linear relation in the following form:

\[
w_n = pw_{n-1} + qw_{n-2} + (p + q - 1)(tn + j), \quad n \geq 2, \ t, j \in \mathbb{Z},
\]

where \( w_0, w_1, p, q \), with \( p + q \neq 1 \), are given constants. Then

\[
w_n = w_n(w_0, w_1, p, q, 0, 0) + \left(j - \frac{t(p + 2q)}{1 - p - q}\right) (w_n(1, 1, p, q, 0, 0) - 1) + t (w_n(0, 1, p, q, 0, 0) - n).
\]

Proof. First we consider the homogeneous part

\[
w_n(w_0, w_1, p, q, 0, 0) = pw_{n-1}(w_0, w_1, p, q, 0, 0) + qw_{n-2}(w_0, w_1, p, q, 0, 0).
\]

Then the characteristic equation

\[x^2 = px + q\]

gives

\[x = \frac{p \pm \sqrt{p^2 + 4q}}{2}.
\]
Let \( \alpha = \frac{v + \sqrt{v^2 + 4q}}{2} \) and \( \beta = \frac{v - \sqrt{v^2 + 4q}}{2} \). Then the homogeneous solution of (15) is
\[
w_n(w_0, w_1, p, q, 0, 0) = c_1\alpha^n + c_2\beta^n.
\]

Suppose \( \alpha \neq \beta \). Assume the particular solution is of the form
\[
w_p^n = An + B,
\]
where \( A = A(t, j) \) and \( B = B(t, j) \). Then we have
\[
An + B = p(A(n - 1) + B) + q(A(n - 2) + B) + (p + q - 1)(tn + j).
\]
Solving for \( A \) and \( B \), we have
\[
A = -t,
\]
\[
B = \frac{t(p + 2q)}{1 - p - q} - j.
\]
Then
\[
w_n = c_1\alpha^n + c_2\beta^n - tn - j + \frac{t(p + 2q)}{1 - p - q}.
\]
Using the initial conditions \( w_0 \) and \( w_1 \), we have
\[
\begin{align*}
w_0 &= c_1 + c_2 - j + \frac{t(p + 2q)}{1 - p - q}, \\
w_1 &= c_1\alpha + c_2\beta - j + \frac{t(p + 2q)}{1 - p - q}.
\end{align*}
\]
Multiplying the first equation by \( \alpha \) and subtract with the second, we have
\[
w_0\alpha - w_1 = c_2(\alpha - \beta) + \left( -j + \frac{t(p + 2q)}{1 - p - q} \right)\alpha + t + j + \frac{t(p + 2q)}{1 - p - q}
\]
\[
\implies c_2 = \frac{w_0\alpha - w_1}{\alpha - \beta} - \frac{t}{\alpha - \beta} + \left( -j + \frac{t(p + 2q)}{1 - p - q} \right) \left( \frac{1 - \alpha}{\alpha - \beta} \right).
\]
Then
\[
c_1 = \frac{w_1 - w_0\beta}{\alpha - \beta} + \frac{t}{\alpha - \beta} + \left( -j + \frac{t(p + 2q)}{1 - p - q} \right) \left( \frac{\beta - 1}{\alpha - \beta} \right).
\]
Thus, the general solution is
\[
w_n = \left[ \frac{w_1 - w_0\beta}{\alpha - \beta} + \frac{t}{\alpha - \beta} + \left( -j + \frac{t(p + 2q)}{1 - p - q} \right) \left( \frac{\beta - 1}{\alpha - \beta} \right) \right] \alpha^n
\]
\[
+ \left[ \frac{w_0\alpha - w_1}{\alpha - \beta} - \frac{t}{\alpha - \beta} + \left( -j + \frac{t(p + 2q)}{1 - p - q} \right) \left( \frac{1 - \alpha}{\alpha - \beta} \right) \right] \beta^n
\]
\[
- tn - j + \frac{t(p + 2q)}{1 - p - q}.
\]
We can rewrite it as
\[
w_n = \left( \frac{w_1 - w_0\beta}{\alpha - \beta} \right) \alpha^n - \left( \frac{w_1 - w_0\alpha}{\alpha - \beta} \right) \beta^n - \left( j - \frac{t(p + 2q)}{1 - p - q} \right) \left( \frac{(\beta - 1)\alpha^n + (1 - \alpha)\beta^n}{\alpha - \beta} \right)
\]
\[
+ t \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) - tn - j + \frac{t(p + 2q)}{1 - p - q}.
\]

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Using (10), (11), and (12), we have
\[
w_n = w_n(w_0, w_1, p, q, 0, 0) + \left( j - \frac{t(p + 2q)}{1 - p - q} \right) (w_n(1, 1, p, q, 0, 0) - 1) \\
+ t \left( w_n(0, 1, p, q, 0, 0) - n \right).
\] (18)

for \( \alpha \neq \beta \).

Now, if \( \alpha = \beta \), we have
\[
w_n = (c_1 + c_2n) \alpha^n.
\]

The solution is
\[
w_n = (c_1 + c_2n) \alpha^n - tn - j + \frac{t(p + 2q)}{1 - p - q}.
\]

Using the initial conditions,
\[
w_0 = c_1 - j + \frac{t(p + 2q)}{1 - p - q}
\]
\[
w_1 = c_1\alpha + c_2\beta - t - j + \frac{t(p + 2q)}{1 - p - q}
\]

Then
\[
c_1 = w_0 + j - \frac{t(p + 2q)}{1 - p - q}
\]
\[
c_2 = \frac{w_1}{\alpha} - w_0 - j + \frac{t(p + 2q)}{1 - p - q} + \frac{1}{\alpha} \left( t + j - \frac{t(p + 2q)}{1 - p - q} \right)
\]

\[
w_n = \left( w_0 + j - \frac{t(p + 2q)}{1 - p - q} \right) \alpha^n + \left[ \frac{w_1}{\alpha} - w_0 - j + \frac{t(p + 2q)}{1 - p - q} + \frac{1}{\alpha} \left( t + j - \frac{t(p + 2q)}{1 - p - q} \right) \right] n\alpha^n
\]
\[
- tn - j + \frac{t(p + 2q)}{1 - p - q}
\]
\[
= \left( w_0 + j - \frac{t(p + 2q)}{1 - p - q} \right) (\alpha^n - n\alpha^n) + \left( w_1 + t + j - \frac{t(p + 2q)}{1 - p - q} \right) n\alpha^{n-1}
\]
\[
- tn - j + \frac{t(p + 2q)}{1 - p - q}
\]
\[
= w_1n\alpha^{n-1} - w_0(n - 1)\alpha^n + \left( j - \frac{t(p + 2q)}{1 - p - q} \right) (n\alpha^{n-1} - (n - 1)\alpha^n - 1) + tn\alpha^{n-1} - tn.
\]

Thus, if \( \alpha = \beta \), the solution is
\[
w_n = w_1n\alpha^{n-1} - w_0(n - 1)\alpha^n + \left( j - \frac{t(p + 2q)}{1 - p - q} \right) (n\alpha^{n-1} - (n - 1)\alpha^n - 1) + tn\alpha^{n-1} - tn.
\]

Using (10), (11), and (12), then
\[
w_n = w_n(w_0, w_1, p, q, 0, 0) + \left( j - \frac{t(p + 2q)}{1 - p - q} \right) (w_n(1, 1, p, q, 0, 0) - 1) \\
+ t(w_n(0, 1, p, q, 0, 0) - n).
\]

The results for both cases are the same. \( \square \)
Corollary 3.3 (Bicknell-Johnson et al. [3]). Consider the Leonardo-like sequence $C_n(a, b, k)$ defined in (6). Using Horadam’s notation, we have

$$w_n = w_n(b - a - k, a, 1, 1, 0, k).$$

Then

$$w_n(b - a - k, a, 1, 1, 0, k) = aF_{n-2} + bF_{n-1} + k(F_n - 1).$$  \hfill (19)

**Proof.** By Theorem 3.2,

$$w_n(b - a - k, a, 1, 1, 0, k) = w_n(b - a - k, a, 1, 1, 0, 0) + k(w_n(1, 1, 1, 1, 0, 0) - 1)$$

$$= w_n(b - a - k, a, 1, 1, 0, 0) + k(F_{n+1} - 1).$$

Since $p = q = 1$ and $t = j = 0$, we can use Theorem 3.1, with $\alpha = \phi$ and $\beta = \psi$:

$$w_n(b - a - k, a, 1, 1, 0, 0) = a - \psi(b - a - k)\frac{\phi^n}{\phi - \psi} - a - \phi(b - a - k)\frac{\psi^n}{\phi - \psi}$$

$$= a\left(\frac{\phi^n - \psi^n}{\phi - \psi}\right) + (b - a - k)\left(\frac{\phi^{n-1} - \psi^{n-1}}{\phi - \psi}\right)$$

$$= aF_n + (b - a - k) F_{n-1} = aF_{n-2} + (b - k) F_{n-1}.$$  

Hence, by Theorem 3.2,

$$w_n(b - a - k, a, 1, 1, 0, k) = aF_{n-2} + (b - k) F_{n-1} + k(F_{n+1} - 1)$$

$$= aF_{n-2} + bF_{n-1} + k(F_n - 1).$$  \hfill $\Box$

**Corollary 3.4.** Consider the general Leonardo sequence $\{w_n(1, 1, 1, 1, t, j)\}$. Then

$$w_n(1, 1, 1, 1, t, j) = (1 + 3t + j)F_{n+1} + t(F_n - n - 3) - j.$$  \hfill (20)

**Proof.** Let $p = q = 1$ and $w_0 = w_1 = 1$ in (16) of Theorem 3.2. Recall that the Fibonacci sequence $\{F_n\}$ satisfies the second order linear recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$  \hfill (21)

where $F_0 = 0$ and $F_1 = 1$. By (11), we have

$$w_n(0, 1, 1, 1, 0, 0) = F_n, \ w_n(1, 1, 1, 1, 0, 0) = F_{n+1}$$

where $\alpha = \phi$ and $\beta = \psi$. Then

$$w_n(1, 1, 1, 1, t, j) = w_n(1, 1, 1, 1, 0, 0) + \left(j - \frac{t(1 + 2)}{1 - 1 - 1}\right) (w_n(1, 1, 1, 1, 0, 0) - 1)$$

$$+ t\left(w_n(0, 1, 1, 1, 0, 0) - n\right)$$

$$= F_{n+1} + (j + 3t) (F_{n+1} - 1) + t(F_n - n)$$

$$= (1 + j + 3t)F_{n+1} + tF_n - tn - j - 3t$$

$$= (1 + 3t + j)F_{n+1} + t(F_n - n - 3) - j.$$

$\Box$

**Corollary 3.5** (Shannon et al. [18]). Consider the sequence $\{w_n(1, 1, 1, 1, j)\}$ of order 2 satisfying the non-homogeneous linear recurrence relation (15). Then

$$w_n(1, 1, 1, 1, j) = (4 + j) F_{n+1} + F_n - n - 3 - j.$$  \hfill (22)

**Proof.** Let $t = 1$ in Corollary 3.4 yield the result.  \hfill $\Box$
Corollary 3.6. The closed formula for the generalized Leonardo sequence \( \{L_{k,n}\} \) defined in Definition 2.1 is

\[
L_{k,n} = (1 + k)F_{n+1} - k
\]
as given in Theorem 2.1.

Proof. Let \( t = 0 \) and \( j = k \) in Corollary 3.4 yield the result. \( \square \)

Corollary 3.7 (Shannon et al. [18]). Consider the sequence \( \{w_n(1,1,1,1,0)\} \) of order 2 satisfying the non-homogeneous linear recurrence relation (15). Then

\[
w_n(1,1,1,1,0) = 4F_{n+1} + F_n - n - 3
\]

Proof. Let \( t = 1 \) and \( j = 0 \) in Corollary 3.4 yield the result. \( \square \)

Theorem 3.3. Let \( \{w_n(w_0,w_1,p,q,t,j)\} \) be a sequence of order 2 satisfying the non-homogeneous linear relation in the following form:

\[
w_n = pw_{n-1} + qw_{n-2} + (p + q - 1)(tn + j), \quad n \geq 2, \quad t, j \in \mathbb{Z}, \tag{24}
\]

where \( w_0, w_1, p, q \), with \( p + q \neq 1 \), are given constants. Then

\[
w_n(w_0, w_1, p, q, t, j + 1) - w_n(w_0, w_1, p, q, t, j) = w_n(1, 1, p, q, 0, 0) - 1 \text{ and} \tag{25}
w_n(w_0, w_1, p, q, t, j + k) - w_n(w_0, w_1, p, q, t, j) = k (w_n(1, 1, p, q, 0, 0) - 1). \tag{26}
\]

Proof. Using the result from Theorem 3.2, we have

\[
w_n(w_0, w_1, p, q, t, j + 1) = w_n(w_0, w_1, p, q, 0, 0) + \left(j + 1 - \frac{t(p + 2q)}{1 - p - q}\right)(w_n(1, 1, p, q, 0, 0) - 1)
\]

\[+ t (w_n(0, 1, p, q, 0, 0) - n)
\]

and

\[
w_n(w_0, w_1, p, q, t, j) = w_n(w_0, w_1, p, q, 0, 0) + \left(j - \frac{t(p + 2q)}{1 - p - q}\right)(w_n(1, 1, p, q, 0, 0) - 1)
\]

\[+ t (w_n(0, 1, p, q, 0, 0) - n).
\]

Subtracting the two equations yields

\[
w_n(w_0, w_1, p, q, t, j + 1) - w_n(w_0, w_1, p, q, t, j) = w_n(1, 1, p, q, 0, 0) - 1.
\]

The second result can be obtained by repeating the same process and replacing \( j + 1 \) by \( j + k \). \( \square \)

Corollary 3.8. Consider the Leonardo-like sequence \( \{w_n(b - a - k, a, 1, 1, 0, k)\} \). Then for \( n \geq 2 \),

\[
w_n(b - a - k, a, 1, 1, 0, k + 1) - w_n(b - a - k, a, 1, 1, 0, k) = F_{n+1} - 1.
\]

Proof. By Theorem 3.3,

\[
w_n(b - a - k, a, 1, 1, 0, k + 1) - w_n(b - a - k, a, 1, 1, 0, k) = w_n(1, 1, 1, 1, 0, 0) - 1 = F_{n+1} - 1. \quad \square
\]
Corollary 3.9 (Shannon et al. [17]). Consider the general Leonardo sequence \( \{w_n(1, 1, 1, t, j)\} \).
Then for \( n \geq 2 \),
\[
w_n(1, 1, 1, 1, t, j + 1) - w_n(1, 1, 1, t, j) = F_{n+1} - 1. \tag{27}
\]

Proof. Using Theorem 3.3. We have
\[
w_n(1, 1, 1, 1, t, j + 1) - w_n(1, 1, 1, 1, t, j) = w_n(1, 1, 1, 1, 0, 0) - 1 = F_{n+1} - 1. \]

Note that \( w_n(1, 1, 1, 1, 0, k) = L_{k,n} \). Hence when \( t = 0 \), we have the same result as Corollary 2.2 (ii).

Next, note that this difference is independent of \( t \). A table by Shannon and Deveci [18] for \( t = 1 \) is given here:

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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
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<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>15</td>
<td>27</td>
<td>47</td>
</tr>
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<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>15</td>
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<td>47</td>
<td>80</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>4</td>
<td>8</td>
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<td>28</td>
<td>50</td>
<td>87</td>
<td>147</td>
<td>245</td>
</tr>
</tbody>
</table>

Differences 0 0 1 2 4 7 12 20 33

Table 1. "Extended Leonardo sequence", [18].

We now give two more tables with \( t = 2 \) and \( t = 3 \) to show the Independence of \( t \):

<table>
<thead>
<tr>
<th>( j )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>-3</td>
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<td>29</td>
<td>53</td>
<td>93</td>
<td>159</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>19</td>
<td>36</td>
<td>65</td>
<td>113</td>
<td>192</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>11</td>
<td>23</td>
<td>43</td>
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<td>1</td>
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<td>291</td>
</tr>
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<td>1</td>
<td>1</td>
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<td>39</td>
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<td>125</td>
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<td>357</td>
</tr>
</tbody>
</table>

Differences 0 0 1 2 4 7 12 20 33

Table 2. Extended Leonardo sequence with \( t = 2 \).
Table 3. Extended Leonardo sequence with \( t = 3 \).

<table>
<thead>
<tr>
<th>( j )</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
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</tr>
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<td>1</td>
<td>6</td>
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<td>57</td>
<td>103</td>
<td>179</td>
</tr>
<tr>
<td></td>
<td>−2</td>
<td>1</td>
<td>1</td>
<td>7</td>
<td>16</td>
<td>34</td>
<td>64</td>
<td>115</td>
<td>199</td>
</tr>
<tr>
<td></td>
<td>−1</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>18</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>1</td>
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<td>1</td>
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<td>1</td>
<td>12</td>
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<td>2</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>20</td>
<td>33</td>
</tr>
</tbody>
</table>

**Theorem 3.4.** Let \( \{a_n\} \) be a sequence of order 2 satisfying the non-homogeneous linear relation:

\[
a_n = a_{n-1} + a_{n-2} + Ck^n, \quad n \geq 2,\tag{28}
\]

where \( a_0 = 0, a_1 = 1, C \neq 0, k \neq 0, \) and \( k^2 - k - 1 \neq 0. \) Then

\[
a_n = \left(1 - \frac{Ck^3}{k^2 - k - 1}\right)F_n + \left(1 - \frac{Ck^2}{k^2 - k - 1}\right)F_{n-1} + \frac{Ck^{n+2}}{k^2 - k - 1}.	ag{29}
\]

**Proof.** The homogeneous solution is

\[
a_n = c_1 \phi^n + c_2 \psi^n,
\]

where \( \phi = \frac{1 + \sqrt{5}}{2} \) and \( \psi = \frac{1 - \sqrt{5}}{2} \).

The particular solution can be found using the method of undetermined coefficients. Assume the particular solution is of the form \( a_n^* = Ak^n \), where \( A \) is a constant. Then

\[
Ak^n = Ak^{n-1} + Ak^{n-2} + Ck^n
\]

\( \implies A = \frac{Ck^2}{k^2 - k - 1} \).

Hence, the general solution to (28) is

\[
a_n = c_1 \phi^n + c_2 \psi^n + \frac{Ck^{n+2}}{k^2 - k - 1}.
\]

With \( a_0 = a_1 = 1 \), we have the following system

\[
\begin{align*}
1 &= c_1 + c_2 + \frac{Ck^2}{k^2 - k - 1} \\
1 &= c_1 \phi + c_2 \psi + \frac{Ck^3}{k^2 - k - 1}
\end{align*}
\]

\( \implies \begin{cases} c_1 + c_2 = 1 - \frac{Ck^2}{k^2 - k - 1} \\ c_1 \phi + c_2 \psi = 1 - \frac{Ck^3}{k^2 - k - 1} \end{cases} \).

Then

\[
c_2(\phi - \psi) = \phi - \frac{Ck^2 \phi}{k^2 - k - 1} - 1 + \frac{Ck^3}{k^2 - k - 1}
\]

\( \implies c_2 = \frac{\phi - 1}{\sqrt{5}} + \frac{Ck^3 - Ck^2 \phi}{\sqrt{5}(k^2 - k - 1)} \)

\( = \frac{\psi}{\sqrt{5}} + \frac{Ck^3 - Ck^2 \phi}{\sqrt{5}(k^2 - k - 1)} \).

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Moreover,
\[
c_1 = 1 - \frac{Ck^2}{k^2 - k - 1} + \frac{\psi}{\sqrt{5}} - \frac{Ck^3 - Ck^2 \phi}{\sqrt{5}(k^2 - k - 1)} = \frac{(k^2 - k - 1 - Ck^2)(\phi - \psi) + \psi(k^2 - k - 1) - Ck^3 + Ck^2 \phi}{\sqrt{5}(k^2 - k - 1)} \]
\[
= \frac{-(k + 1)(\phi - \psi) + (k^2 - Ck^2)(\phi - \psi) + k^2 \psi - (k + 1)\psi - Ck^3 + Ck^2 \phi}{\sqrt{5}(k^2 - k - 1)} \]
\[
= \frac{-(k + 1)\phi + k^2 ((1 - C)(\phi - \psi) - \alpha k + C \phi + \psi)}{\sqrt{5}(k^2 - k - 1)} \]
\[
= \frac{-(k + 1)\phi + k^2 (\phi + C \psi - Ck)}{\sqrt{5}(k^2 - k - 1)} = \frac{\phi}{\sqrt{5}} - \frac{Ck^3 - Ck^2 \psi}{\sqrt{5}(k^2 - k - 1)}.
\]

Hence, the general solution to (28) is
\[
a_n = \left( \frac{\phi}{\sqrt{5}} - \frac{Ck^3 - Ck^2 \psi}{\sqrt{5}(k^2 - k - 1)} \right) \phi^n - \left( \frac{\psi}{\sqrt{5}} - \frac{Ck^3 - Ck^2 \phi}{\sqrt{5}(k^2 - k - 1)} \right) \psi^n + \frac{Ck^{n+2}}{k^2 - k - 1} \]
\[
= \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}} - \frac{Ck^3 (\phi^n - \psi^n)}{\sqrt{5}(k^2 - k - 1)} + \frac{Ck^2 \phi \psi (\phi^{n-1} - \psi^{n-1})}{\sqrt{5}(k^2 - k - 1)} + \frac{Ck^{n+2}}{k^2 - k - 1} \]
\[
= F_{n+1} - \frac{Ck^3}{k^2 - k - 1} \frac{F_n - Ck^2}{k^2 - k - 1} F_{n-1} + \frac{Ck^{n+2}}{k^2 - k - 1} \]
\[
= \left( 1 - \frac{Ck^3}{k^2 - k - 1} \right) F_n + \left( 1 - \frac{Ck^2}{k^2 - k - 1} \right) F_{n-1} + \frac{Ck^{n+2}}{k^2 - k - 1}. \]
\]

**Corollary 3.10** (Shannon et al. [18]). Consider a sequence \( \{a_{j,n}\} \) of order 2 satisfying the following non-homogeneous linear recurrence relation:
\[
a_{j,n} = a_{j,n-1} + a_{j,n-2} + (-1)^n j, \quad n \geq 2, \quad j \geq 0,
\]
\[
(30)
\]
where \( a_0 = 0, \quad a_1 = 1. \) Then
\[
a_{j,n} = F_{n+1} + j F_{n-2} + (-1)^n j, \quad n \geq 2.
\]
\[
(31)
\]
**Proof.** Let \( C = j \) and \( k = -1. \) Then
\[
a_n = \left( 1 - \frac{-j}{1} \right) F_n + \left( 1 - \frac{j}{1} \right) F_{n-1} + \frac{(-1)^{n+2} j}{1} \]
\[
= (1 + j) F_n + (1 - j) F_{n-1} + (-1)^n j \]
\[
= F_{n+1} + j F_{n-2} + (-1)^n j. \]
\]

**Corollary 3.11** (Shannon et al. [18]). Consider a sequence \( \{a_n\} \) of order 2 satisfying the following non-homogeneous linear recurrence relation:
\[
a_n = a_{n-1} + a_{n-2} + (-1)^n, \quad n \geq 2,
\]
\[
(32)
\]
where \( a_0 = 0, a_1 = 1. \) Then
\[
a_n = 2 F_n + (-1)^n.
\]
\[
(33)
\]
Proof. Let \( j = 1 \) in the previous corollary. Then
\[
a_n = F_{n+1} + F_{n-2} + (-1)^n = F_n + F_{n-1} + F_n - F_n + (-1)^n = 2F_n + (-1)^n.
\]

Corollary 3.12 (Shannon et al. [18]). Consider a sequence \( \{a_{j,n}\} \) of order 2 satisfying the following non-homogeneous linear recurrence relation: Let
\[
a_{j,n} = a_{j,n-1} + a_{j,n-2} + (-1)^n j, \ n \geq 2, \ j \geq 0,
\]
where \( a_0 = 0, = a_1 = 1 \). Then
\[
a_{j+1,n} - a_{j,n} = F_{n-2} + (-1)^n, \ n \geq 2.
\] (34)

Proof. By Corollary 3.10, we have
\[
a_{j,n} = F_{n+1} + jF_{n-2} + (-1)^n j
\]
and
\[
a_{j+1,n} = F_{n+1} + (j + 1)F_{n-2} + (-1)^n(j + 1).
\]
Then
\[
a_{j+1,n} - a_{j,n} = F_{n-2} + (-1)^n.
\]

<table>
<thead>
<tr>
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<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td></td>
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<td>11</td>
<td>2</td>
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<td>15</td>
<td>48</td>
<td>53</td>
<td>111</td>
<td>154</td>
<td></td>
</tr>
</tbody>
</table>

Differences | 0 | 0 | 1 | 0 | 2 | 1 | 4 | 4 | 9 | 12 |

Table 4. Table of values for Corollary 3.14.

4 Examples

Consider
\[
w_n(w_0, w_1, p, q, 0, 0) = pw_{n-1}(w_0, w_1, p, q, 0, 0) + qw_{n-2}(w_0, w_1, p, q, 0, 0), \ n \geq 2,
\]
where \( w_0, w_1, p, \) and \( q \neq 0 \) are given constants. The following table by Koshy [12] lists some well-known sequences:
<table>
<thead>
<tr>
<th>Sequence of numbers</th>
<th>$w_0$</th>
<th>$w_1$</th>
<th>$p$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fibonacci $F_n$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Lucas $L_n$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Pell $P_n$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Pell–Lucas $Q_n$</td>
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<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Mersenne $M_n$</td>
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<td>1</td>
<td>3</td>
<td>$-2$</td>
</tr>
<tr>
<td>Jacobsthal $J_n$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Jacobsthal–Lucas $J_n$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Balancing $B_n$</td>
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<td>1</td>
<td>6</td>
<td>$-1$</td>
</tr>
<tr>
<td>Lucas-balancing $C_n$</td>
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<td>3</td>
<td>6</td>
<td>$-1$</td>
</tr>
<tr>
<td>M. Ward $W_n$</td>
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<td>1</td>
<td>4</td>
<td>$-1$</td>
</tr>
<tr>
<td>Fermat of the first kind $T_n$</td>
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<td>3</td>
<td>3</td>
<td>$-2$</td>
</tr>
<tr>
<td>Fermat of the second kind $S_n$</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>$-2$</td>
</tr>
</tbody>
</table>

Table 5. Some well-known sequences, [12].

**Example 4.1.** Let $w_0 = 2$, $w_1 = 1$, $p = q = 1$ in (15), i.e.,

$$w_n(2, 1, 1, 1, t, j) = w_{n-1}(2, 1, 1, 1, t, j) + w_{n-2}(2, 1, 1, 1, t, j) + tn + j, \quad n \geq 2, \quad t \in \mathbb{Z}.$$  

Then

1. $w_n(2, 1, 1, 1, t, j) = L_n + (j + 3t)(F_{n+1} - 1) + t(F_n - n)$;

2. $w_n(2, 1, 1, 1, t, j + 1) - w_n(2, 1, 1, 1, t, j) = F_{n+1} - 1$.

**Proof.** Since $p = q = 1$, $\alpha = \frac{1 + \sqrt{5}}{2}$, $\beta = \frac{1 - \sqrt{5}}{2}$, $w_n(w_0, w_1, p, q, 0, 0) = w_n(2, 1, 1, 1, 0, 0) = L_n$, $w_n(0, 1, p, q, 0, 0) = w_n(0, 1, 1, 1, 0, 0) = F_n$, and $w_n(1, 1, p, q, 0, 0) = w_n(1, 1, 1, 1, 0, 0) = F_{n+1}$. Then by Theorem 3.2,

$$w_n(2, 1, 1, 1, t, j) = w_n(2, 1, 1, 1, 0, 0) + \left( j - \frac{t(1+2)}{1-1-1} \right)(w_n(1, 1, 1, 1, 0, 0) - 1)$$

$$+ t(w_n(0, 1, 1, 1, 0, 0) - n)$$

$$= L_n + (j + 3t)(F_{n+1} - 1) + t(F_n - n).$$

We use Theorem 3.3 to obtain the second result. We have

$$w_n(2, 1, 1, 1, t, j + 1) - w_n(2, 1, 1, 1, t, j) = w_n(1, 1, 1, 1, 0, 0) - 1$$

$$= F_{n+1} - 1. \quad \square$$

We give three tables to show this difference.
For $t = 1$:

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>11</td>
<td>20</td>
<td>35</td>
<td>60</td>
</tr>
<tr>
<td>$-2$</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>10</td>
<td>18</td>
<td>32</td>
<td>55</td>
<td>93</td>
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<tr>
<td>$-1$</td>
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<td>1</td>
<td>4</td>
<td>7</td>
<td>14</td>
<td>25</td>
<td>44</td>
<td>75</td>
<td>126</td>
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<tr>
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<td>2</td>
<td>1</td>
<td>5</td>
<td>9</td>
<td>18</td>
<td>32</td>
<td>56</td>
<td>95</td>
<td>159</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>11</td>
<td>22</td>
<td>39</td>
<td>68</td>
<td>115</td>
<td>192</td>
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<tr>
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<td>1</td>
<td>7</td>
<td>13</td>
<td>26</td>
<td>46</td>
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<td>135</td>
<td>225</td>
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<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>8</td>
<td>15</td>
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<td>53</td>
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<td>155</td>
<td>258</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>20</td>
<td>33</td>
</tr>
</tbody>
</table>

Table 6. Values of $w_n(2, 1, 1, 1, j)$.

For $t = 2$:

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>8</td>
<td>17</td>
<td>32</td>
<td>58</td>
<td>101</td>
<td>172</td>
</tr>
<tr>
<td>$-2$</td>
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<td>1</td>
<td>5</td>
<td>10</td>
<td>21</td>
<td>39</td>
<td>70</td>
<td>121</td>
<td>205</td>
</tr>
<tr>
<td>$-1$</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>12</td>
<td>25</td>
<td>46</td>
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</tr>
<tr>
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<td>1</td>
<td>7</td>
<td>14</td>
<td>29</td>
<td>53</td>
<td>94</td>
<td>161</td>
<td>271</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>8</td>
<td>16</td>
<td>33</td>
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<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>20</td>
<td>33</td>
</tr>
</tbody>
</table>

Table 7. Values of $w_n(2, 1, 1, 2, j)$.

For $t = 3$:

<table>
<thead>
<tr>
<th>$j$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>13</td>
<td>28</td>
<td>53</td>
<td>96</td>
<td>167</td>
<td>284</td>
</tr>
<tr>
<td>$-2$</td>
<td>2</td>
<td>1</td>
<td>7</td>
<td>15</td>
<td>32</td>
<td>60</td>
<td>108</td>
<td>187</td>
<td>317</td>
</tr>
<tr>
<td>$-1$</td>
<td>2</td>
<td>1</td>
<td>8</td>
<td>17</td>
<td>36</td>
<td>67</td>
<td>120</td>
<td>207</td>
<td>350</td>
</tr>
<tr>
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<td>2</td>
<td>1</td>
<td>9</td>
<td>19</td>
<td>40</td>
<td>74</td>
<td>132</td>
<td>227</td>
<td>383</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>21</td>
<td>44</td>
<td>81</td>
<td>144</td>
<td>247</td>
<td>416</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>11</td>
<td>23</td>
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<td>156</td>
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<td>2</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>20</td>
<td>33</td>
</tr>
</tbody>
</table>

Table 8. Values of $w_n(2, 1, 1, 3, j)$.

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We can see that the difference resembles the sequence \( \{F_{n+1} - 1\} \).

**Example 4.2.** Let \( w_0 = 0, w_1 = 1, p = 2, \) and \( q = 1 \) in (15), i.e.,

\[
\begin{align*}
w_n(0, 1, 2, 1, t, j) &= 2w_{n-1}(0, 1, 2, 1, t, j) + w_{n-2}(0, 1, 2, 1, t, j) + 2(tn + j), \quad n \geq 2, \quad t \in \mathbb{Z}.
\end{align*}
\]

Then

(1) \( w_n(0, 1, 2, 1, t, j) = (1 + t)P_n + (j + 2t)(P_{n+1} - P_n - 1) - tn; \)

(2) \( w_n(0, 1, 2, 1, t, j + 1) - w_n(0, 1, 2, 1, t, j) = P_{n+1} - P_n - 1. \)

**Proof.** Since \( p = 2 \) and \( q = 1, \) \( \alpha = 1+\sqrt{2}, \beta = 1-\sqrt{2}, \)

\[
\begin{align*}
w_n(w_0, w_1, p, q, 0, 0) &= w_n(0, 1, 2, 1, 0, 0) = P_n. \quad \text{Then by Theorem 3.2,}
\end{align*}
\]

\[
\begin{align*}
w_n(0, 1, 2, 1, t, j) &= w_n(0, 1, 2, 1, 0, 0) + \left( j - \frac{t(2 + 2)}{1 - 2 - 1} \right) (w_n(1, 1, 2, 1, 0, 0) - 1) \\
&\quad + t (w_n(0, 1, 2, 1, 0, 0) - n) \\
&= P_n + (j + 2t)(P_{n+1} - P_n - 1) + tP_n - tn \\
&= (1 + t)P_n + (j + 2t)(P_{n+1} - P_n - 1) - tn.
\end{align*}
\]

We use Theorem 3.3 to obtain the second result. Then

\[
w_n(0, 1, 2, 1, t, j + 1) - w_n(0, 1, 2, 1, t, j) = w_n(1, 1, 2, 1, 0, 0) - 1 = P_{n+1} - P_n - 1. \quad \square
\]

**Remark 4.1.** In Example 4.2, the following identity is used:

\[
w_n(1, 1, 2, 1, 0, 0) = w_{n+1}(0, 1, 2, 1, 0, 0) - w_n(0, 1, 2, 1, 0, 0).
\]

**Proof.**

\[
\begin{align*}
w_n(0, 1, 2, 1, 0, 0) &= \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}} \\
w_{n+1}(0, 1, 2, 1, 0, 0) &= \frac{(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}}{2\sqrt{2}} \\
w_{n+1}(0, 1, 2, 1, 0, 0) - w_n(0, 1, 2, 1, 0, 0) &= \frac{(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}}{2\sqrt{2}} \\
&\quad - \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}} \\
&= \frac{\sqrt{2}(1 + \sqrt{2})^n + \sqrt{2}(1 - \sqrt{2})^n}{2\sqrt{2}} \\
&= \frac{(1 + \sqrt{2})^n + (1 - \sqrt{2})^n}{2} \\
&= w_n(1, 1, 2, 1, 0, 0). \quad \square
\end{align*}
\]

**Example 4.3.** Let \( w_0 = 2, w_1 = 2, p = 2, \) and \( q = 1 \) in (15), i.e.,

\[
\begin{align*}
w_n(2, 2, 2, 1, t, j) &= 2w_{n-1}(2, 2, 2, 1, t, j) + w_{n-2}(2, 2, 2, 1, t, j) + 2(tn + j), \quad n \geq 2, \quad t \in \mathbb{Z}.
\end{align*}
\]

Then

(1) \( w_n(2, 2, 2, 1, t, j) = Q_n + (j + 2t)(P_{n+1} - P_n - 1) + t(P_n - n); \)

(2) \( w_n(2, 2, 2, 1, t, j + 1) - w_n(2, 2, 2, 1, t, j) = P_{n+1} - P_n - 1. \)

\[766\]
Proof. Similar to the last example, \( \alpha = 1 + \sqrt{2}, \beta = 1 - \sqrt{2}, \) \( w_n(w_0, w_1, p, q, 0, 0) = w_n(2, 2, 2, 1, 0, 0) = Q_n \) and \( w_n(0, 1, 2, 1, 0, 0) = P_n. \) Then by Theorem 3.2,
\[
\begin{align*}
w_n(2, 2, 2, 1, t, j) &= w_n(2, 2, 2, 1, 0, 0) + \left( j - \frac{t(2 + 2)}{1 - 2} \right) (w_n(1, 1, 2, 1, 0, 0) - 1) \\
&+ t \left( w_n(0, 1, 2, 1, 0, 0) - n \right) \\
&= Q_n + (j + 2t) (P_n - 1) + t(P_n - n).
\end{align*}
\]

The second result is the same as the last example.

Example 4.4. Let \( w_0 = 0, \ w_1 = 1, \ p = 1, \) and \( q = 2 \) in (15), i.e.,
\[
w_n(0, 1, 1, 2, t, j) = w_{n-1}(0, 1, 1, 2, t, j) + 2w_{n-2}(0, 1, 1, 2, t, j) + 2(tn + j), \ n \geq 2, \ t \in \mathbb{Z}.
\]

Then
\[
\begin{align*}
(1). \quad w_n(0, 1, 1, 2, t, j) &= (1 + t)J_n + \left( j + \frac{5t}{2} \right) (J_n - 1) - tn; \\
(2). \quad w_n(0, 1, 1, 2, t, j + 1) - w_n(0, 1, 1, 2, t, j) &= J_n - 1.
\end{align*}
\]

Proof. Since \( p = 1 \) and \( q = 2, \) we have \( \alpha = -1, \ \beta = 2, \) and \( w_n(w_0, w_1, p, q, 0, 0) = w_n(0, 1, 1, 2, 0, 0) = J_n. \) Then by Theorem 3.2,
\[
\begin{align*}
w_n(0, 1, 1, 2, t, j) &= w_n(0, 1, 1, 2, 0, 0) + \left( j - \frac{t(1 + 2)}{1 - p - q} \right) (w_n(1, 1, 1, 2, 0, 0) - 1) \\
&+ t \left( w_n(0, 1, 1, 2, 0, 0) - n \right) \\
&= J_n + \left( j + \frac{5t}{2} \right) (J_n - 1) + tJ_n - tn \\
&= (1 + t)J_n + \left( j + \frac{5t}{2} \right) (J_n - 1) - tn.
\end{align*}
\]

We use Theorem 3.3 to obtain the second result. Then
\[
w_n(0, 1, 1, 2, t, j + 1) - w_n(0, 1, 1, 2, t, j) = w_n(1, 1, 2, 1, 0, 0) - 1 = J_n - 1.
\]

Remark 4.2. In Example 4.4, the following identity is used:
\[
w_{n+1}(0, 1, 1, 2, 0, 0) = w_n(1, 1, 1, 2, 0, 0).
\]

Proof.
\[
w_{n+1}(0, 1, 1, 2, 0, 0) = \frac{1}{3} \left( (-1)^{n+2} + 2^{n+1} \right) = \frac{1}{3} \left( (-1)^n + 2^{n+1} \right) \\
= w_n(1, 1, 1, 2, 0, 0) = J_{n+1} = J_n.
\]

Example 4.5. Let \( w_0 = 1, \ w_1 = 1, \ p = 1, \) and \( q = 2 \) in (15), i.e.,
\[
w_n(1, 1, 1, 2, t, j) = w_{n-1}(1, 1, 1, 2, t, j) + 2w_{n-2}(1, 1, 1, 2, t, j) + tn + j, \ n \geq 2, \ t \in \mathbb{Z}.
\]

Then
\[
\begin{align*}
(1). \quad w_n(1, 1, 1, 2, t, j) &= J_n + \left( j + \frac{5t}{2} \right) (J_n - 1) + t(J_n - n); \\
(2). \quad w_n(1, 1, 1, 2, t, j + 1) - w_n(1, 1, 1, 2, t, j) &= J_n - 1.
\end{align*}
\]

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Then by Cassini’s identity.

\[ w_n(1, 1, 1, 2, t, j) = w_n(1, 1, 1, 2, 0, 0) + \left( j - \frac{t(1 + 2 \cdot 2)}{1 - 1 - 2} \right) (w_n(1, 1, 1, 2, 0, 0) - 1) + t (w_n(0, 1, 1, 2, 0, 0) - n) \]
\[ = J_n + \left( j + \frac{5t}{2} \right) (J_n - 1) + t(J_n - n). \]

The second result is the same as the previous example. \hfill \square

**Remark 4.3.** In Examples 4.2, 4.3, 4.4, 4.5, the homogeneous parts are Pell sequence \{P_n\}, Pell–Lucas sequence \{Q_n\}, Jacobsthal sequence \{J_n\}, and Jacobsthal–Lucas sequence \{J_n\}, respectively.

## 5 Some identities involving the generalized Leonardo sequence

**Theorem 5.1.** Let \{L_{k,n}\} denote the generalized Leonardo sequence. Then

1. (Shattuck [19]) \[ L_{k,n}^2 - L_{k,n-1}L_{k,n+1} = (-1)^n (k + 1)^2 + k(k + 1)F_{n-2}; \]

2. (Kuhapanakul [13]) \[ L_{k,m}L_{k,n-1} + L_{k,m-1}L_{k,n} = L_{k,m+1}L_{k,n+1} - (k + 1)L_{k,m+n} - k. \]

**Proof.** By Theorem 2.1, we can write the generalized Leonardo sequence as

\[ L_{k,n} = (1 + k)F_{n+1} - k. \] (35)

Then

\[ L_{k,n}^2 - L_{k,n-1}L_{k,n+1} = (1 + k)^2 F_{n+1}^2 - 2k(1 + k)F_{n+1} + k^2 - ((1 + k)F_n - k)((1 + k)F_{n+2} - k) \]
\[ = (1 + k)^2 (F_{n+1} - F_n F_{n+2}) - k(k + 1)(2F_{n+1} - F_n - F_{n+2}) \]
\[ = (1 + k)^2 (-1)^n - k(1 + k)(2F_{n+1} - F_n - F_{n+1} - F_n) \]
\[ = (1 + k)^2 (-1)^n - k(1 + k)F_{n-2}, \]

by Cassini’s identity.

For the second result, we first note Honsberger’s identity

\[ F_{n-1}F_m + F_n F_{m+1} = F_{m+n}. \]

Then

\[ L_{k,m}L_{k,n-1} = (1 + k)^2 F_{m+1}F_n - k(1 + k)(F_{m+1} + F_n) + k^2, \]
\[ L_{k,m-1}L_{k,n} = (1 + k)^2 F_m F_{n+1} - k(1 + k)(F_m + F_{n+1}) + k^2, \]
\[ L_{k,m+1}L_{k,n+1} = (1 + k)^2 F_{m+2}F_{n+2} - k(1 + k)(F_{m+2} + F_{n+2}) + k^2 \]
\[ = (1 + k)^2 (F_{m+1}F_{n+1} + F_{m+1}F_n + F_{n+1}F_m + F_m F_n) \]
\[ - k(1 + k)(F_{m+1} + F_m + F_{n+1} + F_n) + k^2, \]
\[ L_{k,m+n} = (1 + k)F_{m+n+1} - k \]

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Proof. Since
\[ \mathcal{L}_{k,m} \mathcal{L}_{k,n-1} \mathcal{L}_{k,m-1} \mathcal{L}_{k,n} - \mathcal{L}_{k,m+1} \mathcal{L}_{k,n+1} = k^2 - (1 + k)^2(F_{m+1}F_{n+1} + F_mF_n) = k^2 - (1 + k)^2F_{m+n+1}. \]
Finally,
\[ \mathcal{L}_{k,m} \mathcal{L}_{k,n-1} \mathcal{L}_{k,m-1} \mathcal{L}_{k,n} - \mathcal{L}_{k,m+1} \mathcal{L}_{k,n+1} + (1 + k)\mathcal{L}_{k,m+n} + k = 0. \]

**Theorem 5.2.** Let
\[ a_0F_{n+t} + a_1F_{n+t-1} + \cdots + a_tF_n = 0, \] (36)
where \( a_0 + a_1 + \cdots + a_t = 0, a_i \in \mathbb{Z}, (i = 0, 1, 2, \ldots, t), t \) is a fixed positive integer. Then
\[ a_0\mathcal{L}_{k,n+t-1} + a_1\mathcal{L}_{k,n+t-2} + \cdots + a_t\mathcal{L}_{k,n-1} = 0. \] (37)

**Proof.** Since \( \mathcal{L}_{k,n} = (1 + k)F_{n+1} - k \), we have
\[
\begin{align*}
    a_0\mathcal{L}_{k,n+t-1} &+ a_1\mathcal{L}_{k,n+t-2} + \cdots + a_t\mathcal{L}_{k,n-1} \\
    &= a_0[(1 + k)F_{n+t} - k] + a_1[(1 + k)F_{n+t-1} - k] + \cdots + a_t[(1 + k)F_n - k] \\
    &= (1 + k)[a_0F_{n+t} + a_1F_{n+t-1} + \cdots + a_tF_n] - k[a_0 + a_1 + \cdots + a_t] \\
    &= (1 + k) \cdot 0 - k \cdot 0 = 0. \quad \square
\end{align*}
\]

**Remark 5.1.** (36) can be obtained by computing \((x^2 - x - 1)x^n(x - 1)p(x)\), where \(p(x)\) is a polynomial over \(\mathbb{Z}\) first, then replace each \(x^{n+i}\) by \(F_{n+i}\).

**Algorithm 1** Obtaining this identity

**Input:** A polynomial \(p(x)\) over \(\mathbb{Z}\)

**Output:** An identity with generalized Leonard sequence

1. \(g(x) \leftarrow (x^2 - x - 1) \cdot x^n \cdot (x - 1) \cdot p(x)\)
2. Replace each \(x^{n+i}\) by \(F_{n+i}\)
3. Verify the coefficients of \(F_{n+i}\) sums to zero
4. Replace each \(F_{n+i}\) by \(\mathcal{L}_{n+i-1}\)
5. Output the identity

**Example 5.1.** It is known that
\[ F_n + F_{n+1} + F_{n+6} - 3F_{n+4} = 0. \]
Hence \(a_0 = 1, a_1 = 0, a_2 = -3, a_3 = a_4 = 0, a_5 = 1, a_6 = 1\), i.e. \(\sum a_i = 0\). Then
\[ \mathcal{L}_{k,n+5} - 3\mathcal{L}_{k,n+3} + \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} = 0, \]
or
\[ \mathcal{L}_{k,n+5} + \mathcal{L}_{k,n} + \mathcal{L}_{k,n-1} = 3\mathcal{L}_{k,n+3}. \] (38)

**Example 5.2.** Let \(f(x) = (x^2 - x - 1)x^n\) and \(p(x) = (x - 1)(2x^3 + 3x - 1)\). Then \(g(x) = f(x) \cdot p(x) = 2x^{n+6} - 4x^{n+5} + 3x^{n+4} - 5x^{n+3} + 2x^{n+2} + 3x^{n+1} - x^n\). Replacing each \(x^{n+i}\) by \(F_{n+i}\), we have
\[ 2F_{n+6} - 4F_{n+5} + 3F_{n+4} - 5F_{n+3} + 2F_{n+2} + 3F_{n+1} - F_n = 0. \] (39)
The coefficients are
\[ a_0 = 2, \ a_1 = -4, \ a_2 = 3, \ a_3 = -5, \ a_4 = 2, \ a_5 = 3, \ a_6 = -1, \]
which gives
\[ \sum a_i = 0. \]
Then we have
\[ 2L_{k,n+5} - 4L_{k,n+4} + 3L_{k,n+3} - 5L_{k,n+2} + 2L_{k,n+1} + 3L_{k,n} - L_{k,n-1} = 0, \quad n \geq 1. \]

**Example 5.3.** Let \( f(x) = (x^2 - x - 1)x^n \) and let \( p(x) = (x - 1)(2x^2 + x + 1) \). Then \( g(x) = f(x) \cdot p(x) = 2x^{n+5} - 3x^{n+4} - x^{n+3} + x^{n+1} + x^n. \) Replacing each \( x^{n+i} \) by \( F_{n+i} \), we have
\[ 2F_{n+5} - 3F_{n+4} - F_{n+3} + F_{n+1} + F_n = 0. \] (40)

The coefficients are
\[ a_0 = 2, \ a_1 = -3, \ a_2 = -1, \ a_3 = 0, \ a_4 = 1, \ a_5 = 1, \]
which gives
\[ \sum a_i = 0. \]
Then we have
\[ 2L_{k,n+4} - 3L_{k,n+3} - L_{k,n+2} + L_{k,n} + L_{k,n-1} = 0, \quad n \geq 1. \]

6 **Combinatorial conclusion**

Jarden [10] has also considered Leonardo sequences from the point of view of the following variation of the Leonardo equation related to equation (5):
\[ a_n = a_{n-1} + a_{n-2} \mp 1, \quad n \geq 2, \] (41)
and the associated 3\textsuperscript{rd} order linear recurrence
\[ b_n = 2b_{n-1} - b_{n-3}, \quad n \geq 3, \] (42)
to which the Leonardo sequences conform as in equation (5) with \( k = \mp 1 \). In fact, Jarden considers the sequences in Tables 1, 2, and 3, which can bring out the corresponding relations with the Fibonacci and Lucas sequences. \( \{u_n\} \) is the sequence of differences, and is related to the generalized Fibonacci numbers of Jarden in Table 9 [10] and the hyper-Fibonacci and hyper-Lucas numbers in Table 10 [6] with further generalized and extended Leonardo numbers.

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Table 9. Jarden’s example of equation (5) with \( k = \mp 1 \).
Table 10 below is copied from Table 1 [1]. It shows the interested reader the salient features of these sequences, both horizontally and vertically, as well as diagonally. Further properties to be investigated include intersections between sequences [8] and step functions within sequences [5]. The last of these leads to s-Pascal triangles, as in Table 11.

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Table 10. Hyper-Fibonacci and hyper-Lucas numbers.

Table 11. A simple s-Pascal triangle.

If we then add along the leading diagonals in Table 11, we seem to arrive at the Tribonacci numbers, which can generate third-order Leonardo numbers.

In a different, but somewhat similar manner, Lind [14] defined $L(n, r)$ the $r$-th order nonlinear binomial sum as the sum of the first $r$ terms of the $(n-1)$-th row of the ordinary Pascal’s triangle plus the terms of the rising stair-step (or rising) diagonal originating at the $r$-th term, which can be applied to any of these tables. For example, in Table 11, we can have

$L(1, 3) = 1, L(2, 3) = 3, L(3, 3) = 6, L(4, 3) = 12, L(4, 4) = 18.$

All of these can provide a nexus between the numerical results in this paper and the recent combinatorial work of Shattuck [19], who provided a framework for these and other identities satisfied by the Leonardo numbers in the notation of section 3 and other generalized and extended Fibonacci numbers. The initial step in extending Corollary 3.12 is

$$w_n = w_{n-1} + w_{n-2} + tn + j, \quad n \geq 2, \quad j > -4,$$

and

$$w_n = w_{n-1} + F_{n+1} - 1.$$  \hfill (43)
One can then extend the process to other second order sequences [15] or to other orders and other dimensions [16] for further related combinatorial properties. In this way, one can relate

\[ w_n = w_{n-1} + w_{n-2} + tn + j, \quad n \geq 2, \quad t \geq 1, \]

and

\[ w_n = w_{n-1} + F_n^{[k]}, \quad (44) \]

in which \( F_n^{[k]} \) is hyper-Fibonacci sequence, as in Table 10, the rows of which as \( k \) increases can be seen as staked on top of one another for a third dimension. These can be developed further [2].

We note the neat recurrence relation

\[ F_n^{[k]} = F_{n-1}^{[k]} + F_{n-1}^{[k-1]}, \quad k, n > 0, \quad (45) \]

with boundary conditions \( F_n^{[0]} = F_n \) and \( F_0^{[k]} = 0 \); and with an elegant characteristic polynomial

\[ (x^2 - x - 1)(x - 1)^k, \]

so that

\[ F_n^{[k]} = \sum_{j=1}^{n} \binom{k + n - j - 1}{k - 1} F_j; \quad (46) \]

see [11] for details, including their relation to the infinite matrix in which \( F_n^{[k]} \) is the entry in the \( n \)-th row and \( k \)-th column, and from there to Stirling numbers of the first kind.

References


