Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 4, 717–723
DOI: 10.7546/nntdm.2023.29.4.717-723

Note on the general monic quartic equation

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Received: 22 September 2023 **Accepted:** 27 October 2023 **Online First:** 21 November 2023

Abstract: In this paper we present a new approach to solving the general monic quartic equation. Moreover, we show that each quartic equation could be considered as a quasi-reciprocal equation, after a suitable translation of the variable.
Keywords: Quartic equation.
2020 Mathematics Subject Classification: 65Hxx.

1 Introduction and historical data

Let

$$f(x) = x^4 + ax^3 + bx^2 + cx + d \in \mathbb{C}[x].$$

The question for solving the equation

$$f(x) = 0 \tag{1}$$

has a long history. There are known three fundamental ways to solve (1). They are based on Vieta's substitution $x = z - \frac{a}{4}$, which transforms (1) into the depressed equation

$$h(z) = z^4 + pz^2 + qz + r = 0, \tag{2}$$

where the coefficients p, q, r depend on a, b, c, d.

The Italian mathematician Ludovico Ferrari (1522–1565) solved (2) with the help of the method known as Ferrari's method (see [4]). After Ferrari, the French mathematician René

Descartes (1596–1650) proposed another method for solving (2), using factorisation of $h(z)$ with the help of two quadratic polynomials (see [9] and [7], p. 361). The third way for solving (2) was proposed by the Swiss mathematician Leonhard Euler (1707–1783). His method is a generalization of Hüdde's method for solving the trinomial cubic equation

$$x^3 + px + q = 0$$

(see [7], pp. 358–360).

We shall describe the above three ways for solving (2) in the appendix of this paper.

A general approach to solving (1) and other algebraic equations (using symmetric functions of their roots) was developed by the French mathematician Joseph-Louis Lagrange (1736–1813) (see [7], pp. 363–368, and [5]).

More details about solving (1) are contained in [10] and [11].

The topic for the quartic equation and its solving is still actual (see for example [1–3, 6]). In [8] a review of the known algorithms for solving (1) is made and a universal algorithm that generalizes them is proposed.

2 Our approach to solving (1)

An essential difference between our approach to solving (1) and the ways mentioned above is that we keep the term y^3, which plays a fundamental role. First, we substitute in (1)

$$x = y + \lambda, \tag{3}$$

where $\lambda \in \mathbb{C}$ is an arbitrary parameter, and we obtain

$$g(y) = y^4 + A(\lambda)y^3 + B(\lambda)y^2 + C(\lambda)y + D(\lambda) = 0 \tag{4}$$

with:

$$D(\lambda) = f(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d; \tag{5}$$

$$C(\lambda) = \frac{f'(\lambda)}{1!} = f'(\lambda) = 4\lambda^3 + 3a\lambda^2 + 2b\lambda + c; \tag{6}$$

$$B(\lambda) = \frac{f''(\lambda)}{2!} = 6\lambda^2 + 3a\lambda + b; \tag{7}$$

$$A(\lambda) = \frac{f'''(\lambda)}{3!} = 4\lambda + a. \tag{8}$$

Let λ satisfy the equation:

$$(4\lambda + a^2)f(\lambda) = (f'(\lambda))^2,$$

which is equivalent to the equation

$$(A(\lambda))^2 D(\lambda) = (C(\lambda))^2. \tag{9}$$

Using (5), (6), (8), after computation (9) yields the following cubic equation with respect to λ:

$$(a^3 - 4ab + 8c)\lambda^3 + (a^2b - 4b^2 + 2ac + 16d)\lambda^2 + (a^2c + 8ad - 4bc)\lambda + a^2d - c^2 = 0. \tag{10}$$

Now we shall consider some cases:

1) $\lambda_0 = -\frac{a}{4}$ is a root with multiplicity 3 of (10).

Hence, $A(\lambda_0) = 0$ and (9) implies $C(\lambda_0) = 0$. Then for $\lambda = \lambda_0$ (4) yields

$$y^4 + B(\lambda_0)y^2 + D(\lambda_0) = 0. \tag{11}$$

Since (11) is a biquadratic equation, it is trivial to be solved. Let $y_i, i = 1, 2, 3, 4$ are all roots of (11). Then, from (3) we obtain that

$$x_i = -\frac{a}{4} + y_i,\ i = 1, 2, 3, 4,$$

are all roots of (1);

2) The Equation (10) has a root λ^* different from $-\frac{a}{4}$.

Then $A(\lambda^*) \neq 0$.

Now we consider the following subcases.

2.1) For each root λ of the Equation (10), such that $\lambda \neq -\frac{a}{4}$, $D(\lambda) = 0$ (in particular, $D(\lambda^*) = 0$).

Then (9) implies $C(\lambda^*) = 0$ and (4) for $\lambda = \lambda^*$ yields

$$y^4 + A(\lambda^*)y^3 + B(\lambda^*)y^2 = 0. \tag{12}$$

The Equation (12) has roots $y_1 = 0, y_2 = 0$, while its roots y_3 and y_4 satisfy the quadratic equation

$$y^2 + A(\lambda^*)y + B(\lambda^*) = 0.$$

Then (3) implies that

$$x_i = y_i + \lambda^*,\ i = 1, 2, 3, 4,$$

are all roots of (1).

The main subcase of Case 2) is:

2.2) There exists a root $\tilde{\lambda}$ of the Equation (10), such that $\tilde{\lambda} \neq -\frac{a}{4}$ (i.e., $A(\tilde{\lambda}) \neq 0$) and $C(\tilde{\lambda}) \neq 0$.

Then for $\lambda = \tilde{\lambda}$ we rewrite (9) in the form

$$D(\tilde{\lambda}) = \frac{(C(\tilde{\lambda}))^2}{(A(\tilde{\lambda}))^2}. \tag{13}$$

Since $C(\tilde{\lambda}) \neq 0$, then $D(\tilde{\lambda}) \neq 0$. Therefore, $y = 0$ is not a root of the equation

$$y^4 + A(\tilde{\lambda})y^3 + B(\tilde{\lambda})y^2 + C(\tilde{\lambda})y + D(\tilde{\lambda}) = 0. \tag{14}$$

This gives us the possibility to divide both sides of (14) by y^2 (like in the case of reciprocal equations of degree 4) and obtain

$$y^2 + D(\tilde{\lambda})\frac{1}{y^2} + A(\tilde{\lambda})\left(y + \frac{C(\tilde{\lambda})}{A(\tilde{\lambda})}\frac{1}{y}\right) + B(\tilde{\lambda}) = 0. \tag{15}$$

We must note that the Equation (14) is not a reciprocal equation, but it is similar, since the method for its solving is almost the same. For this reason we call (14) quasi-reciprocal equation. To solve (15), we set

$$z = y + \frac{C(\tilde{\lambda})}{A(\tilde{\lambda})} \cdot \frac{1}{y}. \tag{16}$$

Hence, (16) yields

$$z^2 = y^2 + \frac{(C(\tilde{\lambda}))^2}{(A(\tilde{\lambda}))^2} \cdot \frac{1}{y^2} + 2\frac{C(\tilde{\lambda})}{A(\tilde{\lambda})}$$

and from (13) we obtain

$$z^2 = y^2 + D(\tilde{\lambda})\frac{1}{y^2} + 2\frac{C(\tilde{\lambda})}{A(\tilde{\lambda})}.$$

Hence,

$$y^2 + D(\tilde{\lambda})\frac{1}{y^2} = z^2 - 2\frac{C(\tilde{\lambda})}{A(\tilde{\lambda})}.$$

Thus (15) yields the quadratic equation

$$z^2 + A(\tilde{\lambda})z + B(\tilde{\lambda}) - 2\frac{C(\tilde{\lambda})}{A(\tilde{\lambda})} = 0. \tag{17}$$

Let the roots of (17) be z_i, $i = 1, 2$. Then from (16) we obtain

$$z_i = y + \frac{C(\tilde{\lambda})}{A(\tilde{\lambda})} \cdot \frac{1}{y},\ i = 1, 2. \tag{18}$$

Obviously (18) gives us two quadratic equations for y. Let us denote their roots by y_1, y_2, y_3 and y_4. Then from (3) we obtain that

$$x_i = y_i + \tilde{\lambda},\ i = 1, 2, 3, 4,$$

are all roots of (1).

3 Conclusion

Our method for solving (1) shows that for every monic quartic equation there exists a translation of its variable, which turns it into a quasi-reciprocal equation. Moreover, the number of these translations for our method is not greater than 3 (because of (10)).

An open problem is if there exists a similar method for solving algebraic equations of degree $n \geq 5$.

4 Appendix

4.1 A short description of Ferrari's method for solving (2)

Ferrari started from the identity

$$z^4 + pz^2 + qz + r = (u_\alpha(z))^2 - v_\alpha(z),$$

where:
$$u_\alpha(z) = z^2 + \frac{p}{2} + \alpha;\ v_\alpha(z) = 2\alpha z^2 - qz + \left(\alpha^2 + p\alpha - r + \frac{p^2}{4}\right)$$
and $\alpha \in \mathbb{C}$ is an arbitrary parameter. After that Ferrari chose such α that the discriminant of $v_\alpha(z)$ is 0, i.e.,
$$q^2 - 4.2\alpha\left(\alpha^2 + p\alpha - r + \frac{p^2}{4}\right) = 0.$$

The above resolvent equation is a qubic equation with respect to α. Let λ is one of its roots. Then
$$v_\lambda(z) = 2\lambda(z-\lambda)^2 = \left(\sqrt{2\lambda}(z-\lambda)\right)^2 =: (w_\lambda(z))^2.$$
Therefore,
$$z^4 + pz^2 + qz + r = (u_\lambda(z))^2 - v_\lambda(z) = (u_\lambda(z))^2 - (w_\lambda(z))^2.$$
Hence,
$$z^4 + pz^2 + qz + r = (u_\lambda(z) + w_\lambda(z))(u_\lambda(z) - w_\lambda(z)).$$
Therefore, (2) is equivalent to the following two quadratic equations:
$$u_\lambda(z) + w_\lambda(z) = 0; \quad u_\lambda(z) - w_\lambda(z) = 0.$$

4.2 A short description of Descartes' method for solving (2)

To solve (2), Descartes set
$$z^4 + pz^2 + qz + r = (z^2 + uz + v)(z^2 - uz + t), \tag{19}$$
where u, v, t are unknown coefficients. Hence,
$$z^4 + pz^2 + qz + r = z^4 + (t + v - u^2)z^2 + (ut - uv)z + vt.$$
Therefore,
$$t + v = p + u^2;\ t - v = \frac{q}{u};\ vt = r.$$
Hence,
$$t = \frac{1}{2}(p + u^2 + \frac{q}{u}); \tag{20}$$
$$v = \frac{1}{2}(p + u^2 - \frac{q}{u}); \tag{21}$$
$$vt = r.$$

After eliminitation of t and v from the above three equalities, Descartes obtained the resolvent equation
$$u^6 + 2pu^4 + (p^2 - 4r)u^2 - q^2 = 0.$$
The last after substituton $u^2 = y$ yields the cubic equation
$$y^3 + 2py^2 + (p^2 - 4r)y - q^2 = 0.$$

For an arbitrary root y of the above equation the corresponding u is given by $u = \sqrt{y}$ and then the corresponding t and v are obtained from (20) and (21). For these u, v, t (19) shows that (2) is reduced to the following two quadratic equations:

$$z^2 + uz + v = 0;\; z^2 - uz + t = 0.$$

4.3 A short description of Euler's method for solving (2)

To solve (2), Euler used Hüdde's substitution (but for three variables):

$$z = u + v + w, \tag{22}$$

where u, v, w are unknown variables. Hence,

$$\begin{aligned} z^2 &= u^2 + v^2 + w^2 + 2(uv + uw + vw); \\ z^4 &= (u^2 + v^2 + w^2)^2 + 4(u^2 + v^2 + w^2)(uv + uw + vw) + 4(u^2v^2 + u^2w^2 + v^2w^2) \\ &\quad + 8uvw(u + v + w). \end{aligned}$$

Eliminating the expression $uv + uw + vw$ from the above two equalities and using (22), Euler obtained

$$z^4 - 2(u^2 + v^2 + w^2)z^2 - 8uvwz + (u^2 + v^2 + w^2)^2 - 4(u^2v^2 + u^2w^2 + v^2w^2) = 0.$$

Comparing the last equation with (2) Euler received:

$$u^2 + v^2 + w^2 = -\frac{p}{2}; \quad uvw = -\frac{q}{8}; \quad u^2v^2 + u^2w^2 + v^2w^2 = \frac{p^2 - 4r}{16}.$$

Hence,

$$\begin{aligned} u^2 + v^2 + w^2 &= -\frac{p}{2}; \\ u^2v^2 + u^2w^2 + v^2w^2 &= \frac{p^2 - 4r}{16}; \\ u^2v^2w^2 &= \frac{q^2}{64}. \end{aligned}$$

Obviously, the above three equalities are Vieta's formulae for the cubic (resolvent) equation

$$t^3 + \frac{p}{2}t^2 + \frac{p^2 - 4r}{16}t - \frac{q^2}{64} = 0$$

with roots: $t_1 = u^2$, $t_2 = v^2$; $t_3 = w^2$.

Thus, using (22), Euler finally obtained that the roots of (2) are given by

$$z = \sqrt{t_1} + \sqrt{t_2} + \sqrt{t_3},$$

where $\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3}$ are chosen so that

$$\sqrt{t_1}\sqrt{t_2}\sqrt{t_3} = -\frac{q}{8}.$$

Acknowledgements

The author is thankful to Assoc. Prof. Silvia Boumova, PhD, for her support and valuable advices. The author would like to thank the reviewers for their useful comments and suggestions.
The research of the author was supported, in part, by the Science Foundation of Sofia University under contract 80-10-64/22.03.2021.

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