Note on the general monic quartic equation

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Abstract: In this paper we present a new approach to solving the general monic quartic equation. Moreover, we show that each quartic equation could be considered as a quasi-reciprocal equation, after a suitable translation of the variable.

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1 Introduction and historical data

Let
\[ f(x) = x^4 + ax^3 + bx^2 + cx + d \in \mathbb{C}[x].\]

The question for solving the equation
\[ f(x) = 0 \]
has a long history. There are known three fundamental ways to solve (1). They are based on Vieta’s substitution \( x = z - \frac{a}{4}, \) which transforms (1) into the depressed equation
\[ h(z) = z^4 + pz^2 + qz + r = 0, \]
where the coefficients \( p, q, r \) depend on \( a, b, c, d. \)

The Italian mathematician Ludovico Ferrari (1522–1565) solved (2) with the help of the method known as Ferrari’s method (see [4]). After Ferrari, the French mathematician René
Descartes (1596–1650) proposed another method for solving (2), using factorisation of $h(z)$ with the help of two quadratic polynomials (see [9] and [7], p. 361). The third way for solving (2) was proposed by the Swiss mathematician Leonhard Euler (1707–1783). His method is a generalization of Hübde’s method for solving the trinomial cubic equation

$$x^3 + px + q = 0$$

(see [7], pp. 358–360).

We shall describe the above three ways for solving (2) in the appendix of this paper.

A general approach to solving (1) and other algebraic equations (using symmetric functions of their roots) was developed by the French mathematician Joseph-Louis Lagrange (1736–1813) (see [7], pp. 363–368, and [5]).

More details about solving (1) are contained in [10] and [11].

The topic for the quartic equation and its solving is still actual (see for example [1–3, 6]). In [8] a review of the known algorithms for solving (1) is made and a universal algorithm that generalizes them is proposed.

2 Our approach to solving (1)

An essential difference between our approach to solving (1) and the ways mentioned above is that we keep the term $y^3$, which plays a fundamental role. First, we substitute in (1)

$$x = y + \lambda,$$

where $\lambda \in \mathbb{C}$ is an arbitrary parameter, and we obtain

$$g(y) = y^4 + A(\lambda)y^3 + B(\lambda)y^2 + C(\lambda)y + D(\lambda) = 0$$

with:

$$D(\lambda) = f(\lambda) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d;$$

$$C(\lambda) = \frac{f'(\lambda)}{1!} = f'(\lambda) = 4\lambda^3 + 3a\lambda^2 + 2b\lambda + c;$$

$$B(\lambda) = \frac{f''(\lambda)}{2!} = 6\lambda^2 + 3a\lambda + b;$$

$$A(\lambda) = \frac{f'''(\lambda)}{3!} = 4\lambda + a.$$

Let $\lambda$ satisfy the equation:

$$(4\lambda + a)^2 f(\lambda) = (f'(\lambda))^2;$$

which is equivalent to the equation

$$(A(\lambda))^2 D(\lambda) = (C(\lambda))^2.$$  

(9)

Using (5), (6), (8), after computation (9) yields the following cubic equation with respect to $\lambda$:

$$(a^3 - 4ab + 8c)\lambda^3 + (a^2b - 4b^2 + 2ac + 16d)\lambda^2 + (a^2c + 8ad - 4bc)\lambda + a^2d - c^2 = 0.$$  

(10)

Now we shall consider some cases:
1) \(\lambda_0 = -\frac{a}{4}\) is a root with multiplicity 3 of (10).

Hence, \(A(\lambda_0) = 0\) and (9) implies \(C(\lambda_0) = 0\). Then for \(\lambda = \lambda_0\) (4) yields

\[ y^4 + B(\lambda_0)y^2 + D(\lambda_0) = 0. \tag{11} \]

Since (11) is a biquadratic equation, it is trivial to be solved. Let \(y_i, i = 1, 2, 3, 4\) are all roots of (11). Then, from (3) we obtain that

\[ x_i = -\frac{a}{4} + y_i, \ i = 1, 2, 3, 4, \]

are all roots of (1);

2) The Equation (10) has a root \(\lambda^*\) different from \(-\frac{a}{4}\).

Then \(A(\lambda^*) \neq 0\).

Now we consider the following subcases.

2.1) For each root \(\lambda\) of the Equation (10), such that \(\lambda \neq -\frac{a}{4}\), \(D(\lambda) = 0\) (in particular, \(D(\lambda^*) = 0\)).

Then (9) implies \(C(\lambda^*) = 0\) and (4) for \(\lambda = \lambda^*\) yields

\[ y^4 + A(\lambda^*)y^3 + B(\lambda^*)y^2 = 0. \tag{12} \]

The Equation (12) has roots \(y_1 = 0, y_2 = 0\), while its roots \(y_3\) and \(y_4\) satisfy the quadratic equation

\[ y^2 + A(\lambda^*)y + B(\lambda^*) = 0. \]

Then (3) implies that

\[ x_i = y_i + \lambda^*, \ i = 1, 2, 3, 4, \]

are all roots of (1).

The main subcase of Case 2) is:

2.2) There exists a root \(\tilde{\lambda}\) of the Equation (10), such that \(\tilde{\lambda} \neq -\frac{a}{4}\) (i.e., \(A(\tilde{\lambda}) \neq 0\)) and \(C(\tilde{\lambda}) \neq 0\).

Then for \(\lambda = \tilde{\lambda}\) we rewrite (9) in the form

\[ D(\tilde{\lambda}) = \frac{(C(\tilde{\lambda}))^2}{(A(\lambda))^2}. \tag{13} \]

Since \(C(\tilde{\lambda}) \neq 0\), then \(D(\tilde{\lambda}) \neq 0\). Therefore, \(y = 0\) is not a root of the equation

\[ y^4 + A(\tilde{\lambda})y^3 + B(\tilde{\lambda})y^2 + C(\tilde{\lambda})y + D(\tilde{\lambda}) = 0. \tag{14} \]

This gives us the possibility to divide both sides of (14) by \(y^2\) (like in the case of reciprocal equations of degree 4) and obtain

\[ y^2 + D(\tilde{\lambda})\frac{1}{y^2} + A(\tilde{\lambda})\left(y + \frac{C(\tilde{\lambda})}{A(\lambda)}\frac{1}{y}\right) + B(\tilde{\lambda}) = 0. \tag{15} \]
We must note that the Equation (14) is not a reciprocal equation, but it is similar, since
the method for its solving is almost the same. For this reason we call (14) quasi-reciprocal
equation. To solve (15), we set
\[ z = y + C(\tilde{\lambda}) \cdot \frac{1}{y}. \]  
(16)
Hence, (16) yields
\[ z^2 = y^2 + \frac{(C(\tilde{\lambda}))^2}{(A(\lambda))^2} \cdot \frac{1}{y^2} + 2\frac{C(\tilde{\lambda})}{A(\lambda)}, \]
and from (13) we obtain
\[ z^2 = y^2 + D(\tilde{\lambda}) \frac{1}{y^2} + 2\frac{C(\tilde{\lambda})}{A(\lambda)}. \]
Hence,
\[ y^2 + D(\tilde{\lambda}) \frac{1}{y^2} = z^2 - 2\frac{C(\tilde{\lambda})}{A(\lambda)}. \]
Thus (15) yields the quadratic equation
\[ z^2 + A(\tilde{\lambda})z + B(\tilde{\lambda}) - 2\frac{C(\tilde{\lambda})}{A(\lambda)} = 0. \]  
(17)
Let the roots of (17) be \( z_i, i = 1, 2 \). Then from (16) we obtain
\[ z_i = y + C(\tilde{\lambda}) \cdot \frac{1}{y}, i = 1, 2. \]  
(18)
Obviously (18) gives us two quadratic equations for \( y \). Let us denote their roots by \( y_1, y_2, y_3 \) and \( y_4 \). Then from (3) we obtain that
\[ x_i = y_i + \tilde{\lambda}, i = 1, 2, 3, 4, \]
are all roots of (1).

3 Conclusion

Our method for solving (1) shows that for every monic quartic equation there exists a translation
of its variable, which turns it into a quasi-reciprocal equation. Moreover, the number of these
translations for our method is not greater than 3 (because of (10)).

An open problem is if there exists a similar method for solving algebraic equations of degree
\( n \geq 5 \).

4 Appendix

4.1 A short description of Ferrari’s method for solving (2)
Ferrari started from the identity
\[ z^4 + p z^2 + q z + r = (u_a(z))^2 - u_a(z), \]
where:
\[ u_\alpha(z) = z^2 + \frac{p}{2} + \alpha; \quad v_\alpha(z) = 2\alpha z^2 - qz + \left(\alpha^2 + p\alpha - r + \frac{p^2}{4}\right) \]
and \( \alpha \in \mathbb{C} \) is an arbitrary parameter. After that Ferrari chose such \( \alpha \) that the discriminant of \( v_\alpha(z) \) is 0, i.e.,
\[ q^2 - 4.2\alpha \left(\alpha^2 + p\alpha - r + \frac{p^2}{4}\right) = 0. \]
The above resolvent equation is a cubic equation with respect to \( \alpha \). Let \( \lambda \) is one of its roots.
\[ v_\lambda(z) = 2\lambda(z - \lambda)^2 = (\sqrt{2\lambda(z - \lambda)})^2 \Rightarrow (w_\lambda(z))^2. \]
Therefore, \( z^4 + pz^2 + qz + r = (u_\lambda(z))^2 - v_\lambda(z) = (u_\lambda(z))^2 - (w_\lambda(z))^2. \)
Hence, \( z^4 + pz^2 + qz + r = (u_\lambda(z) + w_\lambda(z))(u_\lambda(z) - w_\lambda(z)). \)
Therefore, (2) is equivalent to the following two quadratic equations:
\[ u_\lambda(z) + w_\lambda(z) = 0; \quad u_\lambda(z) - w_\lambda(z) = 0. \]

4.2 A short description of Descartes’ method for solving (2)
To solve (2), Descartes set
\[ z^4 + pz^2 + qz + r = (z^2 + uz + v)(z^2 - uz + t), \]
where \( u, v, t \) are unknown coefficients. Hence,
\[ z^4 + pz^2 + qz + r = z^4 + (t + v - u^2)z^2 + (ut - uv)z + vt. \]
Therefore,
\[ t + v = p + u^2; \quad t - v = \frac{q}{u}; \quad vt = r. \]
Hence,
\[ t = \frac{1}{2}(p + u^2 + \frac{q}{u}); \quad (20) \]
\[ v = \frac{1}{2}(p + u^2 - \frac{q}{u}); \quad (21) \]
\[ vt = r. \]
After elimination of \( t \) and \( v \) from the above three equalities, Descartes obtained the resolvent equation
\[ u^6 + 2pu^4 + (p^2 - 4r)u^2 - q^2 = 0. \]
The last after substituting \( u^2 = y \) yields the cubic equation
\[ y^3 + 2py^2 + (p^2 - 4r)y - q^2 = 0. \]
For an arbitrary root \( y \) of the above equation the corresponding \( u \) is given by \( u = \sqrt{y} \) and then the corresponding \( t \) and \( v \) are obtained from (20) and (21). For these \( u, v, t \) (19) shows that (2) is reduced to the following two quadratic equations:

\[
z^2 + uz + v = 0; \quad z^2 - uz + t = 0.
\]

### 4.3 A short description of Euler’s method for solving (2)

To solve (2), Euler used Hülde’s substitution (but for three variables):

\[ z = u + v + w, \quad \text{(22)} \]

where \( u, v, w \) are unknown variables. Hence,

\[
z^2 = u^2 + v^2 + w^2 + 2(uv + uw + vw); \quad z^4 = (u^2 + v^2 + w^2)^2 + 4(u^2 + v^2 + w^2)(uv + uw + vw) + 4(u^2v^2 + u^2w^2 + v^2w^2) + 8uvw(u + v + w).
\]

Eliminating the expression \( uv + uw + vw \) from the above two equalities and using (22), Euler obtained

\[
z^4 - 2(u^2 + v^2 + w^2)z^2 - 8uvwz + (u^2 + v^2 + w^2)^2 - 4(u^2v^2 + u^2w^2 + v^2w^2) = 0.
\]

Comparing the last equation with (2) Euler received:

\[
\begin{align*}
  u^2 + v^2 + w^2 &= -\frac{p^2}{2}; \\
  uvw &= -\frac{q^8}{8}; \\
  u^2v^2 + u^2w^2 + v^2w^2 &= \frac{p^2 - 4r}{16}.
\end{align*}
\]

Hence,

\[
\begin{align*}
  u^2 + v^2 + w^2 &= -\frac{p^2}{2}; \\
  u^2v^2 + u^2w^2 + v^2w^2 &= \frac{p^2 - 4r}{16}; \\
  u^2v^2w^2 &= \frac{q^2}{64}.
\end{align*}
\]

Obviously, the above three equalities are Vieta’s formulae for the cubic (resolvent) equation

\[
t^3 + \frac{p}{2}t^2 + \frac{p^2 - 4r}{16}t - \frac{q^2}{64} = 0
\]

with roots: \( t_1 = u^2, t_2 = v^2; t_3 = w^2. \)

Thus, using (22), Euler finally obtained that the roots of (2) are given by

\[ z = \sqrt{t_1} + \sqrt{t_2} + \sqrt{t_3}, \]

where \( \sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3} \) are chosen so that

\[ \sqrt{t_1}\sqrt{t_2}\sqrt{t_3} = -\frac{q}{8}. \]
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