Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 4, 713–716
DOI: 10.7546/nntdm.2023.29.4.713-716

Lower bounds on expressions dependent on functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$

Stoyan Dimitrov

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bulgarian Academy of Sciences
Acad. G. Bonchev Str. Bl. 105, Sofia-1113, Bulgaria
e-mail: xyzstoyan@gmail.com

Received: 10 May 2023 **Revised:** 20 July 2023
Accepted: 26 October 2023 **Online First:** 17 November 2023

Abstract: The inequalities

$$\varphi^2(n) + \psi^2(n) + \sigma^2(n) \geq 3n^2 + 2n + 3,$$
$$\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \geq 3n^2 + 2n - 1$$

connecting $\varphi(n)$, $\psi(n)$ and $\sigma(n)$-functions are formulated and proved.
Keywords: Arithmetic functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$, Inequalities.
2020 Mathematics Subject Classification: 11A25.

1 Notations and formulas

The letter p with or without subscript will always denote prime number. Let $n > 1$ be positive integer with prime factorization

$$n = p_1^{a_1} \cdots p_k^{a_k}.$$

The function $\Omega(n)$ counts the total number of prime factors of n honoring their multiplicity. We have

$$\Omega(n) = \sum_{i=1}^{k} a_i \quad \text{and} \quad \Omega(1) = 0\,.$$

We denote by $\varphi(n)$ the Euler totient function which is defined as the number of positive integers not greater than n that are coprime to n. We have

$$\varphi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1) \quad \text{and} \quad \varphi(1) = 1\,.$$

We define the Dedekind function $\psi(n)$ by the formula

$$\psi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i + 1) \quad \text{and} \quad \psi(1) = 1\,.$$

The function $\sigma(n)$ denotes the sum of the positive divisors of n. We have

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1} \quad \text{and} \quad \sigma(1) = 1\,.$$

2 Main results

In 2013 Atanassov [1] proved that for every natural number $n \geq 2$ the lower bound

$$\varphi(n)\psi(n)\sigma(n) \geq n^3 + n^2 - n - 1$$

holds. Subsequently Sándor [2] sharpened Atanassov's theorem proving that for all $n \geq 1$ one has the inequalities

$$\varphi(n)\psi(n)\sigma(n) \geq \varphi^*(n)(\sigma^*(n))^2 \geq n^3 + n^2 - n - 1\,,$$

where $\psi^*(n)$ and $\sigma^*(n)$ are the unitary analogues of the functions $\psi(n)$ and $\sigma(n)$. We refer to [4, 3] for definitions, properties and references. Inspired by the elegant results of Atanassov and Sándor and using their methods we prove the following two theorems.

Theorem 1. *For every natural number $n \geq 2$ the lower bound*

$$\varphi^2(n) + \psi^2(n) + \sigma^2(n) \geq 3n^2 + 2n + 3 \tag{1}$$

holds.

Theorem 2. *For every natural number $n \geq 2$ the lower bound*

$$\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \geq 3n^2 + 2n - 1 \tag{2}$$

holds.

3 Proof of Theorem 1

Consider several cases.

Case 1. $\Omega(n) = 1$. Bearing in mind that n is a prime number we write

$$\varphi^2(n) + \psi^2(n) + \sigma^2(n) = (n-1)^2 + 2(n+1)^2 = 3n^2 + 2n + 3 .$$

Case 2. $\Omega(n) = 2$, $n = pq$, where p and q are distinct primes. Then

$$\begin{aligned}\varphi^2(n) + \psi^2(n) + \sigma^2(n) &= (p-1)^2(q-1)^2 + 2(p+1)^2(q+1)^2 \\ &= 3p^2q^2 + 2pq + 3 + 2p^2q + 2pq^2 + 3p^2 + 3q^2 + 10pq + 2p + 2q \\ &> 3n^2 + 2n + 3 .\end{aligned}$$

Case 3. $\Omega(n) = 2$, $n = p^2$, where p is a prime. Then

$$\begin{aligned}\varphi^2(n) + \psi^2(n) + \sigma^2(n) &= p^2(p-1)^2 + p^2(p+1)^2 + (p^2+p+1)^2 \\ &= 3p^4 + 2p^2 + 3 + 2p^3 + 3p^2 + 2p - 2 \\ &> 3n^2 + 2n + 3 .\end{aligned}$$

Now we assume that (1) is true for every natural number n with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. Using that $\varphi(n) < n$ we obtain

$$\begin{aligned}\varphi^2(np) + \psi^2(np) + \sigma^2(np) &= \varphi^2(n)(p-1)^2 + \psi^2(n)(p+1)^2 + \sigma^2(n)(p+1)^2 \\ &= \left[\varphi^2(n) + \psi^2(n) + \sigma^2(n)\right](p+1)^2 - 4p\varphi^2(n) \\ &\geq (3n^2 + 2n + 3)(p+1)^2 - 4n^2p \\ &= 3n^2p^2 + 4np + 3 + 2n^2p + 2np^2 + 3n^2 + 2n + 3p^2 + 6p \\ &> 3n^2p^2 + 2np + 3 .\end{aligned}$$

Case B. $p \mid n$. Using that $\sigma(np) > p\sigma(n)$ we get

$$\begin{aligned}\varphi^2(np) + \psi^2(np) + \sigma^2(np) &> \left[\varphi^2(n) + \psi^2(n) + \sigma^2(n)\right]p^2 \\ &\geq (3n^2 + 2n + 3)p^2 \\ &= 3n^2p^2 + 2np + 3 + 2np^2 - 2np + 3p^2 - 3 \\ &> 3n^2p^2 + 2np + 3 .\end{aligned}$$

This completes the proof of Theorem 1. □

4 Proof of Theorem 2

Consider several cases.

Case 1. $\Omega(n) = 1$. Taking into account that n is a prime number we deduce

$$\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) = 2(n^2-1) + (n+1)^2 = 3n^2 + 2n - 1 .$$

Case 2. $\Omega(n) = 2$, $n = pq$, where p and q are distinct primes. Then

$$\begin{aligned}
&\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \\
&= 2(p^2 - 1)(q^2 - 1) + (p + 1)^2(q + 1)^2 \\
&= 3p^2q^2 + 2pq - 1 + p^2(2q - 1) + q^2(2p - 1) + 2pq + 2p + 2q + 4 \\
&> 3n^2 + 2n - 1\,.
\end{aligned}$$

Case 3. $\Omega(n) = 2$, $n = p^2$, where p is a prime. Then

$$\begin{aligned}
\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) &= p^2(p^2 - 1) + p(p^3 - 1) + p(p + 1)(p^2 + p + 1) \\
&= 3p^4 + 2p^2 - 1 + 2p^3 - p^2 + 1 \\
&> 3n^2 + 2n - 1\,.
\end{aligned}$$

Let us assume that (2) is true for every natural number n with $\Omega(n) = m$ for some natural number $m \geq 2$. Let p be a prime number. Then $\Omega(np) = \Omega(n) + 1$.

Case A. $p \nmid n$. Using that $\psi(n) \geq n + 1$ and $\sigma(n) \geq n + 1$ we derive

$$\begin{aligned}
&\varphi(np)\psi(np) + \varphi(np)\sigma(np) + \sigma(np)\psi(np) \\
&= \varphi(n)\psi(n)(p^2 - 1) + \varphi(n)\sigma(n)(p^2 - 1) + \psi(n)\sigma(n)(p + 1)^2 \\
&= \big[\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n)\big](p^2 - 1) + \psi(n)\sigma(n)(2p + 2) \\
&\geq (3n^2 + 2n - 1)(p^2 - 1) + (n + 1)^2(2p + 2) \\
&= 3n^2p^2 + 4np - 1 + (2n - 1)(p^2 - 1) + n^2(2p - 1) + 4n + 2p + 3 \\
&> 3n^2p^2 + 2np - 1\,.
\end{aligned}$$

Case B. $p \,|\, n$. Using that $\sigma(np) > p\sigma(n)$ and $p \geq 2$ we establish

$$\begin{aligned}
&\varphi(np)\psi(np) + \varphi(np)\sigma(np) + \sigma(np)\psi(np) \\
&> \big[\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n)\big]p^2 \\
&\geq (3n^2 + 2n - 1)p^2 \\
&= 3n^2p^2 + 2np - 1 + np(p - 2) + p^2(n - 1) + 1 \\
&> 3n^2p^2 + 2np - 1\,.
\end{aligned}$$

This completes the proof of Theorem 2. □

References

[1] Atanassov, K. (2013). Note on φ, ψ and σ-functions. Part 6. *Notes on Number Theory and Discrete Mathematics*, 19(1), 22–24.

[2] Sándor, J. (2014). On certain inequalities for φ, ψ, σ and related functions. *Notes on Number Theory and Discrete Mathematics*, 20(2), 52–60.

[3] Sándor, J., Mitrinović, D. S., & Crstici, B. (2006). *Handbook of Number Theory I*. Springer.

[4] Sándor, J., & Tóth., L. (1990). On certain number-theoretic inequalities. *The Fibonacci Quarterly*, 28(3), 255–258.