Lower bounds on expressions dependent on functions $\varphi(n)$, $\psi(n)$ and $\sigma(n)$

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Abstract: The inequalities

$$\varphi^2(n) + \psi^2(n) + \sigma^2(n) \geq 3n^2 + 2n + 3,$$
$$\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \geq 3n^2 + 2n - 1$$

connecting $\varphi(n)$, $\psi(n)$ and $\sigma(n)$-functions are formulated and proved.

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1 Notations and formulas

The letter $p$ with or without subscript will always denote prime number. Let $n > 1$ be positive integer with prime factorization

$$n = p_1^{a_1} \cdots p_k^{a_k}.$$
The function $\Omega(n)$ counts the total number of prime factors of $n$ honoring their multiplicity. We have

$$\Omega(n) = \sum_{i=1}^{k} a_i \quad \text{and} \quad \Omega(1) = 0.$$ 

We denote by $\varphi(n)$ the Euler totient function which is defined as the number of positive integers not greater than $n$ that are coprime to $n$. We have

$$\varphi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1) \quad \text{and} \quad \varphi(1) = 1.$$ 

We define the Dedekind function $\psi(n)$ by the formula

$$\psi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i + 1) \quad \text{and} \quad \psi(1) = 1.$$ 

The function $\sigma(n)$ denotes the sum of the positive divisors of $n$. We have

$$\sigma(n) = \prod_{i=1}^{k} p_i^{a_i+1} - \frac{1}{p_i - 1} \quad \text{and} \quad \sigma(1) = 1.$$ 

## 2 Main results

In 2013 Atanassov [1] proved that for every natural number $n \geq 2$ the lower bound

$$\varphi(n)\psi(n)\sigma(n) \geq n^3 + n^2 - n - 1$$

holds. Subsequently Sándor [2] sharpened Atanassov’s theorem proving that for all $n \geq 1$ one has the inequalities

$$\varphi(n)\psi(n)\sigma(n) \geq \varphi^*(n)(\sigma^*(n))^2 \geq n^3 + n^2 - n - 1,$$

where $\psi^*(n)$ and $\sigma^*(n)$ are the unitary analogues of the functions $\psi(n)$ and $\sigma(n)$. We refer to [4, 3] for definitions, properties and references. Inspired by the elegant results of Atanassov and Sándor and using their methods we prove the following two theorems.

**Theorem 1.** For every natural number $n \geq 2$ the lower bound

$$\varphi^2(n) + \psi^2(n) + \sigma^2(n) \geq 3n^2 + 2n + 3 \quad (1)$$

holds.

**Theorem 2.** For every natural number $n \geq 2$ the lower bound

$$\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \geq 3n^2 + 2n - 1 \quad (2)$$

holds.
3 Proof of Theorem 1

Consider several cases.

Case 1. \( \Omega(n) = 1 \). Bearing in mind that \( n \) is a prime number we write
\[
\varphi^2(n) + \psi^2(n) + \sigma^2(n) = (n - 1)^2 + 2(n + 1)^2 = 3n^2 + 2n + 3.
\]

Case 2. \( \Omega(n) = 2, n = pq \), where \( p \) and \( q \) are distinct primes. Then
\[
\varphi^2(n) + \psi^2(n) + \sigma^2(n) = (p - 1)^2(q - 1)^2 + 2(p + 1)^2(q + 1)^2
= 3p^2q^2 + 2pq + 3 + 2p^2q + 2pq^2 + 3p^2 + 3q^2 + 10pq + 2p + 2q
> 3n^2 + 2n + 3.
\]

Case 3. \( \Omega(n) = 2, n = p^2 \), where \( p \) is a prime. Then
\[
\varphi^2(n) + \psi^2(n) + \sigma^2(n) = p^2(p - 1)^2 + p^2(p + 1)^2 + (p^2 + p + 1)^2
= 3p^4 + 2p^2 + 3 + 2p^3 + 3p^2 + 2p - 2
> 3n^2 + 2n + 3.
\]

Now we assume that (1) is true for every natural number \( n \) with \( \Omega(n) = m \) for some natural number \( m \geq 2 \). Let \( p \) be a prime number. Then \( \Omega(np) = \Omega(n) + 1 \).

Case A. \( p \nmid n \). Using that \( \varphi(n) < n \) we obtain
\[
\varphi^2(np) + \psi^2(np) + \sigma^2(np) = \varphi^2(n)(p - 1)^2 + \psi^2(n)(p + 1)^2 + \sigma^2(n)(p + 1)^2
= [\varphi^2(n) + \psi^2(n) + \sigma^2(n)](p + 1)^2 - 4p\varphi^2(n)
\geq (3n^2 + 2n + 3)(p + 1)^2 - 4n^2p
= 3n^2p^2 + 4np + 3 + 2n^2p + 2np^2 + 3n^2 + 2n + 3p^2 + 6p
> 3n^2p^2 + 2np + 3.
\]

Case B. \( p \mid n \). Using that \( \sigma(np) > p\sigma(n) \) we get
\[
\varphi^2(np) + \psi^2(np) + \sigma^2(np) > [\varphi^2(n) + \psi^2(n) + \sigma^2(n)]p^2
\geq (3n^2 + 2n + 3)p^2
= 3n^2p^2 + 2np + 3 + 2np^2 - 2np + 3p^2 - 3
> 3n^2p^2 + 2np + 3.
\]

This completes the proof of Theorem 1.

4 Proof of Theorem 2

Consider several cases.

Case 1. \( \Omega(n) = 1 \). Taking into account that \( n \) is a prime number we deduce
\[
\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) = 2(n^2 - 1) + (n + 1)^2 = 3n^2 + 2n - 1.
\]
Case 2. \( \Omega(n) = 2, n = pq \), where \( p \) and \( q \) are distinct primes. Then
\[
\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \\
= 2(p^2 - 1)(q^2 - 1) + (p + 1)^2(q + 1)^2 \\
= 3p^2q^2 + 2pq - 1 + p^2(2q - 1) + q^2(2p - 1) + 2pq + 2p + 2q + 4 \\
> 3n^2 + 2n - 1.
\]

Case 3. \( \Omega(n) = 2, n = p^2 \), where \( p \) is a prime. Then
\[
\varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) = p^2(p^2 - 1) + p(p^3 - 1) + p(p + 1)(p^2 + p + 1) \\
= 3p^4 + 2p^2 - 1 + 2p^3 - p^2 + 1 \\
> 3n^2 + 2n - 1.
\]

Let us assume that (2) is true for every natural number \( n \) with \( \Omega(n) = m \) for some natural number \( m \geq 2 \). Let \( p \) be a prime number. Then \( \Omega(np) = \Omega(n) + 1 \).

Case A. \( p \nmid n \). Using that \( \psi(n) \geq n + 1 \) and \( \sigma(n) \geq n + 1 \) we derive
\[
\varphi(np)\psi(np) + \varphi(np)\sigma(np) + \sigma(np)\psi(np) \\
= \varphi(n)\psi(n)(p^2 - 1) + \varphi(n)\sigma(n)(p^2 - 1) + \psi(n)\sigma(n)(p + 1)^2 \\
= \left[ \varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \right] (p^2 - 1) + \psi(n)\sigma(n)(2p + 2) \\
\geq (3n^2 + 2n - 1)(p^2 - 1) + (n + 1)^2(2p + 2) \\
= 3n^2p^2 + 4np - 1 + (2n - 1)(p^2 - 1) + n^2(2p - 1) + 4n + 2p + 3 \\
> 3n^2p^2 + 2np - 1.
\]

Case B. \( p \mid n \). Using that \( \sigma(np) > p\sigma(n) \) and \( p \geq 2 \) we establish
\[
\varphi(np)\psi(np) + \varphi(np)\sigma(np) + \sigma(np)\psi(np) \\
> \left[ \varphi(n)\psi(n) + \varphi(n)\sigma(n) + \sigma(n)\psi(n) \right] p^2 \\
\geq (3n^2 + 2n - 1)p^2 \\
= 3n^2p^2 + 2np - 1 + np(p - 2) + p^2(n - 1) + 1 \\
> 3n^2p^2 + 2np - 1.
\]

This completes the proof of Theorem 2. \( \square \)

References


