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On sums with generalized harmonic numbers via Euler's transform

Neşe Ömür[1], Sibel Koparal[2] and Laid Elkhiri[3]

[1] Department of Mathematics, University of Kocaeli
41380 İzmit, Kocaeli, Turkey
e-mail: neseomur@gmail.com

[2] Department of Mathematics, University of Bursa Uludağ
16059 Nilufer, Bursa, Turkey
e-mail: sibelkoparal@uludag.edu.tr

[3] Faculty of Material Sciences, University of Tiaret
Algeria
e-mail: laid@univ-tiaret.dz

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Abstract: In this paper, we define the generalized hyperharmonic numbers of order r, $H_n^r(\sigma)$ and get some identities involving these numbers by using Euler's transform.
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1 Introduction

The harmonic numbers, denoted by H_n, are defined by

$$H_0 = 0,\ H_n = \sum_{k=1}^{n} \frac{1}{k} \text{ for } n \geq 1,$$

and their generating function is

$$\sum_{n=0}^{\infty} H_n t^n = \frac{-\ln(1-t)}{1-t}.$$

From [14], it is known that

$$\sum_{k=0}^{n} \frac{H_k}{n+1-k} = H_{n+1}^2 - H_{n+1,2}. \tag{1.1}$$

Harmonic numbers are interesting research objects. Recently, these numbers have been generalized by several authors. There are a lot of works involving harmonic numbers and generalizations of them (see [5, 9, 12, 13]).

Guo and Cha [9] defined generalized harmonic numbers by

$$H_0(\sigma) = 0 \text{ and } H_n(\sigma) = \sum_{k=1}^{n} \frac{\sigma^k}{k} \text{ for } n \in \mathbb{N},$$

where σ is an appropriate parameter, and their generating function is

$$\sum_{n=0}^{\infty} H_n(\sigma) t^n = \frac{-\ln(1-\sigma t)}{1-t}. \tag{1.2}$$

When $\sigma = 1/\alpha$ for $\alpha \in \mathbb{R}^+$, $H_n(1/\alpha) := \sum_{k=1}^{n} \frac{1}{k\alpha^k}$ are called the generalized harmonic numbers by Genčev [8].

The Daehee numbers of order r, showed by D_n^r, are defined by the generating functions to be

$$\sum_{n=0}^{\infty} D_n^r \frac{t^n}{n!} = \left(\frac{\ln(1+t)}{t} \right)^r.$$

For $r = 1$, $D_n^1 = D_n$ are called Daehee numbers. It is clear that

$$D_0 = 1, D_1 = -\frac{1}{2}, \cdots, D_n = (-1)^n \frac{n!}{n+1}. \tag{1.3}$$

By using Euler's transform for power series, some authors work various binomial identities with harmonic numbers (see [2, 4, 6, 7, 10, 15]). For example, in [4], the author proved the identity as follows: for $n \in \mathbb{N}$ and $\lambda, \mu \in \mathbb{C}$,

$$\sum_{k=1}^{n} \binom{n}{k} \mu^k \lambda^{n-k} H_k = (\lambda+\mu)^n H_n - \left(\lambda (\lambda+\mu)^{n-1} + \frac{\lambda^2}{2} (\lambda+\mu)^{n-2} + \cdots + \frac{\lambda^n}{n} \right).$$

This formula allows to derive the following identity

$$\sum_{k=1}^{n} 2^{k-1} H_{n-k} = 2^n H_n(2) - H_n.$$

The author gave the following expansion in a neighborhood of zero

$$\sum_{n=1}^{\infty} \beta^n H_n \left(\frac{\sigma}{\beta} \right) t^n = \frac{-\ln(1-\sigma t)}{1-\beta t}, \tag{1.4}$$

where σ, β are appropriate parameters. When $\beta = 1$ in (1.4), (1.2) is obtained.

Also there is the generating function given by

$$\sum_{n=1}^{\infty}\left(\beta\sum_{k=0}^{n-1}(\sigma+\beta)^{n-k-1}\sigma^{k}H_{k}+\sigma^{n}H_{n}\right)t^{n}=\frac{-\ln(1-\sigma t)}{1-(\sigma+\beta)t}.$$

For $a, b \in \mathbb{Z}^{+}$, it is known that

$$\sum_{n=0}^{\infty}\binom{a+n-b}{n-b}t^{n}=\frac{t^{b}}{(1-t)^{a+1}}. \tag{1.5}$$

In [7], Frontczak proved a new expression for binomial sums with harmonic numbers. His derivation was based on elementary analysis of the Euler's transform of these sums. The author discovered some known identities involving skew-harmonic, and Fibonacci and Lucas numbers. For example, for positive integer n,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\mu^{k}\lambda^{n-k}H_{k}^{-} &= (\mu-\lambda)^{n}H_{n}^{-}+\lambda^{n}H_{n}+\mu\sum_{k=0}^{n-1}(\mu+\lambda)^{k}\lambda^{n-k-1}H_{n-k-1}\\ &\quad+2\lambda\sum_{k=0}^{n-1}(\mu+\lambda)^{k}(\mu-\lambda)^{n-k-1}H_{n-k-1}^{-},\end{aligned}$$

where $H_{k}^{-}=\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}$ with $H_{0}^{-}=0$ are skew-harmonic numbers.

In [3], Batır and Sofo proved some general combinatorics formulas. Applying these formulas, they obtained some new identities and reproved some known identities included in the works of Frontczak and Boyadzhiev. For example, for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$,

$$\sum_{k=0}^{n}\binom{n}{k}\lambda^{k}H_{k}^{2}=(1+\lambda)^{n}\left(H_{n}^{2}-\sum_{k=1}^{n}\frac{H_{n}-2H_{k}+H_{n-k}}{k(1+\lambda)^{k}}-2\sum_{k=1}^{n}\frac{1}{k^{2}(1+\lambda)^{k}}\right).$$

Abel's partial summation formula asserts that for every pair of families $(a_k)_{k=1}^{n}$ and $(b_k)_{k=1}^{n}$ of complex numbers, there is the relation (see [1, 10])

$$\sum_{k=1}^{n}a_{k}b_{k}=\sum_{k=1}^{n-1}\left((a_{k}-a_{k+1})\sum_{j=1}^{k}b_{j}\right)+a_{n}\sum_{j=1}^{n}b_{j}. \tag{1.6}$$

2 Main results

In this section, we will define a new generalization for the generalized harmonic numbers $H_n(\sigma)$ and then give some identities with these numbers by using Euler's transform.

Definition 2.1. *For the generalized harmonic numbers $H_n(\sigma)$, the generalized hyperharmonic numbers of order r, $H_n^r(\sigma)$, are defined by*

$$H_{n}^{r}(\sigma)=\begin{cases}\sum_{k=1}^{n}H_{k}^{r-1}(\sigma), & \text{if } n,\ r\geq 1,\\ \frac{\sigma^{n}}{n}, & \text{if } r=0 \text{ and } n>0,\\ 0, & \text{if } r<0 \text{ or } n\leq 0,\end{cases}$$

where σ is an appropriate parameter.

Specifically, when $\sigma = 1/\alpha$, $\alpha \in \mathbb{R}^{+}$, for the generalized harmonic numbers $H_n(1/\alpha)$, the generalized hyperharmonic numbers of order r are $H_n^r(1/\alpha)$ [12].

The proof of Theorem 2.1 is similar to the proof of Theorem 1 in [12].

Theorem 2.1. *For any positive integers n and r, we have*

$$H_n^r(\sigma) = \sum_{k=1}^{n} \binom{n+r-k-1}{r-1} \frac{\sigma^k}{k}.$$

Theorem 2.2. *For any positive integer r, we have*

$$\sum_{n=1}^{\infty} H_n^r(\sigma)\, t^n = \frac{-\ln(1-\sigma t)}{(1-t)^r}.$$

Proof. Consider that

$$\begin{aligned} \frac{-\ln(1-\sigma t)}{(1-t)^r} &= -\left(\sum_{n=1}^{\infty} \frac{\sigma^n}{n} t^n\right)\left(\sum_{n=0}^{\infty} \binom{r-1+n}{r-1} t^n\right) \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} \binom{n+r-k-1}{r-1} \frac{\sigma^k}{k} t^n = \sum_{n=1}^{\infty} H_n^r(\sigma)\, t^n, \end{aligned}$$

as claimed. □

Lemma 2.1. *For any positive integer $r \geq 2$, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \beta^n \binom{r-2+k}{k} H_{n-k}(\sigma/\beta)\, t^n = \frac{-\ln(1-\sigma t)}{(1-\beta t)^r},$$

where σ, β are appropriate parameters.

Proof. By (1.4) and (1.5), using Cauchy's product rule for power series, we get

$$\begin{aligned} \frac{-\ln(1-\sigma t)}{(1-\beta t)^r} &= \frac{1}{(1-\beta t)^{r-1}} \times \frac{-\ln(1-\sigma t)}{1-\beta t} \\ &= \sum_{n=0}^{\infty} \beta^n \binom{r-2+n}{n} t^n \times \sum_{n=0}^{\infty} \beta^n H_n(\sigma/\beta)\, t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \beta^n \binom{r-2+k}{k} H_{n-k}(\sigma/\beta)\, t^n, \end{aligned}$$

as claimed. □

For example, putting $\beta = 1$ in Lemma 2.1, the following expression is valid:

$$H_n^r(\sigma) = \sum_{k=0}^{n} \binom{r-2+k}{k} H_{n-k}(\sigma).$$

Lemma 2.2. [4] *Let a function analytical on the unit disk be $f(t) = \sum_{n=0}^{\infty} f_n t^n$. The Euler's transform can be given as*

$$\frac{1}{(1-\lambda t)^m} f\left(\frac{\mu t}{1-\lambda t}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{m-1+n}{n-k} \mu^k \lambda^{n-k} f_k t^n,$$

where λ, μ are appropriate parameters.

Theorem 2.3. *Let n be any positive integer, we have*

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\mu}{\lambda}\right)^{k}H_{k}\left(\sigma\right)=\left(\frac{\lambda+\mu}{\lambda}\right)^{n}\left(H_{n}\left(\frac{\lambda+\sigma\mu}{\lambda+\mu}\right)-H_{n}\left(\frac{\lambda}{\lambda+\mu}\right)\right) \tag{2.1}$$

and for $r\geq 2$,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n+r-1}{n-k}\left(\frac{\mu}{\lambda}\right)^{k}H_{k}^{r}\left(\sigma\right) \\ &= \sum_{k=0}^{n}\sum_{i=0}^{n-k}\binom{k+r-2}{k}\binom{n-k}{i}\left(\frac{\mu}{\lambda}\right)^{i}\left(\frac{\lambda+\mu}{\lambda}\right)^{k}H_{i}\left(\sigma\right),\end{aligned} \tag{2.2}$$

where λ and μ are as above.

Proof. For $f(t)=\frac{-\ln\left(1-\sigma t\right)}{\left(1-t\right)^{r}}$, by applying Lemma 2.2, the right-hand side becomes

$$\begin{aligned}\frac{1}{\left(1-\lambda t\right)^{r}}f\left(\frac{\mu t}{1-\lambda t}\right) &= \frac{1}{\left(1-\lambda t\right)^{r}}\frac{-\ln\left(1-\sigma\frac{\mu t}{1-\lambda t}\right)}{\left(1-\frac{\mu t}{1-\lambda t}\right)^{r}} \\ &= -\frac{\ln\left(1-\left(\lambda+\sigma\mu\right)t\right)-\ln\left(1-\lambda t\right)}{\left(1-\left(\lambda+\mu\right)t\right)^{r}} \\ &= -\frac{\ln\left(1-\left(\lambda+\sigma\mu\right)t\right)}{\left(1-\left(\lambda+\mu\right)t\right)^{r}}-\frac{-\ln\left(1-\lambda t\right)}{\left(1-\left(\lambda+\mu\right)t\right)^{r}}.\end{aligned}$$

Lemma 2.1 yields that

$$\begin{aligned}\frac{1}{\left(1-\lambda t\right)^{r}}f\left(\frac{\mu t}{1-\lambda t}\right) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(\lambda+\mu)^{n}\binom{k+r-2}{k}H_{n-k}\left(\frac{\lambda+\sigma\mu}{\lambda+\mu}\right)t^{n} \\ &\quad -\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\lambda+\mu\right)^{n}\binom{k+r-2}{k}H_{n-k}\left(\frac{\lambda}{\lambda+\mu}\right)t^{n} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\lambda+\mu\right)^{n}\binom{k+r-2}{k}\left(H_{n-k}\left(\frac{\lambda+\sigma\mu}{\lambda+\mu}\right)-H_{n-k}\left(\frac{\lambda}{\lambda+\mu}\right)\right)t^{n}.\end{aligned} \tag{2.3}$$

At the same time, by using Euler's transform, the left hand side becomes

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+r-1}{n-k}\mu^{k}\lambda^{n-k}H_{k}^{r}\left(\sigma\right)t^{n}. \tag{2.4}$$

Thus, by using (2.1), comparing coefficients in (2.3) and (2.4), the proof of (2.2) is obtained. Similarly, taking $f(t)=\frac{-\ln\left(1-\sigma t\right)}{1-t}$ and using Lemma 2.2 and (1.4), we have the proof of (2.1). $\square$

For example, setting $\lambda=\mu=\sigma=1$ in Theorem 2.3, we have the well-known identities

$$\sum_{k=1}^{n}\binom{n}{k}H_{k}=2^{n}\left(H_{n}-H_{n}\left(1/2\right)\right),$$

and for any integer $r\geq 2$,

$$\sum_{k=1}^{n}\binom{n+r-1}{n-k}H_{k}^{r}=\sum_{k=0}^{n}\sum_{i=1}^{n-k}\binom{k+r-2}{k}\binom{n-k}{i}2^{k}H_{i}.$$

Corollary 2.1. *For any positive integer n, we have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k 2^{n-k}H_k(\sigma)=H_n(2-\sigma)-H_n(2),$$

and for any integer $r\geq 2$,

$$\sum_{k=0}^{n}\binom{n+r-1}{n-k}(-1)^k 2^{-k}H_k^r(\sigma)=\sum_{k=0}^{n}\sum_{i=0}^{n-k}\binom{k+r-2}{k}\binom{n-k}{i}(-1)^i 2^{-k-i}H_i(\sigma).$$

Proof. Putting $\lambda=-2\mu$ in Theorem 2.3, the desired results are given. □

Theorem 2.4. *For any positive integers n and m, we have*

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k\binom{n+m}{k+m}H_k^r(\sigma) &= H_n^{m-r+1}(1-\sigma)-H_n^{m-r+1}\\ &= \sum_{k=1}^{n-1}(-1)^{n-k}\binom{2r-m-1}{n-k-1}\left(H_k^r(1-\sigma)-H_k^r\right)\\ &= \sum_{k=1}^{n}\frac{(-1)^k}{n-k+1}\binom{r-m-1}{k-1}\left(1-(1-\sigma)^{n-k+1}\right).\end{aligned}$$

Proof. For the proof, we take

$$f(t)=-\frac{\ln(1-\sigma t)}{(1-t)^r}$$

and by applying Lemma 2.2, the left-hand side becomes

$$\begin{aligned}\frac{1}{(1+t)^{m+1}}f\left(\frac{t}{1+t}\right) &= -\frac{1}{(1+t)^{m+1}}\frac{\ln\left(1-\sigma\frac{t}{1+t}\right)}{\left(1-\frac{t}{1+t}\right)^r}\\ &= -\frac{1}{(1+t)^{m-r+1}}\left(\ln(1+t(1-\sigma))-\ln(1+t)\right)\\ &= \sum_{n=0}^{\infty}(-1)^n\left(H_n^{m-r+1}(1-\sigma)-H_n^{m-r+1}\right)t^n, \qquad (2.5)\end{aligned}$$

$$\begin{aligned}\frac{1}{(1+t)^{m+1}}f\left(\frac{t}{1+t}\right) &= -\frac{1}{(1+t)^{m+1}}\frac{\ln\left(1-\sigma\frac{t}{1+t}\right)}{\left(1-\frac{t}{1+t}\right)^r}\\ &= -(1+t)^{r-m-1}t\left((1-\sigma)\frac{\ln(1+t(1-\sigma))}{(1-\sigma)t}-\frac{\ln(1+t)}{t}\right)\\ &= -\sum_{n=1}^{\infty}\binom{r-m-1}{n-1}t^n\times\sum_{n=0}^{\infty}\frac{D_n}{n!}\left((1-\sigma)^{n+1}-1\right)t^n\\ &= -\sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{r-m-1}{k-1}\frac{D_{n-k}}{(n-k)!}\left((1-\sigma)^{n-k+1}-1\right)t^n, \qquad (2.6)\end{aligned}$$

and

$$\frac{1}{(1+t)^{m+1}}f\left(\frac{t}{1+t}\right) = -\frac{1}{(1+t)^{m+1}}\frac{\ln\left(1-\sigma\frac{t}{1+t}\right)}{\left(1-\frac{t}{1+t}\right)^r}$$

$$
\begin{aligned}
&= -(1+t)^{2r-m-1}\, t \left(\frac{\ln(1+t(1-\sigma))}{(1+t)^r} - \frac{\ln(1+t)}{(1+t)^r} \right) \\
&= \sum_{n=1}^{\infty} \binom{2r-m-1}{n-1} t^n \times \sum_{n=0}^{\infty} (-1)^n \left(H_n^r (1-\sigma) - H_n^r \right) t^n \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (-1)^k \binom{2r-m-1}{n-k-1} \left(H_k^r (1-\sigma) - H_k^r \right) t^n, \qquad (2.7)
\end{aligned}
$$

and finally, setting $\lambda = -1$ and $\mu = 1$ in Lemma 2.2, the right hand side becomes

$$
\frac{1}{(1+t)^{m+1}} f\left(\frac{t}{1+t} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+m}{k+m} (-1)^{n-k} H_k^r (\sigma)\, t^n. \qquad (2.8)
$$

Hence, by (1.3), from (2.5)–(2.8), comparing the coefficients of t^n completes the proof. □

Corollary 2.2. *For any positive integers* n *and* r*, we have*

$$
\sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^{n+k+i} \binom{2r-1}{n-k} \binom{k+2r-1}{i+2r-1} H_i^r (\sigma) = \sum_{k=1}^{n} (-1)^k \binom{r-1}{k-1} \frac{1-(1-\sigma)^{n-k+1}}{n-k+1}.
$$

Lemma 2.3. *For any nonnegative integer* n, *we have*

$$
\sum_{k=0}^{n} H_k H_{n-k} = (n+1)(2 - 2H_{n+1} + H_{n+1}^2 - H_{n+1,2}).
$$

Proof. Putting $a_k = H_{n-k}$ and $b_k = H_k$ in (1.6), we write

$$
\begin{aligned}
\sum_{k=0}^{n} H_k H_{n-k} &= \sum_{k=1}^{n-1} \left(\frac{(k+1)\, H_k}{n-k} - \frac{k}{n-k} \right) = \sum_{k=1}^{n-1} \left(\frac{(k-n+n+1)\, H_k}{n-k} + \frac{n-k-n}{n-k} \right) \\
&= \sum_{k=1}^{n-1} \left(-H_k + \frac{(n+1)\, H_k}{n-k} + 1 - \frac{n}{n-k} \right) \\
&= n - 1 - \sum_{k=1}^{n-1} H_k + (n+1) \sum_{k=1}^{n-1} \frac{H_k}{n-k} - n \sum_{k=1}^{n-1} \frac{1}{n-k} \\
&= 2n - 2 - 2nH_{n-1} + (n+1) \sum_{k=1}^{n-1} \frac{H_k}{n-k}.
\end{aligned}
$$

Thus, by using $\sum_{j=1}^{n-1} \frac{H_j}{n-j} = H_{n+1}^2 - H_{n+1,2} - \frac{2}{n+1} H_n$ from (1.1), we have the proof. □

Theorem 2.5. *For any nonnegative integer* n, *we have*

$$
\sum_{k=0}^{n} \binom{n+1}{k+1} A_k = 2^n \left(A_n + \sum_{k=0}^{n} H_{n-k}(1/2) \left(H_k(1/2) - 2H_k \right) \right),
$$

where $A_n = (n+1)\left(2 - 2H_{n+1} + H_{n+1}^2 - H_{n+1,2}\right)$.

Proof. Using Cauchy's product rule for power series, we have

$$\left(\frac{\ln(1-t)}{1-t}\right)^2 = \left(\sum_{n=0}^{\infty} H_n t^n\right)^2 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} H_k H_{n-k}\right) t^n.$$

For $f(t) = \left(\frac{\ln(1-t)}{1-t}\right)^2$, by applying Lemma 2.2, we write

$$\begin{aligned}
&\frac{1}{(1-t)^2} f\left(\frac{t}{1-t}\right) \\
=\ & \left(\frac{\ln(1-2t)-\ln(1-t)}{1-2t}\right)^2 \\
=\ & \left(\frac{\ln(1-2t)}{1-2t}\right)^2 + \left(\frac{\ln(1-t)}{1-2t}\right)^2 - 2\left(\frac{\ln(1-2t)}{1-2t}\right)\left(\frac{\ln(1-t)}{1-2t}\right) \\
=\ & \left(\sum_{n=0}^{\infty} 2^n H_n t^n\right)^2 + \left(\sum_{n=0}^{\infty} 2^n H_n(1/2) t^n\right)^2 - 2\left(\sum_{n=0}^{\infty} 2^n H_n t^n\right)\left(\sum_{n=0}^{\infty} 2^n H_n(1/2) t^n\right) \\
=\ & \sum_{n=0}^{\infty} 2^n \sum_{k=0}^{n} \left(H_{n-k}H_k + H_{n-k}(1/2)H_k(1/2) - 2H_k H_{n-k}(1/2)\right) t^n, \qquad (2.9)
\end{aligned}$$

and

$$\frac{1}{(1-t)^2} f\left(\frac{t}{1-t}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k+1} A_k t^n. \qquad (2.10)$$

Comparing the coefficients of (2.9) and (2.10), the result is given. □

Corollary 2.3. *For any odd prime number p, we have*

$$\sum_{k=1}^{p-1} H_{p-1-k}(1/2)\left(H_k(1/2) - 2H_k\right) \equiv 2^{1-p} p\left(2 - q_p^2(2)\right) - 2p \pmod{p^2},$$

where $q_p(2)$ are the Fermat quotients with base 2.

Proof. From the congruence $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$ for $k = 0, 1, \ldots, p-1$, we have

$$\begin{aligned}
& p\sum_{k=0}^{p-1}\binom{p-1}{k}\left(2 - 2H_{k+1} + H_{k+1}^2 - H_{k+1,2}\right) \\
\equiv\ & p\sum_{k=0}^{p-1}(-1)^k\left(2 - 2H_{k+1} + H_{k+1}^2 - H_{k+1,2}\right) \\
=\ & p\left(2 - 2H_p + H_p^2 - H_{p,2}\right) - p\sum_{k=1}^{p-1}(-1)^k\left(2 - 2H_k + H_k^2 - H_{k,2}\right) \\
=\ & 2p - 2 - 2pH_{p-1} + pH_{p-1}^2 + 2H_{p-1} - pH_{p-1,2} - 2p\sum_{k=1}^{p-1}(-1)^k \\
& +2p\sum_{k=1}^{p-1}(-1)^k H_k - p\sum_{k=1}^{p-1}(-1)^k H_k^2 + p\sum_{k=1}^{p-1}(-1)^k H_{k,2} \pmod{p^2},
\end{aligned}$$

and with the help of the congruences $H_{p-1} \equiv 0 \pmod{p^2}$ and $H_{p-1,2} \equiv 0 \pmod{p}$,

$$\begin{aligned} & p\sum_{k=0}^{p-1}\binom{p-1}{k}\left(2-2H_{k+1}+H_{k+1}^2-H_{k+1,2}\right) \\ \equiv\ & 2p-2+2p\sum_{k=1}^{p-1}(-1)^k H_k - p\left(\sum_{k=1}^{p-1}(-1)^k H_k^2 - \sum_{k=1}^{p-1}(-1)^k H_{k,2}\right) \pmod{p^2}. \end{aligned}$$

By $\sum_{k=1}^{p-1}(-1)^k H_k \equiv q_p(2) \pmod{p}$ and $\sum_{k=1}^{p-1}(-1)^k\left(H_k^2 - H_{k,2}\right) \equiv q_p^2(2) \pmod{p}$ [11], we write

$$\begin{aligned} & p\sum_{k=0}^{p-1}\binom{p-1}{k}\left(2-2H_{k+1}+H_{k+1}^2-H_{k+1,2}\right) \\ \equiv\ & 2p-2-2pq_p(2)-pq_p^2(2) \pmod{p^2}. \end{aligned} \tag{2.11}$$

When $n = p-1$ in Theorem 2.5, from (2.11), we have

$$\begin{aligned} & 2p-2-2pq_p(2)-pq_p^2(2) \\ \equiv\ & 2^{p-1}\left(2p-2pH_p+pH_p^2-pH_{p,2}+\sum_{k=0}^{p-1}H_{p-1-k}(1/2)\left(H_k(1/2)-2H_k\right)\right) \pmod{p^2}. \end{aligned}$$

From here, using the congruences $H_{p-1} \equiv 0 \pmod{p^2}$ and $H_{p-1,2} \equiv 0 \pmod{p}$, and making the necessary arrangements, the desired result is obtained. □

References

[1] Abel, N. H. (1826). Untersuchungen über die Reihe $1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3+\ldots$. *Journal für die reine und angewandte Mathematik*, 1, 311–339.

[2] Batır, N. (2020). Combinatorial identities involving harmonic numbers. *Integers*, #A25, 20(2).

[3] Batır, N., & Sofo, A. (2021). A unified treatment of certain classes of combinatorial identities. *Journal of Integer Sequences*, 24, Article 21.3.2.

[4] Boyadzhiev, K. N. (2009). Harmonic number identities via Euler's transform. *Journal of Integer Sequences*, 12, Article 09.6.1.

[5] Choi, J., & Srivastava, H. M. (2011). Some summation formulas involving harmonic numbers and generalized harmonic numbers. *Mathematical and Computer Modelling*, 54, 2220–2234.

[6] Frontczak, R. (2020). Harmonic sums via Euler's Transform: Complementing the approach of Boyadzhiev. *Journal of Integer Sequences*, 23, Article 20.3.2.

[7] Frontczak, R. (2021). Binomial sums with skew-harmonic numbers. *Palestine Journal of Mathematics*, 10(2), 756–763.

[8] Genčev, M. (2011). Binomial sums involving harmonic numbers. *Mathematica Slovaca*, 61(2), 215–226.

[9] Guo, D., & Chu, W. (2021). Summation formulae involving multiple harmonic numbers. *Applicable Analysis and Discrete Mathematics*, 15(1), 201–212.

[10] Jin, H.-T., & Du, D. K. (2015). Abel's lemma and identities on harmonic numbers. *Integers*, 15, #A22.

[11] Mestrovic, R. (2015). Five curious congruences modulo p^2. *Mathematica Slovaca*, 65(3), 451–462.

[12] Ömür, N., & Bilgin, G. (2018). Some applications of the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$. *Advances and Applications in Mathematical Sciences*, 17(9), 617–627.

[13] Ömür, N., & Koparal, S. (2018). On the matrices with the generalized hyperharmonic numbers of order r. *Asian-European Journal of Mathematics*, 11(3), 1850045.

[14] Spieß, J. (1990). Some identities involving harmonic numbers. *Mathematics of Computation*, 55(192), 839–863.

[15] Thapa, G. B., Lòpez-Bonilla, J., & Lòpez-Vázquez, R. (2017). A note on some applications of Boyadzhiev's formula. *Journal of the Institute of Engineering*, 31(1), 175–177.