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Almost balancers, almost cobalancers, almost Lucas-balancers and almost Lucas-cobalancers

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Abstract: In this work, the general terms of almost balancers, almost cobalancers, almost Lucas-balancers and almost Lucas-cobalancers of first and second type are determined in terms of balancing and Lucas-balancing numbers. Later some relations on all almost balancing numbers and all almost balancers are obtained. Further the general terms of all balancing numbers, Pell numbers and Pell–Lucas number are determined in terms of almost balancers, almost Lucas-balancers, almost cobalancers and almost Lucas-cobalancers of first and second type.
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1 Introduction

Behera and Panda ([2]) defined that a positive integer n is called a balancing number if the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1.1}$$

holds for some positive integer r which is called balancer corresponding to n.

Panda and Ray ([12]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r) \tag{1.2}$$

holds for some positive integer r which is called cobalancer corresponding to n.

Let B_n denote the n-th balancing number and let b_n denote the n-th cobalancing number. Then B_n is a balancing number iff $8B_n^2 + 1$ is a perfect square and b_n is a cobalancing number iff $8b_n^2 + 8b_n + 1$ is a perfect square. Thus $C_n = \sqrt{8B_n^2 + 1}$ and $c_n = \sqrt{8b_n^2 + 8b_n + 1}$ are integers which are called the n-th Lucas-balancing number and n-th Lucas-cobalancing number, respectively. (Here one can note that Lucas-balancers and Lucas-cobalancers are not defined in the literature before. But anyways, one can see that if R_n is the n-th balancer, then $8R_n^2 + 8R_n + 1$ is a perfect square and if r_n is the n-th cobalancer, then $8r_n^2 + 1$ is a perfect square. Thus $CR_n = \sqrt{8R_n^2 + 8R_n + 1}$ and $cr_n = \sqrt{8r_n^2 + 1}$ are integers which may be called n-th Lucas-balancer and the n-th Lucas-cobalancer, respectively). (see also [4, 9, 11, 14]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects (see [5, 6, 7, 8, 15, 16, 17, 20, 21]). Recently, almost balancing numbers defined by Panda and Panda in [13]. A natural number n is called an almost balancing number if the Diophantine equation

$$|[(n + 1) + (n + 2) + \cdots + (n + r)] - [1 + 2 + \cdots + (n - 1)]| = 1 \tag{1.3}$$

holds for some positive integer r which is called the almost balancer. In [10], Panda defined that a positive integer n is called an almost cobalancing number if the Diophantine equation

$$|[(n + 1) + (n + 2) + \cdots + (n + r)] - (1 + 2 + \cdots + n)| = 1 \tag{1.4}$$

holds for some positive integer r which is called an almost cobalancer (see also [18, 19, 23]).

2 Almost balancers and almost Lucas-balancers

In this section, we try to determine the general terms of almost balancers and almost Lucas-balancers of first and second type. From (1.3), we have two cases:

Case 1: If $[(n + 1) + (n + 2) + \cdots + (n + r)] - [1 + 2 + \cdots + (n - 1)] = 1$, then n is called an almost balancing number of first type and r is called an almost balancer of first type and in this case

$$r = \frac{-2n - 1 + \sqrt{8n^2 + 9}}{2} \quad \text{and} \quad n = \frac{2r + 1 + \sqrt{8r^2 + 8r - 7}}{2}. \tag{2.1}$$

Let B_n^* denote the n-th almost balancing number of first type and let R_n^* denote the n-th almost balancer of first type. Then from (2.1), B_n^* is an almost balancing number of first type iff $8(B_n^*)^2 + 9$ is a perfect square and R_n^* is an almost balancer of first type iff $8(R_n^*)^2 + 8R_n^* - 7$ is a perfect square. Thus

$$C_n^* = \sqrt{8(B_n^*)^2 + 9} \quad \text{and} \quad CR_n^* = \sqrt{8(R_n^*)^2 + 8R_n^* - 7} \tag{2.2}$$

are integers which are called the n-th almost Lucas-balancing number of first type and the n-th almost Lucas-balancer of first type, respectively.

Case 2: If $[(n+1)+(n+2)+\cdots+(n+r)]-[1+2+\cdots+(n-1)]=-1$, then n is called an almost balancing number of second type and r is called an almost balancer of second type and in this case

$$r=\frac{-2n-1+\sqrt{8n^2-7}}{2} \quad \text{and} \quad n=\frac{2r+1+\sqrt{8r^2+8r+9}}{2}. \tag{2.3}$$

Let B_n^{**} denote the n-th almost balancing number of second type and let R_n^{**} denote the n-th almost balancer of second type. Then from (2.3), B_n^{**} is an almost balancing number of second type iff $8(B_n^{**})^2-7$ is a perfect square and R_n^{**} is an almost balancer of second type iff $8(R_n^{**})^2+8R_n^{**}+9$ is a perfect square. Thus

$$C_n^{**}=\sqrt{8(B_n^{**})^2-7} \quad \text{and} \quad CR_n^{**}=\sqrt{8(R_n^{**})^2+8R_n^{**}+9} \tag{2.4}$$

are integers which are called the n-th almost Lucas-balancing number of second type and the n-th almost Lucas-balancer of second type, respectively. (Just as Lucas-balancers or almost Lucas-balancers have not been defined before, we have defined almost Lucas-balancer of first and second type for the first time here, as we defined balcobalancing numbers in [22] for the first time before).

From (2.2), we note that R_n^* is an almost balancer of first type iff $8(R_n^*)^2+8R_n^*-7$ is a perfect square. So we set

$$8(R_n^*)^2+8R_n^*-7=y^2$$

for some integer $y\geq 1$. Then

$$2(2R_n^*+1)^2-9=y^2,$$

and taking $x=2R_n^*+1$, we get the Pell equation (see [1])

$$2x^2-y^2=9. \tag{2.5}$$

Similarly from (2.4), we note that R_n^{**} is an almost balancer of second type iff $8(R_n^{**})^2+8R_n^{**}+9$ is a perfect square. So we set

$$8(R_n^{**})^2+8R_n^{**}+9=y^2$$

for some integer $y\geq 1$. Then

$$2(2R_n^{**}+1)^2+7=y^2$$

and taking $x=2R_n^{**}+1$, we get the Pell equation

$$2x^2-y^2=-7. \tag{2.6}$$

Let Ω denote the set of all integer solutions of (2.5) and (2.6). Then

Theorem 2.1. *The set of all integer solutions of* (2.5) *is*

$$\Omega=\{(-6B_n+3C_n,12B_n-3C_n):n\geq 1\},$$

and the set of all integer solutions of (2.6) *is*

$$\Omega=\{(6B_{n-1}+C_{n-1},4B_{n-1}+3C_{n-1}):n\geq 1\}\cup\{(6B_n-C_n,-4B_n+3C_n):n\geq 1\}.$$

Proof. For the Pell equation in (2.5), the indefinite form is $F = (2, 0, -1)$ of discriminant $\Delta = 8$. So $\tau_8 = 3 + 2\sqrt{2}$ and the set of representatives is $\text{Rep} = \{[\pm 3 \quad 3]\}$ (see [3]). Here $[3 \quad -3]M^n$ generates all integer solutions (x_n, y_n) for $n \geq 1$, where $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Since the n-th power of M is $M^n = \begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$, we conclude that $\Omega = \{(-6B_n + 3C_n, 12B_n - 3C_n) : n \geq 1\}$.

For the second Pell equation in (2.6), the set of representatives is $\text{Rep} = \{[\pm 1 \quad 3]\}$. Here $[1 \quad 3]M^{n-1}$ generates all integer solutions (x_{2n-1}, y_{2n-1}) and $[-1 \quad 3]M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$. So

$$\Omega = \{(6B_{n-1} + C_{n-1}, 4B_{n-1} + 3C_{n-1}) : n \geq 1\} \cup \{(6B_n - C_n, -4B_n + 3C_n) : n \geq 1\}.$$

This completes the proof. □

From Theorem 2.1, we can give the following theorem.

Theorem 2.2. *The general terms of almost balancers and almost Lucas-balancers of first type are*

$$R_n^* = \frac{-6B_n + 3C_n - 1}{2}, \; CR_n^* = 12B_n - 3C_n$$

for $n \geq 1$, *and of second type are*

$$\begin{aligned} R_{2n-1}^{**} &= \frac{6B_{n-1} - C_{n-1} - 1}{2}, \; CR_{2n-1}^{**} = -4B_{n-1} + 3C_{n-1} \\ R_{2n}^{**} &= \frac{6B_{n-1} + C_{n-1} - 1}{2}, \; CR_{2n}^{**} = 4B_{n-1} + 3C_{n-1} \end{aligned}$$

for $n \geq 1$.

Proof. We proved in Theorem 2.1 that the set of all integer solutions of (2.5) is

$$\Omega = \{(-6B_n + 3C_n, 12B_n - 3C_n) : n \geq 1\}.$$

Since $x = 2R_n^* + 1$, we get

$$R_n^* = \frac{-6B_n + 3C_n - 1}{2}.$$

Thus from (2.2), we deduce that

$$\begin{aligned} CR_n^* &= \sqrt{8(R_n^*)^2 + 8R_n^* - 7} \\ &= \sqrt{8\left(\frac{-6B_n + 3C_n - 1}{2}\right)^2 + 8\left(\frac{-6B_n + 3C_n - 1}{2}\right) - 7} \\ &= \sqrt{9(C_n^2 - 1) - 72B_nC_n + 18(8B_n^2 + 1) - 9} \\ &= \sqrt{144B_n^2 - 72B_nC_n + 9C_n^2} \\ &= 12B_n - 3C_n. \end{aligned}$$

The others can be proved similarly. □

3 Almost cobalancers and almost Lucas-cobalancers

In this section, we try to determine the general terms of almost cobalancers and almost Lucas-cobalancers of first and second type. From (1.4), we have two cases:

Case 1: If $[(n+1)+(n+2)+\cdots+(n+r)]-(1+2+\cdots+n)=1$, then n is called an almost cobalancing number of first type and r is called an almost cobalancer of first type and in this case

$$r=\frac{-2n-1+\sqrt{8n^2+8n+9}}{2} \quad \text{and} \quad n=\frac{2r-1+\sqrt{8r^2-7}}{2}. \tag{3.1}$$

Let b_n^* denote the n-th almost cobalancing number of first type and let r_n^* denote the n-th almost cobalancer of first type. Then from (3.1), b_n^* is an almost cobalancing number of first type iff $8(b_n^*)^2+8b_n^*+9$ is a perfect square and r_n^* is an almost cobalancer of first type iff $8(r_n^*)^2-7$ is a perfect square. Thus

$$c_n^*=\sqrt{8(b_n^*)^2+8b_n^*+9} \quad \text{and} \quad cr_n^*=\sqrt{8(r_n^*)^2-7} \tag{3.2}$$

are integers which are called the n-th almost Lucas-cobalancing number of first type and the n-th almost Lucas-cobalancer of first type, respectively.

Case 2: If $[(n+1)+(n+2)+\cdots+(n+r)]-(1+2+\cdots+n)=-1$, then n is called an almost cobalancing number of second type and r is called an almost cobalancer of second type and in this case

$$r=\frac{-2n-1+\sqrt{8n^2+8n-7}}{2} \quad \text{and} \quad n=\frac{2r-1+\sqrt{8r^2+9}}{2}. \tag{3.3}$$

Let b_n^{**} denote the n-th almost cobalancing number of second type and let r_n^{**} denote the n-th almost cobalancer of second type. Then from (3.3), b_n^{**} is an almost cobalancing number of second type iff $8(b_n^{**})^2+8b_n^{**}-7$ is a perfect square and r_n^{**} is an almost cobalancer of second type iff $8(r_n^{**})^2+9$ is a perfect square. Thus

$$c_n^{**}=\sqrt{8(b_n^{**})^2+8b_n^{**}-7} \quad \text{and} \quad cr_n^{**}=\sqrt{8(r_n^{**})^2+9} \tag{3.4}$$

are integers which are called the n-th almost Lucas-cobalancing number of second type and the n-th almost Lucas-cobalancer of second type, respectively. (Just as Lucas-cobalancers or almost Lucas-cobalancers have not been defined before, we have defined almost Lucas-cobalancer of first and second type for the first time here).

From (3.2), we note that r_n^* is an almost cobalancer of first type iff $8(r_n^*)^2-7$ is a perfect square. So we set

$$8(r_n^*)^2-7=y^2$$

for some integer $y\geq 1$. Taking $x=r_n^*$, we get the Pell equation

$$8x^2-y^2=7. \tag{3.5}$$

Similarly from (3.4), we note that r_n^{**} is an almost cobalancer of second type iff $8(r_n^{**})^2+9$ is a perfect square. So we set

$$8(r_n^{**})^2+9=y^2$$

for some integer $y\geq 1$. Taking $x=r_n^{**}$, we get the Pell equation

$$8x^2-y^2=-9. \tag{3.6}$$

For the set of all integer solutions of (3.5) and (3.6), we can give the following theorem.

Theorem 3.1. *The set of all integer solutions of* (3.5) *is*

$$\Omega = \{(B_{n-1}+C_{n-1}, 8B_{n-1}+C_{n-1}) : n \geq 1\} \cup \{(-B_n+C_n, 8B_n-C_n) : n \geq 1\},$$

and the set of all integer solutions of (3.6) *is* $\Omega = \{(3B_n, 3C_n) : n \geq 1\}$.

Proof. For the Pell equation in (3.5), the indefinite form is $F = (8, 0, -1)$ of discriminant $\Delta = 32$. So $\tau_{32} = 3+\sqrt{8}$ and the set of representatives is $\text{Rep} = \{[\pm 1 \quad 1]\}$. Here $[1 \quad 1]M^{n-1}$ generates all integer solutions (x_{2n-1}, y_{2n-1}) and $[1 \ -1]M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$, where $M = \begin{bmatrix} 3 & 8 \\ 1 & 3 \end{bmatrix}$. Since the n-th power of M is $M^n = \begin{bmatrix} C_n & 8B_n \\ B_n & C_n \end{bmatrix}$ for $n \geq 1$, we get

$$\Omega = \{(B_{n-1}+C_{n-1}, 8B_{n-1}+C_{n-1}) : n \geq 1\} \cup \{(-B_n+C_n, 8B_n-C_n) : n \geq 1\}.$$

For the second Pell equation in (3.6), the set of representatives is $\text{Rep} = \{[0 \ \ 3]\}$ and $[0 \ \ 3]M^n$ generates all integer solutions (x_n, y_n) for $n \geq 1$. Thus $\Omega = \{(3B_n, 3C_n) : n \geq 1\}$. □

From Theorem 3.1, we can give the following theorem.

Theorem 3.2. *The general terms of almost cobalancers and almost Lucas-cobalancers of first type are*

$$r^*_{2n-1} = B_{n-1}+C_{n-1}, \ cr^*_{2n-1} = 8B_{n-1}+C_{n-1}, r^*_{2n} = -B_n+C_n, \ cr^*_{2n} = 8B_n - C_n$$

for $n \geq 1$, *and of second type are*

$$r^{**}_n = 3B_{n-1}, \ cr^{**}_n = 3C_{n-1}$$

for $n \geq 1$.

Proof. We proved in Theorem 3.1 that the set of all integer solutions of (3.5) is

$$\Omega = \{(B_{n-1}+C_{n-1}, 8B_{n-1}+C_{n-1}) : n \geq 1\} \cup \{(-B_n+C_n, 8B_n-C_n) : n \geq 1\}.$$

Since $x = r^*_n$, we get

$$r^*_{2n-1} = B_{n-1}+C_{n-1}.$$

Thus from (3.2), we observe that

$$\begin{aligned} cr^*_{2n-1} &= \sqrt{8(r^*_{2n-1})^2 - 7} \\ &= \sqrt{8(B_{n-1}+C_{n-1})^2 - 7} \\ &= \sqrt{C^2_{n-1} - 1 + 16B_{n-1}C_{n-1} + 8(8B^2_{n-1}+1) - 7} \\ &= \sqrt{64B^2_{n-1} + 16B_{n-1}C_{n-1} + C^2_{n-1}} \\ &= 8B_{n-1}+C_{n-1}. \end{aligned}$$

The others can be proved similarly. □

4 Balancing numbers and almost balancers, almost cobalancers, almost Lucas-balancers, almost Lucas-cobalancers

We can give the general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers in terms of almost balancers, almost cobalancers, almost Lucas-balancers and almost Lucas-cobalancers of first and second type as follows.

Theorem 4.1. *For the general terms of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers, we have:*

1. *They can be given in terms of almost balancers and almost Lucas-balancers of first type to be*

$$B_n = \frac{2R_n^* + CR_n^* + 1}{6},\ b_n = \frac{R_n^* - 1}{3},\ C_n = \frac{4R_n^* + CR_n^* + 2}{3}$$

for $n \geq 1$ *and*

$$c_n = \frac{2(R_n^* + R_{n-1}^*) + CR_n^* + CR_{n-1}^* + 2}{6}$$

for $n \geq 2$.

2. *They can be given in terms of almost balancers and almost Lucas-balancers of second type to be*

$$B_n = \begin{cases} \frac{6R_{2n+1}^{**}+CR_{2n+1}^{**}+3}{14} \\ \frac{6R_{2n+2}^{**}-CR_{2n+2}^{**}+3}{14} \end{cases} \qquad b_n = \begin{cases} \frac{-R_{2n+1}^{**}+CR_{2n+1}^{**}-4}{7} \\ \frac{-5R_{2n+2}^{**}+2CR_{2n+2}^{**}-6}{7} \end{cases}$$

$$C_n = \begin{cases} \frac{4R_{2n+1}^{**}+3CR_{2n+1}^{**}+2}{7} \\ \frac{-4R_{2n+2}^{**}+3CR_{2n+2}^{**}-2}{7} \end{cases} \qquad c_n = \begin{cases} \frac{6(R_{2n+1}^{**}+R_{2n-1}^{**})+CR_{2n+1}^{**}+CR_{2n-1}^{**}+6}{14} \\ \frac{6(R_{2n+2}^{**}+R_{2n}^{**})-(CR_{2n+2}^{**}+CR_{2n}^{**})+6}{14} \end{cases}$$

for $n \geq 1$.

3. *They can be given in terms of almost cobalancers and almost Lucas-cobalancers of first type to be*

$$B_n = \begin{cases} \frac{r_{2n}^*+cr_{2n}^*}{7} \\ \frac{-r_{2n+1}^*+cr_{2n+1}^*}{7} \end{cases} \qquad b_n = \begin{cases} \frac{6r_{2n}^*-cr_{2n}^*-7}{14} \\ \frac{10r_{2n+1}^*-3cr_{2n+1}^*-7}{14} \end{cases} \qquad C_n = \begin{cases} \frac{8r_{2n}^*+cr_{2n}^*}{7} \\ \frac{8r_{2n+1}^*-cr_{2n+1}^*}{7} \end{cases}$$

for $n \geq 1$ *and*

$$c_n = \begin{cases} \frac{r_{2n}^*+r_{2n-2}^*+cr_{2n}^*+cr_{2n-2}^*}{7},\ n \geq 2 \\ \frac{-r_{2n+1}^*-r_{2n-1}^*+cr_{2n+1}^*+cr_{2n-1}^*}{7},\ n \geq 1. \end{cases}$$

4. *They can be given in terms of almost cobalancers and almost Lucas-cobalancers of second type to be*

$$B_n = \frac{r^{**}_{n+1}}{3},\ b_n = \frac{-2r^{**}_{n+1} + cr^{**}_{n+1} - 3}{6},\ C_n = \frac{cr^{**}_{n+1}}{3}\ \text{ and }\ c_n = \frac{r^{**}_{n+1} + r^{**}_n}{3}$$

for $n \geq 1$.

Proof. **(1)** From Theorem 2.2, we notice that $R^*_n = \frac{-6B_n+3C_n-1}{2}$ and $CR^*_n = 12B_n - 3C_n$. Thus from the system of equations $-6B_n + 3C_n = 2R^*_n + 1$ and $12B_n - 3C_n = CR^*_n$, we get

$$B_n = \frac{2R^*_n + CR^*_n + 1}{6} \quad \text{and} \quad C_n = \frac{4R^*_n + CR^*_n + 2}{3}.$$

Since $b_n = \frac{-2B_n+C_n-1}{2}$, we easily get

$$b_n = \frac{-2B_n + C_n - 1}{2} = \frac{-2(\frac{2R^*_n+CR^*_n+1}{6}) + (\frac{4R^*_n+CR^*_n+2}{3}) - 1}{2} = \frac{R^*_n - 1}{3}$$

and since $c_n = B_n + B_{n-1}$ we conclude that

$$c_n = B_n + B_{n-1} = \frac{2(R^*_n + R^*_{n-1}) + CR^*_n + CR^*_{n-1} + 2}{6}$$

as we wanted. The others can be proved similarly. □

Recall that every balancing number is a cobalancer and every cobalancing number is a balancer, that is,

$$B_n = r_{n+1} \quad \text{and} \quad R_n = b_n$$

for $n \geq 1$. Similarly we can give the following result.

Theorem 4.2. *For the integer sequences mentioned above, we have*

$$B^*_n = r^{**}_{n+1},\ b^*_n = R^{**}_{n+2},\ C^*_n = cr^{**}_{n+1},\ c^*_n = CR^{**}_{n+2}$$
$$B^{**}_n = r^*_n,\ b^{**}_n = R^*_n,\ C^{**}_n = cr^*_n,\ c^{**}_n = CR^*_n$$

for $n \geq 1$.

Proof. Since $B^*_n = 3B_n$ and also $r^{**}_n = 3B_{n-1}$ by Theorem 3.2, we get $B^*_n = r^{**}_{n+1}$. Since $b^*_{2n} = 2b_{n+1} - b_n$, we easily get

$$\begin{aligned}
b^*_{2n} &= 2b_{n+1} - b_n \\
&= 2(\frac{\alpha^{2n+1} - \beta^{2n+1}}{4\sqrt{2}} - \frac{1}{2}) - (\frac{\alpha^{2n-1} - \beta^{2n-1}}{4\sqrt{2}} - \frac{1}{2}) \\
&= \alpha^{2n}(\frac{6}{8\sqrt{2}} + \frac{1}{4}) + \beta^{2n}(\frac{-6}{8\sqrt{2}} + \frac{1}{4}) - \frac{1}{2} \\
&= \frac{6(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}) + \frac{\alpha^{2n}+\beta^{2n}}{2} - 1}{2} \\
&= \frac{6B_n + C_n - 1}{2} \\
&= R^{**}_{2n+2}
\end{aligned}$$

by Theorem 2.2. Similarly it can be shown that $b^*_{2n-1} = 4b_n - b_{n-1} + 1 = \frac{6B_n-C_n-1}{2} = R^{**}_{2n+1}$. Thus $b^*_n = R^{**}_{n+2}$ for every $n \geq 1$. The other cases can be proved similarly. □

Further we can give the general terms of almost balancers, almost Lucas-balancers, almost cobalancers and almost Lucas-cobalancers of first type in terms of almost balancers and almost Lucas-balancers of second type as follows:

Theorem 4.3. *The general terms of almost balancers, almost Lucas-balancers, almost cobalancers and almost Lucas-cobalancers of first type are*

$$R_n^* = \begin{cases} \frac{-3R_{2n+1}^{**}+3CR_{2n+1}^{**}-5}{7} \\ \frac{-15R_{2n+2}^{**}+6CR_{2n+2}^{**}-11}{7} \end{cases} \qquad CR_n^* = \begin{cases} \frac{48R_{2n+1}^{**}-6CR_{2n+1}^{**}+24}{14} \\ \frac{48R_{2n+2}^{**}-15CR_{2n+2}^{**}+24}{7} \end{cases}$$

$$r_{2n-1}^* = \begin{cases} \frac{14R_{2n-1}^{**}+7CR_{2n-1}^{**}+7}{14} \\ \frac{-2R_{2n}^{**}+5CR_{2n}^{**}-1}{14} \end{cases} \qquad r_{2n}^* = \begin{cases} \frac{2R_{2n+1}^{**}+5CR_{2n+1}^{**}+1}{14} \\ \frac{-14R_{2n+2}^{**}+7CR_{2n+2}^{**}-7}{14} \end{cases}$$

$$cr_{2n-1}^* = \begin{cases} \frac{28R_{2n-1}^{**}+7CR_{2n-1}^{**}+14}{7} \\ \frac{20R_{2n}^{**}-CR_{2n}^{**}+10}{7} \end{cases} \qquad cr_{2n}^* = \begin{cases} \frac{20R_{2n+1}^{**}+CR_{2n+1}^{**}+10}{7} \\ \frac{28R_{2n+2}^{**}-7CR_{2n+2}^{**}+14}{7} \end{cases}$$

for $n \geq 1$.

Proof. Note that $B_n = \frac{6R_{2n+1}^{**}+CR_{2n+1}^{**}+3}{14}$ and $C_n = \frac{4R_{2n+1}^{**}+3CR_{2n+1}^{**}+2}{7}$ by (2) of Theorem 4.1. Thus from Theorem 2.2, we get

$$\begin{aligned} R_n^* &= \frac{-6B_n+3C_n-1}{2} \\ &= \frac{-6(\frac{6R_{2n+1}^{**}+CR_{2n+1}^{**}+3}{14})+3(\frac{4R_{2n+1}^{**}+3CR_{2n+1}^{**}+2}{7})-1}{2} \\ &= \frac{-3R_{2n+1}^{**}+3CR_{2n+1}^{**}-5}{7} \end{aligned}$$

as we claimed. The other cases can be proved similarly. □

Conversely, we can give the general terms of almost balancers, almost Lucas-balancers, almost cobalancers and almost Lucas-cobalancers of second type in terms of almost cobalancers and almost Lucas-cobalancers of first type as follows:

Theorem 4.4. *The general terms of almost balancers, almost Lucas-balancers, almost cobalancers and almost Lucas-cobalancers of second type are*

$$R_{2n-1}^{**} = \begin{cases} \frac{-2r_{2n-2}^*+5cr_{2n-2}^*-7}{14}, & n \geq 2 \\ \frac{-14r_{2n-1}^*+7cr_{2n-1}^*-7}{14}, & n \geq 1 \end{cases} \qquad R_{2n}^{**} = \begin{cases} \frac{14r_{2n-2}^*+7cr_{2n-2}^*-7}{14}, & n \geq 2 \\ \frac{2r_{2n-1}^*+5cr_{2n-1}^*-7}{14}, & n \geq 1 \end{cases}$$

$$CR_{2n-1}^{**} = \begin{cases} \frac{20r_{2n-2}^*-cr_{2n-2}^*}{7}, & n \geq 2 \\ \frac{28r_{2n-1}^*-7cr_{2n-1}^*}{7}, & n \geq 1 \end{cases} \qquad CR_{2n}^{**} = \begin{cases} \frac{28r_{2n-2}^*+7cr_{2n-2}^*}{7}, & n \geq 2 \\ \frac{20r_{2n-1}^*+cr_{2n-1}^*}{7}, & n \geq 1 \end{cases}$$

$$r_n^{**} = \begin{cases} \frac{3r_{2n-2}^*+3cr_{2n-2}^*}{7}, & n \geq 2 \\ \frac{-3r_{2n-1}^*+3cr_{2n-1}^*}{7}, & n \geq 1 \end{cases} \qquad cr_n^{**} = \begin{cases} \frac{24r_{2n-2}^*+3cr_{2n-2}^*}{7}, & n \geq 2 \\ \frac{24r_{2n-1}^*-3cr_{2n-1}^*}{7}, & n \geq 1. \end{cases}$$

Proof. It can be proved as in the same way that Theorem 4.3 was proved. □

Thus from Theorems 4.3 and 4.4, we construct a one to one correspondece between almost balancers, almost Lucas-balancers, almost cobalancers, almost Lucas-cobalancers of first type and of second type.

5 Pell numbers, Pell–Lucas numbers and almost balancers, almost cobalancers, almost Lucas-balancers, almost Lucas-cobalancers

Let $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ be the roots of the characteristic equation for Pell and Pell–Lucas numbers which are the numbers defined by $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ and $Q_0 = Q_1 = 2, Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$. Ray ([14]) derived some nice results on balancing numbers and Pell numbers his Ph.D. thesis. He proved that the general terms of even and odd ordered Pell numbers can be given in terms of balancing and cobalancing numbers, namely, $P_{2n} = 2B_n$ and $P_{2n-1} = 2b_n + 1$. Similarly we can give the following theorem.

Theorem 5.1. *The general terms of even and odd ordered Pell numbers can be given in terms of almost balancers, almost cabalancers, almost Lucas-balancers and almost Lucas-cobalancers of first type to be*

$$P_{2n-1} = \begin{cases} \frac{2R_n^*+1}{3} \\ \frac{10r_{2n+1}^*-3cr_{2n+1}^*}{7} \end{cases} \qquad P_{2n} = \begin{cases} \frac{2R_n^*+CR_n^*+1}{3} \\ \frac{-2r_{2n+1}^*+2cr_{2n+1}^*}{7} \end{cases}$$

for $n \geq 1$, *or of second type to be*

$$P_{2n-1} = \begin{cases} R_{2n+1}^{**} - R_{2n}^{**} \\ \frac{-2r_{n+1}^{**}+cr_{n+1}^{**}}{3} \end{cases} \qquad P_{2n} = \begin{cases} \frac{6R_{2n+1}^{**}+CR_{2n+1}^{**}+3}{7} \\ \frac{2r_{n+1}^{**}}{3} \end{cases}$$

for $n \geq 1$.

Proof. Since $P_n = \frac{\alpha^n-\beta^n}{2\sqrt{2}}$, we deduce that

$$\begin{aligned} P_{2n-1} &= \frac{\alpha^{2n-1} - \beta^{2n-1}}{2\sqrt{2}} \\ &= \frac{\alpha^{2n}(-1+\sqrt{2}) + \beta^{2n}(1+\sqrt{2})}{2\sqrt{2}} \\ &= \alpha^{2n}(\frac{-1}{2\sqrt{2}} + \frac{1}{2}) + \beta^{2n}(\frac{1}{2\sqrt{2}} + \frac{1}{2}) \\ &= -2(\frac{\alpha^{2n} - \beta^{2n}}{4\sqrt{2}}) + \frac{\alpha^{2n} + \beta^{2n}}{2} \\ &= \frac{2(\frac{-6B_n+3C_n-1}{2}) + 1}{3} \\ &= \frac{2R_n^* + 1}{3} \end{aligned}$$

by Theorem 2.2. The other cases can be proved similarly. □

Also the general terms of even and odd ordered Pell–Lucas numbers can be given in terms of balancing and cobalancing numbers, namely, $Q_{2n} = 4B_n + 4b_n + 2$ and $Q_{2n-1} = 4B_n - 4b_n - 2$. Similarly we can give the following theorem.

Theorem 5.2. *The general terms of even and odd ordered Pell–Lucas numbers can be given in terms of almost balancers, almost cabalancers, almost Lucas-balancers and almost Lucas-cobalancers of first type to be*

$$Q_{2n-1} = \begin{cases} \frac{2(R_n^*+R_{n-1}^*+1)+CR_n^*+CR_{n-1}^*}{3} \\ \frac{-2(r_{2n+1}^*+r_{2n-1}^*-cr_{2n+1}^*-cr_{2n-1}^*)}{7} \end{cases} \quad Q_{2n} = \begin{cases} \frac{2(R_{n+1}^*+R_n^*+1)}{3} \\ \frac{10(r_{2n+3}^*+r_{2n+1}^*)-3(cr_{2n+3}^*+cr_{2n+1}^*)}{7} \end{cases}$$

for $n \geq 1, 2$ *or of second type to be*

$$Q_{2n-1} = \begin{cases} \frac{6(R_{2n+1}^{**}+R_{2n-1}^{**}+1)+CR_{2n+1}^{**}+CR_{2n-1}^{**}}{7} \\ \frac{2(r_{n+1}^{**}+r_n^{**})}{3} \end{cases} \quad Q_{2n} = \begin{cases} R_{2n+3}^{**} - R_{2n+2}^{**} + R_{2n+1}^{**} - R_{2n}^{**} \\ \frac{-2(r_{n+2}^{**}+r_{n+1}^{**})+cr_{n+2}^{**}+cr_{n+1}^{**}}{3} \end{cases}$$

for $n \geq 1$.

Proof. It can be proved as in the same way that Theorem 5.1 was proved. □

6 Sums of almost balancers, almost Lucas-balancers, almost cobalancers and almost Lucas-cobalancers

Theorem 6.1. *The sums of first* n*-terms of almost balancers and almost Lucas-balancers of first and second type are*

$$\sum_{i=1}^{n} R_i^* = \frac{3B_n - n}{2},\ n \geq 1$$

$$\sum_{i=1}^{n} CR_i^* = \frac{9B_n - 3B_{n-1} - 3}{2},\ n \geq 1$$

$$\sum_{i=1}^{n} R_i^{**} = \begin{cases} \frac{15B_{\frac{n-2}{2}} - 3B_{\frac{n-4}{2}} - n - 3}{2}, & n \geq 2 \text{ even} \\ \frac{34B_{\frac{n-3}{2}} - 6B_{\frac{n-5}{2}} - n - 3}{2}, & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n} CR_i^{**} = \begin{cases} 21B_{\frac{n-2}{2}} - 3B_{\frac{n-4}{2}} + 3, & n \geq 2 \text{ even} \\ 48B_{\frac{n-3}{2}} - 8B_{\frac{n-5}{2}} + 3, & n \geq 1 \text{ odd} \end{cases}$$

and the sums of first n*-terms of almost cobalancers and almost Lucas-cobalancers of first and second type are*

$$\sum_{i=1}^{n} r_i^* = \begin{cases} 18B_{\frac{n-2}{2}} - 3B_{\frac{n-4}{2}}, & n \geq 2 \text{ even} \\ 7B_{\frac{n-1}{2}} - B_{\frac{n-3}{2}}, & n \geq 1 \text{ odd} \end{cases}$$

$$\sum_{i=1}^{n} cr_i^* = \begin{cases} 51B_{\frac{n-2}{2}} - 9B_{\frac{n-4}{2}} - 3, & n \geq 2 \textit{ even} \\ 20B_{\frac{n-1}{2}} - 4B_{\frac{n-3}{2}} - 3, & n \geq 1 \textit{ odd} \end{cases}$$

$$\sum_{i=1}^{n} r_i^{**} = \frac{15B_{n-1} - 3B_{n-2} - 3}{4}, n \geq 1$$

$$\sum_{i=1}^{n} cr_i^{**} = \frac{21B_{n-1} - 3B_{n-2} + 3}{2}, n \geq 1.$$

Proof. Recall that $B_1 + B_2 + \cdots + B_n = \frac{5B_n - B_{n-1} - 1}{4}$ and $C_1 + C_2 + \cdots + C_n = \frac{7B_n - B_{n-1} - 1}{2}$. Thus from Theorem 2.2, we get

$$\begin{aligned} \sum_{i=1}^{n} R_i^* &= R_1^* + R_2^* + \cdots + R_{n-1}^* + R_n^* \\ &= \frac{-6B_1 + 3C_1 - 1}{2} + \frac{-6B_2 + 3C_2 - 1}{2} + \cdots \\ &\quad + \frac{-6B_{n-1} + 3C_{n-1} - 1}{2} + \frac{-6B_n + 3C_n - 1}{2} \\ &= \frac{-6(B_1 + B_2 + \cdots + B_n) + 3(C_1 + C_2 + \cdots + C_n)}{2} - \frac{n}{2} \\ &= \frac{-6(\frac{5B_n - B_{n-1} - 1}{4}) + 3(\frac{7B_n - B_{n-1} - 1}{2})}{2} - \frac{n}{2} \\ &= \frac{3B_n - n}{2}. \end{aligned}$$

The others can be proved similarly. □

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