Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 4, 670–681
DOI: 10.7546/nntdm.29.4.670-681

Identities for Fibonacci and Lucas numbers

George Grossman[1], Aklilu Zeleke[2] and Xinyun Zhu[3]

[1] Department of Mathematics, Central Michigan University
Pearce Hall 217, Mt. Pleasant, MI, USA 48858
e-mail: gross1gw@cmich.edu

[2] Department of Statistics and Probability and the Lyman Briggs College,
Michigan State University
East Lansing MI, USA 48824
e-mail: zeleke@msu.edu

[3] Department of Mathematics, University of Texas of the Permian Basin
Odessa, TX, USA 79762
e-mail: zhu_x@utpb.edu

Received: 10 February 2023 **Revised:** 18 October 2023
Accepted: 30 October 2023 **Online First:** 1 November 2023

Abstract: In this paper several new identities are given for the Fibonacci and Lucas numbers. This is accomplished by by solving a class of non-homogeneous, linear recurrence relations.
Keywords: Recurrence relation, Non-homogeneous, Fibonacci and Lucas numbers.
2020 Mathematics Subject Classification: 11B39, 65Q30.

1 Introduction

Combinatorial identities comprise a vast area of number theory. A comprehensive summary of methods and techniques for finding and proving identities was first given by Riordan in [6]. Fibonacci numbers were studied in ancient and medieval India circa A.D. 600, [7]. Fibonacci and Lucas numbers [1–5] are also closely related to binomial coefficients. A study of Pascal's triangle reveals many patterns that are related to such numbers, including the identity

$$F_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}.$$

In $[1, 4]$ results were given for Fibonacci numbers in terms of binomial coefficients. In [2] identities with Fibonacci and Lucas numbers in terms of sums of binomial coefficients were also given.

The motivation for the present work begins with the linear recurrence relation for the Fibonacci numbers

$$F_{n+2} - F_{n+1} - F_n = 0, \quad n = 0, 1, 2, ..., \tag{1.1}$$

with $F_0 = 0, F_1 = 1$. Binet's formula gives

$$F_n = (\phi^n - (-1/\phi)^n)/\sqrt{5}, \tag{1.2}$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. Similarly, for the Lucas numbers

$$L_n = \phi^n + (-1/\phi)^n, \tag{1.3}$$

where $L_0 = 2,\ L_1 = 1$.

The approach of the present paper is to add a non-homogeneous term to (1.1). The goal of the paper is to find combinatorial identities involving the Fibonacci and Lucas numbers.

In the present work we consider the linear non-homogeneous recurrence equation with a binomial coefficient:

$$cw_{n+2} - bw_{n+1} - w_n = \binom{k+n+2}{k}, \quad n, k = 0, 1, 2, ..., \tag{1.4}$$

where b and c are nonzero real numbers. In [3], a solution for (1.4) that depends on k, n was derived with certain initial conditions. The generating function for (1.4) was also found. Examples and the theory of generating functions can be found in [8].

Let $b = c = 1$ in (1.4). The following solution is considered

$$w_n = \alpha_1 r^n + \alpha_2 s^n + p_k(n) \tag{1.5}$$

where r, s are the zeros of the characteristic equation, α_1, α_2 are real constants and $p_k(n)$ is the particular solution. Combinatorial identities are presented in section 2 for the Fibonacci numbers that are used to compute α_1, α_2 in (1.5). An expression for $p_k(n)$, denoted by $p_{k,n}$ for convenience, is given in [9]. A solution is formulated which combined with the solution from [3] yields the following

$$F_{2k+n+3} = \sum_{i=0}^{n} F_{i+1}\binom{n+k-i}{k} + \sum_{i=1}^{k} F_{2i}\binom{n+k-i+2}{n+2} + \binom{n+k+2}{k}. \tag{1.6}$$

Other identities include

$$L_{2k+3} = \sum_{i=1}^{k} F_{2i}(3k+4-3i) + 3k+4, \tag{1.7}$$

and

$$F_{2k+n+3} = \sum_{i=1}^{k} F_{2i}[F_{n+2}k + F_{n+3} - iF_{n+2}] + F_{n+2}k + F_{n+3}$$

with a corresponding expression for L_{2k+n+3}.

In Section 2 results are formulated for the Fibonacci numbers. These identities are used, along with an explicit expression for $p_{k,n}$, in solving (1.4) with $b = c = 1$ in section 3. Several lemmas related to binomial coefficients are derived in Section 3. In Section 4, a theorem is proved by mathematical induction and several identities for the Fibonacci and Lucas numbers are given.

2 Basic results

In this section we present results that give representation for the odd and even indexed Fibonacci numbers. Lemma 2.1 and Lemma 2.3 give two different representations for the odd indexed Fibonacci numbers while Lemma 2.4 gives two representations for the even indexed Fibonacci numbers. This leads to the fact that such representations are not unique and leaves the door open for finding other forms.

Lemma 2.1.

$$F_{2k+3} = \sum_{i=1}^{k} F_{2i}(k + 2 - i) + k + 2. \tag{2.1}$$

Proof. The result is true for $k = 0$. Assume it is true for k. Then for $k + 1$ we have

$$\begin{aligned}
\sum_{i=1}^{k+1} F_{2i}(k + 3 - i) + k + 3 &= \sum_{i=1}^{k+1} F_{2i}(k + 2 - i + 1) + k + 2 + 1 \\
&= \sum_{i=1}^{k} F_{2i}(k + 2 - i) + k + 2 + F_{2k+2} + \sum_{i=1}^{k+1} F_{2i} + 1 \\
&= F_{2(k+1)+2} + \sum_{i=1}^{k+1} F_{2i} + 1 = F_{2(k+1)+2} + F_{2(k+1)+1} \\
&= F_{2(k+1)+3}.
\end{aligned}$$

□

Lemma 2.2.

$$1 + \frac{(k+2)(k-1)}{2} = \sum_{i=1}^{k} F_{2i} \left[\binom{k+2-i}{k+1-i} - \binom{k+2-i}{k-i} \right]. \tag{2.2}$$

Proof. The result is true for $k = 1$. Assume it is true for k. The left hand side simplifies as

$$\begin{aligned}
1 + \frac{(k+3)k}{2} &= 1 + \frac{(k+2+1)k - 1 + 1}{2} \\
&= 1 + \frac{(k+2)(k-1)}{2} + \frac{k+2+k-1+1}{2} \\
&= 1 + \frac{(k+2)(k-1)}{2} + k + 1 \\
&= \sum_{i=1}^{k} F_{2i} \left[\binom{k+2-i}{k+1-i} - \binom{k+2-i}{k-i} \right] + k + 1.
\end{aligned} \tag{2.3}$$

The last result follows from applying the induction assumption on (2.3). Replace now k by $k+1$ on the right hand side of (2.2) to find that

$$\begin{aligned}\sum_{i=1}^{k+1} F_{2i}\left[\binom{k+3-i}{k+2-i}-\binom{k+3-i}{k+1-i}\right] &= \sum_{i=1}^{k+1} F_{2i} - \sum_{i=1}^{k} F_{2i}\binom{k+2-i}{k-i} \\ &= F_{2(k+1)+1} - 1 - \sum_{i=1}^{k} F_{2i}\binom{k+2-i}{k-i}.\end{aligned} \tag{2.4}$$

The equivalence between (2.3) and (2.4) is shown.

$$\begin{aligned}F_{2(k+1)+1} - 1 - \sum_{i=1}^{k} F_{2i}\binom{k+2-i}{k-i} = \sum_{i=1}^{k} F_{2i}\left[\binom{k+2-i}{k+1-i}-\binom{k+2-i}{k-i}\right] \\ + k + 1.\end{aligned}$$

Simplifying the last expression yields

$$F_{2(k+1)+1} = \sum_{i=1}^{k} F_{2i}(k+2-i) + k + 2 \tag{2.5}$$

which follows from Lemma 2.1. □

Lemma 2.3.

$$F_{2k+3} = 1 + \binom{k+2}{k} + \sum_{i=1}^{k} F_{2i}\binom{k+2-i}{k-i}. \tag{2.6}$$

Proof. By Lemma 2.1, it suffices to show,

$$\sum_{i=1}^{k} F_{2i}(k+2-i) + k + 2 = 1 + \binom{k+2}{k} + \sum_{i=1}^{k} F_{2i}\binom{k+2-i}{k-i}. \tag{2.7}$$

Rearranging terms in (2.7) gives

$$\begin{aligned}\sum_{i=1}^{k} F_{2i}\left[\binom{k+2-i}{k+1-i}-\binom{k+2-i}{k-i}\right] &= 1 + \binom{k+2}{k} - (k+2) \\ &= 1 + \frac{(k+1)(k+2)}{2} - \frac{2(k+2)}{2} \\ &= 1 + \frac{(k+2)(k-1)}{2}.\end{aligned}$$

Apply Lemma 2.2. □

Lemma 2.4.

$$F_{2(k+1)} = \binom{k+3}{k} + \sum_{i=1}^{k} F_{2i}\left[\binom{k+3-i}{k-i}-\binom{k+2-i}{k+1-i}\right], \tag{2.8}$$

$$F_{2(k+2)} = k + 2 + \binom{k+3}{k} + \sum_{i=1}^{k} F_{2i}\binom{k+3-i}{k-i}. \tag{2.9}$$

Proof. The difference between (2.9) and (2.8) yields Lemma 2.1. Hence it suffices to prove (2.8) by employing the recursive definition

$$F_{2(k+2)} = F_{2k+3} + F_{2(k+1)}.$$

Using induction on (2.8): the result is true for $k = 1$. Assume it is true for k. With $k + 1$ in place of k in (2.8) yields

$$\binom{k+4}{k+1} + \sum_{i=1}^{k+1} F_{2i} \left[\binom{k+4-i}{k+1-i} - \binom{k+3-i}{k+2-i} \right]. \tag{2.10}$$

Applying the induction hypothesis and (2.9) yields

$$\begin{aligned} k+2 \quad & + \binom{k+3}{k} + \sum_{i=1}^{k} F_{2i} \binom{k+3-i}{k-i} \\ & = \binom{k+4}{k+1} + \sum_{i=1}^{k+1} F_{2i} \left[\binom{k+4-i}{k+1-i} - \binom{k+3-i}{k+2-i} \right]. \end{aligned}$$

Simplifying and combining terms gives

$$\begin{aligned} k+2 \quad & + \binom{k+3}{k} - \binom{k+4}{k+1} \\ & = \sum_{i=1}^{k} F_{2i} \left[\binom{k+4-i}{k+1-i} - \binom{k+3-i}{k-i} - \binom{k+3-i}{k+2-i} \right] - F_{2(k+1)} \end{aligned}$$

or,

$$\begin{aligned} k+2 \quad & + \frac{(k+3)(k+2)\left[k+1-(k+4)\right]}{6} \\ & = \sum_{i=1}^{k} F_{2i} \left[\binom{k+4-i}{k+1-i} - \binom{k+3-i}{k-i} - \binom{k+3-i}{k+2-i} \right] - F_{2(k+1)}. \end{aligned}$$

The left-hand side simplifies to $-\frac{1}{2}(k+2)(k+1) = -\binom{k+2}{k}$. Hence

$$-\binom{k+2}{k} = \sum_{i=1}^{k} F_{2i} \left[\binom{k+4-i}{k+1-i} - \binom{k+3-i}{k-i} - \binom{k+3-i}{k+2-i} \right] - F_{2(k+1)} \tag{2.11}$$

Using the identity

$$\binom{k+4-i}{k+1-i} = \binom{k+3-i}{k+1-i} + \binom{k+3-i}{k-i}$$

(2.11) can be written as

$$\binom{k+2}{k} = \sum_{i=1}^{k} F_{2i} \left[\binom{k+3-i}{k+2-i} - \binom{k+3-i}{k+1-i} \right] + F_{2(k+1)}. \tag{2.12}$$

The binomial coefficient difference on the right-hand side of (2.12) can be simplified to

$$\binom{k+3-i}{k+2-i} - \binom{k+3-i}{k+1-i} = \binom{k+2-i}{k+2-i} + \binom{k+2-i}{k+1-i}$$
$$-\binom{k+2-i}{k+1-i} - \binom{k+2-i}{k-i} = 1 - \binom{k+2-i}{k-i}.$$

Thus, (2.12) becomes

$$\begin{aligned}\binom{k+2}{k} &= \sum_{i=1}^{k} F_{2i}\left[1 - \binom{k+2-i}{k-i}\right] + F_{2(k+1)} \\ &= \sum_{i=1}^{k+1} F_{2i} - \sum_{i=1}^{k} F_{2i}\binom{k+2-i}{k-i} \\ &= F_{2(k+1)+1} - \sum_{i=1}^{k} F_{2i}\binom{k+2-i}{k-i} - 1\end{aligned}$$

or, by Lemma 2.1

$$\binom{k+2}{k} - (k+1) = \frac{1}{2}k(k+1) = \sum_{i=1}^{k} F_{2i}\left[\binom{k+2-i}{k+1-i} - \binom{k+2-i}{k-i}\right] \tag{2.13}$$

which is true by Lemma 2.2. □

3 Lemmas and solution

In this section, (1.4) is recast as a non-homogeneous, doubly indexed, second order linear recurrence relation whose right hand side is a binomial coefficient denoted as $b_{k,n}$. Denote also $w_n := w_{k,n}$ in (1.4) and (1.5) to show dependence of w_n on k. Several properties of $b_{k,n},\ w_{k,n}$ are also proved. Let

$$b_{k,n} = \binom{n+k+2}{k},\ n, k \in \{0\} \cup \mathbb{N} \tag{3.1}$$

Define

$$w_{k,n} = \sum_{i=0}^{n} w_{k-1,i},\ n \geq 0, k \geq 1,\ w_{k,0} = 1, \forall k \geq 0, w_{0,1} = 2. \tag{3.2}$$

Consider

$$w_{0,n+2} - w_{0,n+1} - w_{0,n} = b_{0,n} = 1, n \geq 0. \tag{3.3}$$

Recurrence relations and identities satisfied by $b_{k,n}$ and $w_{k,n}$ are also given in the results that follow.

Lemma 3.1. *Let $b_{k,n}$ be defined as in (3.1). Then the following hold true.*

$$b_{k+1,n+1} = b_{k,n+1} + b_{k+1,n}, \tag{3.4}$$

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{k+n+2}{k} = \binom{n+k+2+1}{k+1} = b_{k+1,n}, \tag{3.5}$$

$$\binom{n+2}{0} + \binom{n+3}{1} + \ldots + \binom{n+k+2}{k} = \binom{n+k+2+1}{k} = b_{k,n+1}. \tag{3.6}$$

Recurrence relations and properties satisfied by $w_{k,n}$ are given in Lemmas 3.2–3.7.

Lemma 3.2. *Let $w_{k,n}$ be defined as in (3.2). Then*

$$w_{k,1} = k+2,\ k \geq 0. \tag{3.7}$$

Proof. This follows immediately from (3.1) and (3.2),

$$w_{k,1} = w_{k-1,0} + w_{k-1,1} = k+2.$$
□

Lemma 3.3.

$$w_{k,\, n+1} = w_{k,n} + w_{k-1,n+1},\ n \geq 0, k \geq 1. \tag{3.8}$$

Proof. From (3.2)

$$w_{k,n+1} = \sum_{i=0}^{n+1} w_{k-1,i} = \sum_{i=0}^{n} w_{k-1,i} + w_{k-1,n+1} = w_{k,n} + w_{k-1,n+1}.$$
□

Lemma 3.4.

$$w_{k,n} = w_{k,\, n-1} + w_{k-1,n},\ n,k \geq 1. \tag{3.9}$$

Proof. This follows from (3.2), and note that

$$w_{k+1,n-1} + w_{k,n} = \sum_{i=1}^{n-1} w_{k,i} + w_{k,n} = w_{k+1,n} + w_{k,n} - w_{k,n} = w_{k+1,n}.$$
□

Lemma 3.5. $b_{k,n}$ *is well-defined, that is*

$$w_{k+1,n+2} - w_{k+1,n+1} - w_{k+1,n} = b_{k+1,n}. \tag{3.10}$$

Proof. Consider the following

$$\sum_{i=0}^{n+2} w_{k,i} - \sum_{i=0}^{n+1} w_{k,i} - \sum_{i=0}^{n} w_{k,i} = w_{k,0} + w_{k,1} - w_{k,0} + \sum_{i=0}^{n} b_{k,i}. \tag{3.11}$$

From (3.1), (3.2) and (3.5), (3.11) becomes

$$w_{k+1,n+2} - w_{k+1,n+1} - w_{k+1,n} = b_{k+1,n}. \tag{3.12}$$

This completes the proof. □

A non-homogeneous recurrence relation for $w_{k,n}$ is given in Lemmas 3.6 and 3.7.

Lemma 3.6.

$$w_{k,n+3} - w_{k,n+2} - w_{k,n+1} = b_{k,n+1}. \tag{3.13}$$

Proof. Employing Lemma 3.5 successively yields

$$\begin{aligned}
w_{k,n+3} - w_{k,n+2} - w_{k,n+1} &= w_{k-1,n+3} - w_{k-1,n+2} - w_{k-1,n+1} + b_{k,n} \\
&= w_{k-2,n+3} - w_{k-2,n+2} - w_{k-2,n+1} + b_{k-1,n} + b_{k,n} \\
&= \ldots = \\
&= 1 + b_{1,n} + \ldots + b_{k-1,n} + b_{k,n} \\
&= b_{k,n+1}.
\end{aligned}$$

□

Lemma 3.7.

$$w_{k+1,n+3} - w_{k+1,n+2} - w_{k+1,n+1} = b_{k+1,n+1}. \tag{3.14}$$

Proof. Add (3.10), (3.13) and apply Lemma 3.4. □

Hence

$$w_{k,n+2} - w_{k,n+1} - w_{k,n} = b_{k,n}. \tag{3.15}$$

For fixed k, (3.15) has the characteristic polynomial given by $P(x) = x^2 - x - 1$ with zeros $\phi, -1/\phi$ where ϕ is the golden ratio. Thus

$$w_{k,n} = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n - p_{k,n} \tag{3.16}$$

for some polynomial $p_{k,n}$, the negative sign is taken for convenience. The expression for $p_{k,n}$ is given in [9]:

$$p_{k,n} = b_{k,n} + \sum_{i=1}^{k} F_{2i} b_{k-i,n} > 0, \tag{3.17}$$

a polynomial of degree k in n. The technique of undetermined coefficients shows the pattern of particular solutions $p_{k,n}$. A property of $p_{k,n}$ is shown in Section 4.

Take $n = 0$ in (3.16) and employ (3.17) to get

$$w_{k,0} = -p_{k,0} + \alpha_1 + \alpha_2 = -\left(b_{k,0} + \sum_{i=1}^{k} F_{2i} b_{k-i,0}\right) + \alpha_1 + \alpha_2 \tag{3.18}$$

Also $w_{k,0} = 1$. Employing Lemma 1.3, expanding and simplifying (3.18) gives

$$\alpha_1 + \alpha_2 = 1 + \binom{k+2}{2} + F_2 \binom{k+2-1}{k-1} + \cdots + F_{2k} \binom{2}{0} = F_{2k+3}. \tag{3.19}$$

Likewise

$$w_{k,1} = -\left(b_{k,1} + \sum_{i=1}^{k} F_{2i} b_{k-i,1}\right) + \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right). \tag{3.20}$$

Or, by using $w_{k,1} = k + 2$ and expanding (3.20) we obtain,

$$\frac{\alpha_1 + \alpha_2}{2} + \left(\alpha_1 - \alpha_2\right) \frac{\sqrt{5}}{2} = k + 2 + \binom{k+3}{k} + F_2 \binom{k+2}{k-1} + \cdots + F_{2k} \binom{3}{0}. \tag{3.21}$$

Employing Lemma 1.4, (2.9), and (3.19) and the identity $2F_n - F_{n-1} = L_{n-1}$, (3.21) simplifies to

$$\alpha_1 - \alpha_2 = -F_{2k+3} + 2F_{2(k+2)} = L_{2k+3}/\sqrt{5}. \tag{3.22}$$

Solving (3.19), (3.22) for α_1, α_2 gives,

$$\alpha_1 = \frac{1}{2}\left(F_{2k+3} + \frac{L_{2k+3}}{\sqrt{5}}\right) \tag{3.23}$$

$$\alpha_2 = \frac{1}{2}\left(F_{2k+3} - \frac{L_{2k+3}}{\sqrt{5}}\right) \tag{3.24}$$

Employing (3.23), (3.24) in (3.16) yields, for $n, k \geq 0$,

$$w_{k,n} = \frac{1}{2}\left(F_{2k+3} + \frac{L_{2k+3}}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{2}\left(F_{2k+3} - \frac{L_{2k+3}}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n - p_{k,n} \tag{3.25}$$

In Section 4 identities for the Fibonacci and Lucas numbers and a theorem are given.

4 Theorem and identities

In this section, we formulate and prove the main result that connects $w_{k,n}$ and the Fibonacci numbers. By definition of $L_n = \phi^n + (-1/\phi)^n$ and $F_n = 1\sqrt{5}(\phi^n - (-1/\phi)^n)$, (3.25) becomes

$$w_{k,n} = \frac{1}{2}\left(F_{2k+3}L_n + L_{2k+3}F_n\right) - p_{k,n} = F_{2k+n+3} - p_{k,n}. \tag{4.1}$$

The following lemma is required to establish the theorem.

Lemma 4.1.

$$p_{k+1,n+1} = p_{k+1,\,n} + p_{k,n+1},\ \ n, k \geq 0 \tag{4.2}$$

Proof. From (3.17)

$$p_{k+1,\,n} + p_{k,n+1} = b_{k+1,n} + b_{k,n+1} + \sum_{i=1}^{k+1} F_{2i} b_{k+1-i,n} + \sum_{i=1}^{k} F_{2i} b_{k-i,n+1}.$$

We have that $b_{k+1,n+1} = b_{k+1,n} + b_{k,n+1}$ and $b_{k+1-i,n} + b_{k-i,n+1} = b_{k+1-i,n+1}$. □

Theorem 4.1. *Let $w_{k,n}$ and $p_{k,n}$ be defined as in* (3.2) *and* (3.17) *respectively. Then*

$$w_{k,n} + p_{k,n} = F_{2k+n+3}. \tag{4.3}$$

Proof. The proof, which follows by induction on k, n, is included for completeness.
For $n = k = 0$

$$w_{0,0} + p_{0,0} = 2 = F_3.$$

Likewise,

$$w_{0,1} + p_{0,1} = 3 = F_4 \text{ and } w_{1,0} + p_{1,0} = 5 = F_5,$$

and

$$F_6 = F_{2+1+3} = 8 = w_{1,1} + p_{1,1} = w_{0,1} + p_{0,1} + w_{1,0} + p_{1,0} = (w_{0,1} + w_{1,0}) + (p_{1,0} + p_{0,1}).$$

The induction hypothesis is that

$$w_{k+1,n} + p_{k+1,n} = F_{2(k+1)+n+3} \text{ and } w_{k,n+1} + p_{k,n+1} = F_{2k+(n+1)+3}.$$

It is required to show that

$$w_{k+1,n+1} + p_{k+1,n+1} = F_{2(k+1)+(n+1)+3}.$$

Applying the induction hypothesis gives

$$w_{k+1,n} + p_{k+1,n} = F_{2(k+1)+n+3}, \tag{4.4}$$

$$w_{k,n+1} + p_{k,n+1} = F_{2k+n+3+1}. \tag{4.5}$$

Summing (4.4), (4.5) yields

$$w_{k+1,n} + p_{k+1,n} + w_{k,n+1} + p_{k,n+1} = F_{2k+n+3+3} = F_{2(k+1)+(n+1)+3}, \tag{4.6}$$

or, by Lemmas 3.4 and 4.1,

$$w_{k+1,n+1} + p_{k+1,n+1} = F_{2(k+1)+n+4} = F_{2(k+1)+(n+1)+3}. \tag{4.7}$$

This completes the proof. □

From [4], one obtains

$$w_{k,n} = \sum_{i=0}^{n} F_{i+1} \binom{n+k-i}{k}. \tag{4.8}$$

Combining (4.8) with (4.3), (3.17) gives (1.6).

The following lemmas are give representations for the Fibonacci and Lucas numbers.

Lemma 4.2.

$$F_{2k+n+3} = \sum_{i=0}^{n} F_{i+1} \binom{n+k-i}{k} + \sum_{i=1}^{k} F_{2i} \binom{n+k-i+2}{n+2} + \binom{n+k+2}{k}.$$

Adding (2.8), (2.9) obtains

Lemma 4.3.

$$L_{2k+3} = k + 2 + 2\binom{k+3}{k} + \sum_{i=1}^{k} F_{2i} \left[2\binom{k+3-i}{k-i} - \binom{k+2-i}{k+1-i} \right].$$

Replace $k+1 \leftarrow k$ in (2.1), add (2.1) to (2.6). Use identity $L_n = F_{n+1} + F_{n-1}$ to get

Lemma 4.4.

$$L_{2(k+2)} = k + 4 + \binom{k+2}{k} + \sum_{i=1}^{k} F_{2i}\left[\binom{k+3-i}{1} + \binom{k+2-i}{k-i}\right] + 2F_{2(k+1)}.$$

In Lemma 4.2, replacing: $n+1 \leftarrow n,\ n-1 \leftarrow n$ and adding the resultant equations, for $n \geq 1$ yields

Lemma 4.5.

$$\begin{aligned} L_{2k+n+3} &= \sum_{i=0}^{n-1} F_{i+1}\left[\binom{n+k-i+1}{k} + \binom{n+k-(i+1)}{k}\right] + \\ &\sum_{i=1}^{k}\left[F_{2i}\binom{n+k-i+3}{n+3} + \binom{n+k-i+1}{n+1}\right] + \binom{n+k+3}{k} + \\ &\binom{n+k+1}{k} + kF_{n+1} + F_{n+3}. \end{aligned} \tag{4.9}$$

If $n = 0$ in (4.9), then

Corollary 4.1.

$$L_{2k+3} = \sum_{i=1}^{k} F_{2i}\left[\binom{k+3-i}{3} + \binom{k+1-i}{1}\right] + \binom{k+3}{k} + 2k + 3.$$

Algebraic elimination between Corollary 4.1 and Lemma 4.3 gives

Corollary 4.2.

$$L_{2k+3} = \sum_{i=1}^{k} F_{2i}(3k + 4 - 3i) + 3k + 4.$$

Corollary 4.3.

$$F_{2k+n+3} = \sum_{i=1}^{k} F_{2i}[F_{n+2}k + F_{n+3} - iF_{n+2}] + F_{n+2}k + F_{n+3}.$$

Proof. Multiply F_n by terms in Corollary 4.2 and L_n by terms in Lemma 1.1, then sum and simplify. □

The next corollary follows from Corollary 4.3.

Corollary 4.4.

$$L_{2k+n+3} = \sum_{i=1}^{k} F_{2i}\left[L_{n+2}k + L_{n+3} - iL_{n+2}\right] + L_{n+2}k + L_{n+3}.$$

5 Conclusion

The present work contributes to the list of known identities for the Fibonacci and Lucas numbers with original results. An initial-value problem for a class of non-homogeneous linear recurrence relations is solved. Future work involves finding new Fibonacci and Lucas number identities and, a particular solution $p_{k,n}$ in (1.5) that corresponds with a larger or different set of values for b, c in (1.4).

Acknowledgements

The authors are grateful to the editor and the anonymous referees for their helpful suggestions and valuable comments.

References

[1] Azarian, M. K. (2012). Fibonacci identities as binomial sums. *International Journal of Contemporary Mathematical Sciences*, 7(38), 1871–1876.

[2] Azarian, M. K. (2012). Identities involving Lucas or Fibonacci and Lucas numbers as binomial sums. *International Journal of Contemporary Mathematical Sciences*, 7(45), 2221–2227.

[3] Grossman, G., Zeleke, A., Zhu, X., & Zdrahal, T. (2016). Recurrence relation with combinatorial identities. *International Journal of Pure and Applied Mathematics*, 107(4), 939–948.

[4] Gulec, H., & Taskara, N. (2009). On the Properties of Fibonacci numbers with binomial coefficients. *International Journal of Contemporary Mathematical Sciences*, 4(25), 1251–1256.

[5] Koshy, T. (2001). *Fibonacci Numbers and Lucas numbers with Applications*. John Wiley & Sons, Inc.

[6] Riordan. J. (1968). *Combinatorial Identities*. John Wiley & Sons. Inc.

[7] Singh, P. (1985). The so-called Fibonacci numbers in ancient and medieval India. *Historia Mathematica*, 12, 229–244.

[8] Wilf, H. S. (1994). *generatingfunctionology*, Academic Press, Inc.

[9] Zhang, Y., & Grossman, G. (2016). *Polynomial solution for a nonhomogeneous recurrence relation*. Pre-print, Oct. 22, 2016. Available online at: https://www.academia.edu/108731005/ .