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New approaches of (q, k)-Fibonacci–Pell sequences via linear difference equations. Applications

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Abstract: In this paper we establish some explicit formulas of (q, k)-Fibonacci–Pell sequences via linear difference equations of order 2 with variable coefficients, and explore some of their new properties. More precisely, our results are based on two approaches, namely, the determinantal and the nested sums approaches, and their closed relations. As applications, we investigate the q-analogue Cassini identities and examine a pair of Rogers–Ramanujan type identities.
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1 Introduction

The classical sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$ defined by the usual recurrence relation $F_{n+1} = F_n + F_{n-1}$, for $n \geq 1$, with initial conditions $F_0 = 0$, $F_1 = 1$, is one of the most known sequences of integers widely studied, as well as the sequence of Pell numbers $\{P_n\}_{n\geq 0}$ defined by the recurrence relation $P_{n+1} = 2P_n + P_{n-1}$, for $n \geq 1$, and the initial conditions $P_0 = 0$, $P_1 = 1$. What is intriguing about these two sequences are their various applications. In this context, it is natural that several generalizations appear in the literature (see for example, [8, 11, 17, 18, 22, 23]). Specifically, we are interested here in one type of extensions of the q-analogues generalizations (see more in [8, 21, 22]). Among these generalizations, it was considered in [21] the q-Pell sequence $\{P_n(q)\}_{n\geq 0}$, which concerns a class of polynomials in the variable q (*), satisfying the linear difference equation of the second order with variable coefficients, given by

$$P_{n+1}(q) = (1 - q^2 + q^{2n-1})P_n(q) + q^2 P_{n-1}(q), \tag{1}$$

where $P_0(q) = 1$ and $P_1(q) = 1 + q^3$. Note that, for $q = 1$, the family of 1-Pell sequences $\{P_n(1)\}_{n\geq 0}$ described by (1), is nothing else that but the sequence of Fibonacci numbers, namely, $P_n(1) = F_{n+1}$. Another generalization was introduced by Santos and Sills in [22], where the q-Pell sequence $\{P_n(q)\}_{n\geq 0}$ satisfies by the following recursive relation,

$$P_{n+1}(q) = (1 + q^{n+1})P_n(q) + q^n P_{n-1}(q), \tag{2}$$

where $P_0(q) = 1$ and $P_1(q) = 1 + q$. The sequence (2) was a motivation for studying a combinatorial interpretation in terms of weighted tilings of q-Pell sequence, as it was shown in [8]. In addiction, the sequences (1) and (2) own an interesting combinatorial interpretation for some series-product identities, listed by Slater in [24]. These identities are of the so called Rogers–Ramanujan type.

In this study, we are interested in the following generalization of Expressions (1) and (2) with fixed parameters (q, k),

$$F^{(k)}_{n+1}(q) = f^{(k)}_{1,n}(q)F^{(k)}_n(q) + f^{(k)}_{2,n}(q)F^{(k)}_{n-1}(q), \tag{3}$$

where $k \in \mathbb{N}$, $F^{(k)}_0(q)$, $F^{(k)}_1(q)$ are the given initial conditions and $f^{(k)}_{1,n}(q)$, $f^{(k)}_{2,n}(q)$ are variable coefficients in n. Namely, for the purpose of conciseness related to the results in the literature and to clarify our two approaches, we would like to specify that the coefficients $f^{(k)}_{1,n}(q)$ and $f^{(k)}_{2,n}(q)$ are functions that depend on 3 parameters k, q and n. Since the parameters k and q are fixed, then the coefficients $f^{(k)}_{1,n}(q)$ and $f^{(k)}_{2,n}(q)$ are variable in n.

In the sequel, we focus our study on the following two cases, the (q, k)-Fibonacci sequences and the (q, k)-Pell sequences, defined by

$$F^{(k)}_{n+1}(q) = f^{(k)}_{1,n}(q)F^{(k)}_n(q) + f^{(k)}_{2,n}(q)F^{(k)}_{n-1}(q), \tag{4}$$

where $f^{(k)}_{1,n}(q) = 1-q^2+q^{2n+2k-1}$, $f^{(k)}_{2,n}(q) = q^2$, the initial conditions are $F^{(k)}_0(q) = 1$, $F^{(k)}_1(q) = 1 + q^{2k+1}$, and

$$P^{(k)}_{n+1}(q) = f^{(k)}_{1,n}(q)P^{(k)}_n(q) + f^{(k)}_{2,n}P^{(k)}_{n-1}(q), \tag{5}$$

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* In combinatorial context the variable q is an enumeration parameter, however if the purpose is to study convergence of q-analogous series, it is common to use $0 < |q| < 1$.

where $f_{1,n}^{(k)}(q) = 1 + q^{kn+1}$, $f_{2,n}^{(k)}(q) = q^{kn}$ and the initial conditions are $P_0^{(k)}(q) = 1$, $P_1^{(k)}(q) = 1 + q^k$.

Expressions of type (3) represent a linear difference equation of the second order with variable coefficients. Several methods and techniques have been developed to solve linear difference equations with variable coefficients (see, for instance, [1,7,12,14–16,20], and references therein). This topic continues to attract much attention due to their many applications in mathematics and applied sciences (see for instance [1, 2, 6, 7]). In [20], Popenda gave explicit formulas for the general solutions of homogeneous and non-homogeneous second-order linear difference equations, with arbitrarily varying coefficients, using a direct computation. In [15], Mallik gave a complete closed form solution of a second order linear homogeneous difference equation with variable coefficients, solely in terms of the coefficients. He extends his results on the explicit solution of a second-order linear homogeneous to a linear difference equation of unbounded order with variable coefficients in [15], and to the nonhomogeneous difference equation with variable coefficients in [14]. Despite these studies, recently in [1,2] the solutions of the second-order linear homogeneous difference equations with variable coefficients are exhibited under a representation based on nested sum and determinantal approaches.

Our main goal is to study the (q, k)-Fibonacci and (q, k)-Pell sequence, $\{F_n^{(k)}(q)\}_{n\geq 0}$ and $\{P_n^{(k)}(q)\}_{n\geq 0}$, by considering Equations (4) and (5), as a linear difference equation of the second order with variable coefficients. Therefore, some explicit formulas of these sequences of polynomial (in q) $\{F_n^{(k)}(q)\}_{n\geq 0}$ and $\{P_n^{(k)}(q)\}_{n\geq 0}$ are established, using the first approach based on the determinantal form of the tridiagonal matrices, related to the variable coefficients $f_{1,n}^{(k)}(q)$ and $f_{2,n}^{(k)}(q)$. The second approach consists of expressing the determinantal form of $\{F_n^{(k)}(q)\}_{n\geq 0}$ and $\{P_n^{(k)}(q)\}_{n\geq 0}$ in terms of the nested sums and their related properties. As a consequence, other formulas of the usual Fibonacci and Pell numbers are deduced. In addition, properties of some known identities, with closed connections to the q-Pell sequence, are provided. Moreover, the generalized Cassini identities for (q, k)-Fibonacci–Pell sequences are established and a pair of Rogers–Ramanujan type identities are examined.

The content of this paper is organized as follows. In Sections 2 and 3, we establish new explicit formulas of the (q, k)-Fibonacci sequences and the (q, k)-Pell sequences, respectively, using the determinantal tridiagonal approach and the generalized combinatorial nested sums approach. Section 4 concerns the Cassini type identity for the (q, k)-Pell and the (q, k)-Fibonacci sequences, in addition other identities are provided. In Section 5 we study the analytic and combinatorial aspect of two Rogers–Ramanujan identities and the related q-analogous of Fibonacci and Pell sequences. Finally, some concluding remarks and perspectives are furnished in Section 6.

2 Some explicit formulas of the (q, k)-Fibonacci sequence

2.1 Solution of (4) by a determinantal approach

Let us consider the linear difference equation of the second order with variable coefficients (4) defining the (q, k)-Fibonacci sequence, namely,

$$F^{(k)}_{n+1}(q) = f^{(k)}_{1,n}(q)F^{(k)}_n(q) + f^{(k)}_{2,n}(q)F^{(k)}_{n-1}(q),$$

where $F^{(k)}_0(q) = 1$, $F^{(k)}_1(q) = 1 + q^{2k+1}$ and $f^{(k)}_{1,n}(q) = 1 - q^2 + q^{2n+2k-1}$, $f^{(k)}_{2,n}(q) = q^2$, of the linear difference equation (4) is given by

$$\begin{bmatrix} F^{(k)}_n(q) \\ F^{(k)}_{n+1}(q) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ q^2 & 1 - q^2 + q^{2n+2k-1} \end{bmatrix} \begin{bmatrix} F^{(k)}_{n-1}(q) \\ F^{(k)}_n(q) \end{bmatrix}, \tag{6}$$

whose vector of initial conditions is $\begin{bmatrix} 1 \\ 1+q \end{bmatrix}$. Let $\mathbb{M} = \{L^{(k)}_q(n)\}_{n\geq 0}$ be the family of matrices defined by

$$L^{(k)}_q(n) = \begin{bmatrix} 0 & 1 \\ q^2 & 1 - q^2 + q^{2n+2k-1} \end{bmatrix}. \tag{7}$$

Since for a fixed q, k and every positive integer n we have $\det(L^{(k)}_q(n)) = -q^2 \neq 0$, this implies that the matrices $L^{(k)}_q(n)$ are invertible matrices of order 2. Then, we can apply the process of the companion factorization to the matrix $L^{(k)}_q(n) = (l^{k,n}_{i,j})_{1\leq i,j\leq 2}$ improved in [1, 3]. This process consists of exhibiting the compact formulas for the entries of the transition matrix related to the matrices $L^{(k)}_q(n) = (l^{k,n}_{i,j})_{1\leq i,j\leq 2}$, described as follows. First, we denote $l^{k,n}_{1,1} = a_{k,n} = 0, l^{k,n}_{1,2} = b_{k,n} = 1, l^{k,n}_{2,1} = c_{k,n} = q^2$ and $l^{k,n}_{2,2} = d_{k,n} = 1 - q^2 + q^{2n+2k-1}$. Observe that for each matrix $L^{(k)}_q(n)$ we have $a_{k,n} = 0$. Then, the coefficients of the associated difference equation are given by $p^{(k)}_1(n) = d_{k,n}, p^{(k)}_2(n) = -\frac{c_{k,n}}{c_{k,n-1}}, n \neq 0, p^{(k)}_2(0) = c_{k,0}$, namely,

$$\begin{bmatrix} F^{(k)}_{n+1}(q) \\ F^{(k)}_{n+2}(q) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ p^{(k)}_2(n) & p^{(k)}_1(n) \end{bmatrix} \begin{bmatrix} F^{(k)}_n(q) \\ F^{(k)}_{n+1}(q) \end{bmatrix}. \tag{8}$$

In this context, we consider the linear matrix equation $x^{(k)}_q(n + 1) = L^{(k)}_q(n)x^{(k)}_q(n)$, where $x^{(k)}_q(n) = \begin{bmatrix} F^{(k)}_n(q) \\ F^{(k)}_{n+1}(q) \end{bmatrix}$. Thus, the associated transition matrix is given in the form $T^{(k)}_q(n) = L^{(k)}_q(n-1)\cdots L^{(k)}_q(1)L^{(k)}_q(0)$, and in the sequel, it will be denoted as follows

$$T^{(k)}_q(n) = \prod_{h=0}^{*,n-1} L^{(k)}_q(h).$$

Therefore, at step n we have

$$\begin{bmatrix} F^{(k)}_n(q) \\ F^{(k)}_{n+1}(q) \end{bmatrix} = \prod_{h=0}^{*,n-1} L^{(k)}_q(h) \begin{bmatrix} 1 \\ 1 + q^{2k+1} \end{bmatrix} = T^{(k)}_q(n) \begin{bmatrix} 1 \\ 1 + q^{2k+1} \end{bmatrix}, \tag{9}$$

where $T^{(k)}_q(n) = \prod_{j=0}^{*,n-1} \begin{bmatrix} 0 & 1 \\ q^2 & 1 - q^2 + q^{2j+2k-1} \end{bmatrix}$ is a square matrix of order 2, which is not explicitly determined. In order to establish an explicit form of the matrix $T^{(k)}_q(n)$, we apply [1, Lemma 2.1], which allows us to derive that the equation (9) is equivalent to the matrix equation

$$\begin{bmatrix} \dfrac{-F^{(k)}_n(q)}{c_{k,n-1}} \\ F^{(k)}_{n+1}(q) \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{c_{k,n-1}} & 0 \\ 0 & 1 \end{bmatrix} \prod_{i=1}^{*,n} \begin{bmatrix} 0 & 1 \\ -\frac{c_{k,i}}{c_{k,i-1}} & d_{k,i} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ c_{k,0} & d_{k,0} \end{bmatrix} \begin{bmatrix} 1 \\ 1 + q^{2k+1} \end{bmatrix}, \tag{10}$$

where $c_{k,0} = l^0_{2,1} = q^2, -\frac{c_{k,i}}{c_{k,i-1}} = -1$, if $i \neq 0$, and $d_{k,i} = 1 - q^2 + q^{2i+2k-1}$.

Solving Equation (4) is equivalent to solving the matrix equations (8)–(10). This process requires us to determine the explicit formula of the transition matrix $T_q^{(k)}(n)$. Now, in the aim to provide an explicit formula of the right side of Expression (10), we consider the determinantal form of the canonical solutions of (4). More precisely, following [12], the canonical solutions $\phi_n^{(0)}(q,k)$ and $\phi_n^{(1)}(q,k)$ of Equation (4), with initial conditions

$$\phi_0^{(0)}(q,k)=0,\ \ \phi_1^{(0)}(q,k)=1 \text{ and } \ \phi_0^{(1)}(q,k)=1,\ \ \phi_1^{(1)}(q,k)=0,$$

are given under the following determinant of a special tridiagonal matrix

$$\phi_n^{(0)}(q,k)=\det\begin{pmatrix} p_1^{(k)}(0) & p_2^{(k)}(1) & 0 & \cdots & 0 \\ -1 & p_1^{(k)}(1) & p_2^{(k)}(2) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & p_1^{(k)}(n-3) & p_2^{(k)}(n-2) \\ 0 & \cdots & \cdots & -1 & p_1^{(k)}(n-2) \end{pmatrix}, \qquad (11)$$

$$\phi_n^{(1)}(q,k)=\det\begin{pmatrix} p_2^{(k)}(0) & 0 & 0 & \cdots & 0 \\ -1 & p_1^{(k)}(1) & p_2^{(k)}(2) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & p_1^{(k)}(n-3) & p_2^{(k)}(n-2) \\ 0 & \cdots & \cdots & -1 & p_1^{(k)}(n-2) \end{pmatrix}. \qquad (12)$$

Indeed, let us apply the result of [1, Theorem 3.2] to the solutions of Equation (4), with variables coefficients defined by $p_2^{(k)}(0)=q^2$, and $p_2(n)=-1$, $p_1^{(k)}(n)=1-q^2+q^{2n+2k-1}$, if $n\neq 0$. Then, by direct application of [1, Theorem 3.2] (with $a_{k,n}=0$), we get the following matrix representation of Equation (4).

Theorem 2.1. *Under the preceding data, the solution of the matrix Equation* (10)*, is given by*

$$\begin{bmatrix} -\dfrac{F_n^{(k)}(q)}{c_{n-1}} \\ F_{n+1}^{(k)}(q) \end{bmatrix} = \begin{bmatrix} -\dfrac{F_0^{(k)}(q,k)}{c_{n-1}}\phi_n^{(1)}(q,k)-\dfrac{F_1^{(k)}(q,k)}{c_{n-1}}\phi_n^{(0)}(q,k) \\ F_0^{(k)}(q)\phi_{n+1}^{(1)}(q,k)+F_1^{(k)}(q)\phi_{n+1}^{(0)}(q,k) \end{bmatrix}, \qquad (13)$$

where $\phi_n^{(0)}(q,k)$ and $\phi_n^{(1)}(q,k)$ are the canonical solutions of (4) *formulated by Expressions* (11) *and* (12)*.*

As a consequence of Theorem 2.1, we can formulate the following corollary.

Corollary 2.1. (Determinantal form of $F_n^{(k)}(q)$) *The determinantal explicit expression of the $(n+1)$-th term of the (q,k)-Fibonacci sequence defined as in Equation* (4)*, is given by*

$$F_{n+1}^{(k)}(q)=F_0^{(k)}(q)\phi_{n+1}^{(1)}(q,k)+F_1^{(k)}(q)\phi_{n+1}^{(0)}(q,k),$$

where $F_0^{(k)}(q)=1$ and $F_1^{(k)}(q)=1+q^{2k+1}$ are the initial conditions of (4) *and $\phi_n^{(0)}(q,k)$, $\phi_n^{(1)}(q,k)$ are the canonical solutions of* (4) *given by Expressions* (11) *and* (12)*.*

As an application we are considering two special cases. For $k = 0$ we get

$$F_{n+1}^{(0)}(q) = (1 - q^2 + q^{2n-1})F_n^{(0)}(q) + q^2 F_{n-1}^{(0)}(q), \tag{14}$$

with initial conditions $F_0^{(0)}(q) = 1$, $F_1^{(0)}(q) = 1 + q$. For $k = 1$ we derive

$$F_{n+1}^{(1)}(q) = (1 - q^2 + q^{2n+1})F_n^{(1)}(q) + q^2 F_{n-1}^{(1)}(q), \tag{15}$$

with initial conditions $F_0^{(1)}(q) = 1$, $F_1^{(1)}(q) = 1 + q^3$.

Applying Corollary 2.1 for $k = 0$, we obtain the determinantal explicit formula for Expression (14) given by

$$F_{n+1}^{(0)}(q) = F_0^{(0)}(q)\phi_{n+1}^{(1)}(q,0) + F_1^{(k)}(q)\phi_{n+1}^{(0)}(q,0),$$

where $F_0^{(0)}(q) = 1$, $F_1^{(0)}(q) = 1 + q$ are the initial conditions of (4) and $\phi_n^{(0)}(q,0)$, $\phi_n^{(1)}(q,0)$ are the canonical solutions (11) and (12) of Equation (4). Similarly, replacing $k = 1$ in Corollary 2.1 we get the determinantal explicit formula for Expression (15) given by

$$F_{n+1}^{(1)}(q) = F_0^{(1)}(q)\phi_{n+1}^{(1)}(q,1) + F_1^{(k)}(q)\phi_{n+1}^{(0)}(q,1),$$

where $F_0^{(0)}(q) = 1$ and $F_1^{(0)}(q) = 1 + q^3$ are the initial conditions of (4) and $\phi_n^{(0)}(q,1)$, $\phi_n^{(1)}(q,1)$ are the canonical solutions of (4) expressed by (11) and (12).

The determinantal expressions of the canonical solutions (11) and (12) of Equations (4), can be expressed with the aid of the nested sums. Therefore, we can obtain another explicit expression for the n-th term of the (q,k)-Fibonacci sequence, as given in Equation (4).

Theorem 2.1 and Corollary 2.1 allow us to deduce an explicit formula of the entries of the transition matrix in terms of the determinantal approach. Using these results, in the next subsection we will discuss the solution in terms of the combinatorial nested sum.

2.2 Generalized combinatorial approach for solving (4)

Recently, for linear difference equations of the second order with variable coefficients, expressions of type (11) and (12) of the determinantal solutions have been also described using nested sums in [1, 2]. To achieve this connection, with the determinantal solutions of Equation (4), let us consider the following coefficients

$$\alpha(j) \equiv \alpha(k,j) = \frac{p_2^{(k)}(j)}{p_1^{(k)}(j-1)p_1^{(k)}(j)} = \frac{-1}{(1 + q^2 + q^{2j+2k-3})(1 + q^2 + q^{2j+2k-1})}.$$

Then, Expression (11) of the canonical solution $\phi_n^{(0)}(q,k)$, takes the form,

$$\phi_n^{(0)}(q,k) = \left(\prod_{i=0}^{n-2}(1 + q^2 + q^{2k+2i-1})\right) \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \Delta_m^{(0)}(n), \tag{16}$$

where $\Delta_m^{(0)}(n)$ is the nested sum in the form

$$\Delta_m^{(0)}(n) = \sum_{j_1=2(m-1)+1}^{n-2} \alpha(j_1) \left(\sum_{j_2=2(m-2)+1}^{j_1-2} \alpha(j_2) \left(\dots \left(\sum_{j_m=1}^{j_{m-1}-2} \alpha(j_m) \right) \dots \right) \right). \tag{17}$$

In the same way, with the aid of the nested sums, the expansion of the canonical solution (12) of Equation (2), is given by

$$\phi_n^{(1)}(q,k) = q^2\left(\prod_{i=1}^{n-2}(1+q^2+q^{2k+2i-1})\right)\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n), \tag{18}$$

where the nested sum $\Delta_m^{(1)}(n)$ is expressed by

$$\Delta_m^{(1)}(n) = \sum_{j_1=2(m-1)+2}^{n-2}\alpha(j_1)\left(\sum_{j_2=2(m-2)+2}^{j_1-2}\alpha(j_2)\left(\dots\left(\sum_{j_m=2}^{j_{m-1}-2}\alpha(j_m)\right)\dots\right)\right). \tag{19}$$

Therefore, Expression (13) of Theorem 2.1 and Expressions (16), (18), allow us to obtain the following result.

Theorem 2.2. (Generalized combinatorial form of $F_n^{(k)}(q)$) *The* n*-th element of* (q,k)*-Fibonacci sequences in terms of fundamental matrix of the difference equation* (4) *is given by*

$$\begin{aligned} F_n^{(k)}(q) &= q^2\left(\prod_{i=1}^{n-2}(1+q^2+q^{2k+2i-1})\right)\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n) \\ &\quad + (1+q^{2k+1})\left(\prod_{i=0}^{n-2}(1+q^2+q^{2k+2i-1})\right)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n), \end{aligned} \tag{20}$$

where $\Delta_m^{(0)}(n)$ *and* $\Delta_m^{(1)}(n)$ *are the nested sums given as in Expressions* (17) *and* (19)*, respectively.*

Theorem 2.2 is a generalized combinatorial formula for $F_n^{(k)}(q)$. For $k=q=1$, Expressions (14) and (15) allow us to show that $F_n^{(1)}(1)$ takes the form

$$F_n^{(1)}(1) = F_{n+1}.$$

Hence, as a consequence of Expression (20), a new formula for the Fibonacci numbers can be provided in terms of the nested sums (17) and (19), as follows,

$$F_{n+1} = 3^{(n-2)}\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n) + 2\cdot 3^{(n-1)}\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n).$$

Since $q=1$ we obtain $\alpha(j) = \frac{p_2^{(1)}(j)}{p_1^{(1)}(j-1)p_1^{(1)}(j)} = \frac{-1}{9}$, for all $j\geq 1$. Then, for $k=0$ or $k=1$ and $q=1$, the fundamental solutions are given by

$$\phi_n^{(0)}(1,0) = \phi_n^{(0)}(1,1) = 3^{n-1}\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n) = \sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^m\binom{n-1-m}{m}3^{n-1-2m},$$

$$\phi_n^{(1)}(1,0) = \phi_n^{(1)}(1,1) = 3^{n-2}\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n) = \sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^m\binom{n-2-m}{m}3^{n-2-2m}.$$

Moreover, a direct computation permits us to derive the following corollary.

Corollary 2.2. *The formula of the usual Fibonacci numbers F_n, in terms of nested sums and combinatorial identity, is as follows,*

$$F_{n+1} = 3^{(n-2)} \left(\sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} \Delta_m^{(1)}(n) + 6 \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \Delta_m^{(0)}(n) \right), \tag{21}$$

where

$$\begin{aligned} \Delta_m^{(0)}(n) &= \left(\frac{-1}{9}\right)^m \sum_{j_1=2(m-1)+1}^{n-2} \left(\sum_{j_2=2(m-2)+1}^{j_1-2} \left(\dots \left(\sum_{j_m=1}^{j_{m-1}-2} 1 \right) \dots \right) \right) \\ &= \frac{(-1)^m}{9^m} \binom{n-1-m}{m}, \end{aligned}$$

and

$$\begin{aligned} \Delta_m^{(1)}(n) &= \left(\frac{-1}{9}\right)^m \sum_{j_1=2(m-2)+2}^{n-2} \left(\sum_{j_2=2(m-2)+1}^{j_1-2} \left(\dots \left(\sum_{j_m=1}^{j_{m-1}-2} 1 \right) \dots \right) \right) \\ &= \frac{(-1)^m}{9^m} \binom{n-2-m}{m}. \end{aligned}$$

The result of Corollary 2.2 allows us to obtain a new expression of the Fibonacci numbers in terms of nested sums. It seems to us that the formula (21) is not known in the literature.

3 Explicit formulas of the (q,k)-Pell sequence

3.1 Solution of the generalized Equation (5) by a determinantal approach

Let us first recall the Equation (5), which defines the sequence (q,k)-Pell, whose elements are denoted $P_n^{(k)}(q)$, namely,

$$P_{n+1}^{(k)}(q) = f_{1,n}^{(k)}(q) P_n^{(k)}(q) + f_{2,n}^{(k)} P_{n-1}^{(k)}(q),$$

where $f_{1,n}^{(k)}(q) = 1 + q^{kn+1}$ and $f_{2,n}^{(k)}(q) = q^{kn}$, and the initial conditions are $P_0^{(k)}(q) = 1$ and $P_1^{(k)}(q) = 1 + q^k$. Replacing $k = 1$ in Expression (5), we recovered Expression (2), namely,

$$P_{n+1}^{(1)}(q) = (1 + q^{n+1}) P_n^{(1)}(q) + q^n P_{n-1}^{(1)}(q),$$

where $P_0^{(1)}(q) = 1$ and $P_1^{(1)}(q) = 1 + q$ are the initial conditions. The matrix formulation of Equation (5), defining the $P_n^{(k)}(q)$, is presented as follows

$$\begin{bmatrix} P_n^{(k)}(q) \\ P_{n+1}^{(1)}(q) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ q^{kn} & 1 + q^{kn+1} \end{bmatrix} \begin{bmatrix} P_{n-1}^{(k)}(q) \\ P_n^{(k)}(q) \end{bmatrix}, \tag{22}$$

where $\begin{bmatrix} 1 \\ 1+q^k \end{bmatrix}$ is the vector of initial conditions.

For every fixed positive numbers $n, k \in \mathbb{N}$, let $\mathbb{M} = \{L_q^{(k)}(n)\}_{n\geq 0}$ be the family of matrices defined by $L_q^{(k)}(n) = (l_{i,j}^{k,n})_{1\leq i,j\leq 2} = \begin{bmatrix} 0 & 1 \\ q^{kn} & 1+q^{kn+1} \end{bmatrix}$. For studying the matrix Equation (22) we utilize the same method as described in Section 2, by denoting $l_{1,1}^{k,n} = a_{k,n} = 0, l_{1,2}^{k,n} = b_{k,n} = 1$, $l_{2,1}^{k,n} = c_{k,n} = q^{kn}$ and $l_{2,2}^{k,n} = d_{k,n} = 1 + q^{kn+1}$. Then, the coefficients of the associated difference equation are given by $p_1^{(k)}(n) = d_{k,n}, p_2^{(k)}(n) = -\frac{c_{k,n}}{c_{k,n-1}}, n \neq 0, p_2^{(k)}(0) = c_{k,0}$, namely,

$$\begin{bmatrix} P_q^{(k)}(n+1) \\ P_q^{(k)}(n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ p_2^{(k)}(n) & p_1^{(k)}(n) \end{bmatrix} \begin{bmatrix} P_q^{(k)}(n) \\ P_q^{(k)}(n+1) \end{bmatrix}.$$

Therefore, the transition matrix is defined by $T_q^{(k)}(n) = L_q^{(k)}(n-1)\cdots L_q^{(k)}(1)L_q^{(k)}(0)$ and denoted as follows $T_q^{(k)}(n) = \prod_{j=0}^{*,n-1} L_q^{(k)}(j)$. Thus, at step n we have,

$$\begin{bmatrix} P_n^{(k)}(q) \\ P_{n+1}^{(k)}(q) \end{bmatrix} = \prod_{j=0}^{*,n-1} L_q^{(k)}(j) \begin{bmatrix} 1 \\ 1+q^k \end{bmatrix} = T_q^{(k)}(n) \begin{bmatrix} 1 \\ 1+q^k \end{bmatrix}. \tag{23}$$

Applying Lemma 2.1 of [1], we obtain

$$\begin{bmatrix} \dfrac{-P_n^{(k)}(q)}{q^{k(n-1)}} \\ P_{n+1}^{(k)}(q) \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{c_{k,n-1}} & 0 \\ 0 & 1 \end{bmatrix} \prod_{i=1}^{*,n} \begin{bmatrix} 0 & 1 \\ -\frac{c_{k,i}}{c_{k,i-1}} & d_{k,i} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ c_{k,0} & d_{k,0} \end{bmatrix} \begin{bmatrix} 1 \\ 1+q^k \end{bmatrix}, \tag{24}$$

where $c_{k,0} = 1, -\frac{c_{k,i}}{c_{k,i-1}} = -q^k$, if $i \geq 1$, and $d_i = 1 + q^{ki+1}$.

Similarly, solving Equation (5) is equivalent to solving the matrix equations (22)–(24). In the aim to give an explicit formula of the right side of Expression (24), we consider the determinantal form of canonical solutions of (5). More precisely, following [12], the canonical solutions $\phi_n^{(0)}(q,k)$ and $\phi_n^{(1)}(q,k)$ of Equation (5), with initial conditions $\phi_0^{(0)}(q,k) = 0$, $\phi_1^{(0)}(q,k) = 1$ and $\phi_0^{(1)}(q,k) = 1$, $\phi_1^{(1)}(q,k) = 0$, are given under the following determinant of a special tridiagonal matrix, namely,

$$\phi_n^{(0)}(q,k) = \det \begin{pmatrix} p_1^{(k)}(0) & p_2^{(k)}(1) & 0 & \cdots & 0 \\ -1 & p_1^{(k)}(1) & p_2^{(k)}(2) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & p_1^{(k)}(n-3) & p_2^{(k)}(n-2) \\ 0 & \cdots & \cdots & -1 & p_1^{(k)}(n-2) \end{pmatrix}, \tag{25}$$

$$\phi_n^{(1)}(q,k) = \det \begin{pmatrix} p_2^{(k)}(0) & 0 & 0 & \cdots & 0 \\ -1 & p_1^{(k)}(1) & p_2^{(k)}(2) & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & p_1^{(k)}(n-3) & p_2^{(k)}(n-2) \\ 0 & \cdots & \cdots & -1 & p_1^{(k)}(n-2) \end{pmatrix}, \tag{26}$$

with $p_1^{(k)}(j) = 1 + q^{kj+1}, p_2^{(k)}(j) = -q^k, j > 0, p_2^{(k)}(0) = 1$. Indeed, let us apply [1, Theorem 3.2] to the solutions of Equation (5). That is, by direct application of [1, Theorem 3.2], we get the solution of the matrix representation of Equation (5).

Theorem 3.1. *Under the preceding data, the solution of the matrix Equation* (24)*, is given by*

$$\begin{bmatrix} -\dfrac{P_n^{(k)}(q)}{c_{n-1}} \\ P_{n+1}^{(k)}(q) \end{bmatrix} = \begin{bmatrix} -\dfrac{P_0^{(k)}}{c_{n-1}}\phi_n^{(1)}(q,k) - \dfrac{P_1^{(k)}}{c_{n-1}}\phi_n^{(0)}(q,k) \\ P_0^{(k)}\phi_{n+1}^{(1)}(q,k) + P_1^{(k)}\phi_{n+1}^{(0)}(q,k) \end{bmatrix}, \tag{27}$$

where $P_0^{(k)} = 1$, $P_1^{(k)} = 1 + q^k$ *are the initial conditions of* (5) *and* $\phi_n^{(0)}(q,k)$, $\phi_n^{(1)}(q,k)$ *are the canonical solutions of* (5) *expressed under the formulas* (25) *and* (26)*.*

As a consequence of Theorem 3.1, we can formulate the following proposition.

Proposition 3.1. (The determinantal form of the $P_n^{(k)}(q)$) *The determinantal explicit expression of the* $(n+1)$*-th element of* (q,k)*-Pell sequence, defined as in Equation* (5)*, is given by*

$$P_{n+1}^{(k)}(q) = P_0^{(k)}\phi_{n+1}^{(1)}(q,k) + P_1^{(k)}\phi_{n+1}^{(0)}(q,k),$$

where $P_0^{(k)} = 1$ *and* $P_1^{(k)} = 1 + q^k$ *are the initial conditions of* (5) *and* $\phi_n^{(0)}(q,k)$ *and* $\phi_n^{(1)}(q,k)$ *are the canonical solutions of* (5) *expressed under the formulas* (25) *and* (26)*.*

Now, let us apply result Theorem 3.1 to the solutions of Equation (2). Indeed, for $k = 1$, we get the following corollary.

Corollary 3.1. *The determinantal explicit expression of the* $(n+1)$*-th element of* q*-Pell sequence, defined as in Equation* (2)*, is given by*

$$P_{n+1}^{(1)}(q) = P_0^{(1)}\phi_{n+1}^{(1)}(q,1) + P_1^{(1)}\phi_{n+1}^{(0)}(q,1),$$

where $P_0 = 1$, $P_1 = 1 + q$ *are the initial conditions of* (2) *and* $\phi_n^{(0)}(q,1)$, $\phi_n^{(1)}(q,1)$ *are the canonical solutions of* (2) *expressed by* (25) *and* (26) *for* $k = 1$.

The determinantal expressions of the canonical solutions (25) and (26) of Equation (5) can be expressed through nested sums. That is, via Theorem 3.1, we can deduce an explicit formula of the entries of the transition matrix in terms of combinatorial nested sum approach. In addition, we can derive another explicit formula of the n-th element $P_n^{(k)}(q)$ of the (q,k)-Pell sequence. This will be discussed in the next subsection.

3.2 Generalized combinatorial approach for solving (5)

It was established in [1,2] that the expressions of the determinantal solutions of linear difference equations of the second order with variable coefficients can be described using nested sums. To realize this connection, with the determinantal solutions of Equation (5), let us consider the following coefficients

$$\alpha(j) \equiv \alpha(k,j) = \frac{p_2^{(k)}(j)}{p_1^{(k)}(j-1)p_1^{(k)}(j)} = \frac{-q^k}{(1+q^{kj+1})(1+q^{k(j-1)+1})}.$$

Then, the canonical solution (25) of Equation (5) is given by

$$\phi_n^{(0)}(q,k) = \left(\prod_{i=0}^{n-2}(1+q^{ki+1})\right)\sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor}\Delta_m^{(0)}(n), \tag{28}$$

where $\Delta_m^{(0)}(n)$, for every fixed m and n, is the nested sum

$$\Delta_m^{(0)}(n) = \sum_{j_1=2(m-1)+1}^{n-2} \alpha(j_1) \left(\sum_{j_2=2(m-2)+1}^{j_1-2} \alpha(j_2) \left(\dots \left(\sum_{j_m=1}^{j_{m-1}-2} \alpha(j_m) \right) \dots \right) \right). \tag{29}$$

Similarly, with the aid of the nested sums, the expansion of the canonical solution (26) of the Equation (5), is given by

$$\phi_n^{(1)}(q,k) = \left(\prod_{i=1}^{n-2} (1+q^{ki+1}) \right) \sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} \Delta_m^{(1)}(n), \tag{30}$$

where $\Delta_m^{(1)}(n)$, with fixed integers m and n, is the nested sum

$$\Delta_m^{(1)}(n) = \sum_{j_1=2(m-1)+2}^{n-2} \alpha(j_1) \left(\sum_{j_2=2(m-2)+2}^{j_1-2} \alpha(j_2) \left(\dots \left(\sum_{j_m=2}^{j_{m-1}-2} \alpha(j_m) \right) \dots \right) \right). \tag{31}$$

Therefore, Expression (27) of Theorem 3.1 and Expressions (28)-(31), allow us to obtain the follow result.

Theorem 3.2. (The generalized combinatorial form of the $P_n^{(k)}(q)$) *The element $P_n^{(k)}(q)$ of the (q,k)-Pell sequence in terms of fundamental matrix of the difference equation* (5) *is given in terms of nested sums as*

$$P_n^{(k)}(q) = \left(\prod_{i=1}^{n-2} (1+q^{ki+1}) \right) \sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} \Delta_m^{(1)}(n) + (1+q) \left(\prod_{i=0}^{n-2} (1+q^{ki+1}) \right) \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \Delta_m^{(0)}(n), \tag{32}$$

where $\Delta_m^{(0)}(n)$ and $\Delta_m^{(1)}(n)$ are given as in (29) *and* (31)*, respectively.*

Theorem 3.2 is a generalized combinatorial formula for the terms of (q,k)-Pell sequence. For $k = q = 1$, we show that Expression (2) takes the form,

$$P_{n+1}^{(1)}(1) = 2P_n^{(1)}(1) + P_{n-1}^{(1)}(1), \tag{33}$$

with $P_0^{(1)}(1) = 1$, $P_1^{(1)}(1) = 2$, which is nothing else but the usual sequence Pell numbers $P_n = P_n^{(1)}(1)$. As a consequence of Expression (32) a new formula for the usual Pell numbers can be provided in terms of the nested sums (29) and (31), as follows

$$P_n = 2^{n-2} \left(\sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} \Delta_m^{(1)}(n) + 4 \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \Delta_m^{(0)}(n) \right).$$

Since for $q = 1$ we get $\alpha(k) = \dfrac{-1^k}{(1+1^{kj+1})(1+1^{k(j-1)+1})} = -\frac{1}{4}$, therefore, a direct computation allows us to derive the following corollary.

Corollary 3.2. (Another combinatorial form of the usual Pell numbers) *The formula of the usual Pell numbers P_n in terms of nested sums is as follows,*

$$P_n = 2^{n-2} \left(\sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} \Delta_m^{(1)}(n) + 4 \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \Delta_m^{(0)}(n) \right),$$

where

$$\Delta_m^{(0)}(n) = -\frac{(-1)^m}{4^m}\sum_{k_1=2(m-1)+1}^{n-2}\left(\sum_{k_2=2(m-2)+1}^{k_1-2}\left(\cdots\left(\sum_{k_m=1}^{k_{m-1}-2}1\right)\cdots\right)\right),$$

$$\Delta_m^{(1)}(n) = \frac{(-1)^m}{4^m}\sum_{k_1=2(m-2)+2}^{n-2}\left(\sum_{k_2=2(m-2)+1}^{k_1-2}\left(\cdots\left(\sum_{k_m=1}^{k_{m-1}-2}1\right)\cdots\right)\right).$$

It seems to us that the results established in this section are not existing in the literature.

4 Applications

4.1 Cassini type identity for (q, k)-Fibonacci sequence and more identities

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers, generated by the well-known recursive relation $F_{n+1} = F_n + F_{n-1}$, with arbitrary initial conditions $F_0 = \alpha_0$ and $F_1 = \alpha_1$. The Cassini identity of the Fibonacci numbers is defined as the determinant of the related Casoratian matrix, namely,

$$\begin{vmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{vmatrix} = F_{n+1}F_{n-1} - F_n^2 = (-1)^n,$$

(for more details see, for instance, [10, 19]). We can observe that

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix},$$

and $T_n(1) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^n$ is nothing else but the transition matrix, related to the matrix formulation of the Fibonacci numbers. Proceeding in the same way, we can define the element $F_n^{(k)}(q)$ of the (q, k)-Fibonacci sequence as the determinant of the transition matrix related, namely,

$$T_q^{(k)}(1) = \prod_{j=0}^{*,n-1} L_q^{(k)}(j),$$

where the matrices $L_q^{(k)}(j)$ are given by Expression (7) and $L_q^{(k)}(n) = \begin{bmatrix} 0 & 1 \\ q^2 & 1 - q^2 + q^{2n+2k-1} \end{bmatrix}$.

By Theorem 2.1 the formula of the transition matrix of the difference equation (4) is given by

$$T_q^{(k)}(n) = \prod_{j=0}^{*,n-1} L_q^{(k)}(j) = \begin{bmatrix} \dfrac{\phi_n^{(1)}(q,k)}{c_{k,n-1}} & \dfrac{\phi_n^{(0)}(q,k)}{c_{k,n-1}} \\ \phi_{n+1}^{(1)}(q,k) & \phi_{n+1}^{(0)}(q,k) \end{bmatrix}.$$

On the other side, we have $\det(L_q^{(k)}(j)) = -q^2$. Therefore, the property of the determinant of product of matrices, namely, $\det(\prod_{i=1}^{m} A_i) = \prod_{i=1}^{m}\det(A_i)$, permits us to have the following property on the Cassini identity type for the $F_n^{(k)}(q)$.

Proposition 4.1. ((q,k)-Cassini identity for (4): Determinantal approach) *For the n-th term of the (q,k)-Fibonacci sequence* (4)*, we have the following identity,*

$$\begin{aligned}\det T_q^{(k)}(n) &= -\frac{1}{c_{k,n-1}}\left[F_0^{(k)}(q)\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) - F_1^{(k)}(q)\phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k)\right]\\ &= (-1)^n q^{2n},\end{aligned}$$

or equivalently,

$$\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) - (1+q^{2k+1})\phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k) = (-1)^{n+1}q^{2n}c_{k,n-1}, \tag{34}$$

where $c_{k,0} = q^2$, $c_{k,i} = c_{k,i-1}$, and $\phi_n^{(0)}(q,k)$, $\phi_n^{(1)}(q,k)$ are the canonical solutions of (4) *expressed under the determinantal form* (11) *and* (12)*.*

The Identity (34) represents a Cassini identity type for the (q,k)-Fibonacci sequence, it is called the (q,k)-*Cassini identity* related to the (q,k)-Fibonacci sequence. On the other side, Expressions (28) and (31) show that the canonical solutions (11) and (12) of Equation (4), are expressed in terms of the nested sums. Then, we have,

$$\begin{aligned}\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) &= \left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n)\left(\prod_{i=0}^{n-1}(1+q^{i+1})\right)\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\Delta_m^{(0)}(n+1)\\ &= (1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n)\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\Delta_m^{(0)}(n+1),\end{aligned}$$

and

$$\begin{aligned}\phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k) &= \left(\prod_{i=1}^{n-1}(1+q^{i+1})\right)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(1)}(n+1).\left(\prod_{i=0}^{n-2}(1+q^{i+1})\right)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n)\\ &= (1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(1)}(n+1)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n).\end{aligned}$$

Therefore, the (q,k)-Cassini identity related to the (q,k)-Fibonacci sequence can be expressed in terms of the nested sums. Indeed, we have the following result.

Proposition 4.2. *((q,k)-Cassini identity for* (4)*: Nested sum approach) For the (q,k)-Fibonacci sequence* (4)*, we have the following identity*

$$\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) - (1+q)\phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k) = (-1)^{n+1}q^{2n}c_{k,n-1},$$

where $c_{k,0} = q^2$, $c_{k,i} = c_{k,i-1}$, and

$$\begin{aligned}\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) &= (1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n)\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\Delta_m^{(0)}(n+1),\\ \phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k) &= (1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(1)}(n+1)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n).\end{aligned}$$

with $\Delta_m^{(0)}(n)$ *and* $\Delta_m^{(1)}(n)$ *being given by* (17) *and* (19)*, or equivalently,*

$$\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor} \Delta_m^{(1)}(n) \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor} \Delta_m^{(0)}(n+1) - \sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor} \Delta_m^{(1)}(n+1) \sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor} \Delta_m^{(0)}(n)$$
$$= (-1)^{n+1} \frac{q^{\frac{n(n-1)}{2}}}{(1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2} c_{k,n-1}.$$

It seems to us that the identities in Proposition 4.1 and Proposition 4.2, related to the (q,k)-Cassini identities, are not existing in the literature.

4.2 Cassini type identity for (q, k)-Pell sequence and more identities

For the usual sequence of Pell numbers $\{P_n\}_{n\geq 0}$ defined by (33), with arbitrary initial conditions $P_0 = \alpha_0$ and $P_1 = \alpha_1$, the Cassini identity related to the Pell numbers is defined as the determinant of the related Casoratian matrix, namely,

$$\begin{vmatrix} P_{n-1} & P_n \\ P_n & P_{n+1} \end{vmatrix} = P_{n+1}P_{n-1} - P_n^2 = (-1)^n$$

(for more details see, for instance, [10, 19]). We can observe that

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^n = \begin{bmatrix} P_{n-1} & P_n \\ P_n & P_{n+1} \end{bmatrix},$$

and $T_n(1) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^n$ is nothing else but the matrix formulation of the usual Pell numbers (33). Proceeding in the same way, we can define the element $P_n^{(k)}(q)$ of the (q,k)-Pell sequence as the determinant of the transition matrix related the q-Pell sequence (5), namely,

$$T_q(n) = \prod_{j=0}^{*,n-1} \begin{bmatrix} 0 & 1 \\ q^{kj} & 1+q^{kj+1} \end{bmatrix}.$$

Since $\det L_q^{(k)}(j) = \begin{vmatrix} 0 & 1 \\ q^{kj} & 1+q^{kj+1} \end{vmatrix} = -q^{kj}$, we derive that

$$\det T_q^{(k)}(n) = \prod_{j=0}^{*,n-1} \det L_q^{(k)}(j) = (-1)^n q^{\frac{kn(n-1)}{2}}.$$

On the other side, by Theorem 3.1 the formula of the transition matrix of the difference equation (5) is defined by

$$T_q^{(k)}(n) = \prod_{j=0}^{*,n-1} L_q^{(k)}(j) = \begin{bmatrix} -\frac{P_0^{(k)}(q)}{c_{k,n-1}}\phi_n^{(1)}(q,k) & -\frac{P_1^{(k)}(q)}{c_{k,n-1}}\phi_n^{(0)}(q),k \\ \phi_{n+1}^{(1)}(q,k) & \phi_{n+1}^{(0)}(q,k) \end{bmatrix}.$$

Therefore, we have the following property on the Cassini identity for the (q,k)-Pell sequence.

Proposition 4.3. ((q,k)-Cassini identity for (5): Determinantal approach) *For the element $P_n^{(k)}(q)$ of the (q,k)-Pell sequence we have the following identity,*

$$\begin{aligned}\det T_q^{(k)}(n) &= -\frac{1}{c_{k,n-1}}\left[P_0^{(k)}(q)\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) - P_1^{(k)}(q)\phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k)\right] \\ &= (-1)^n q^{\frac{kn(n-1)}{2}},\end{aligned}$$

or equivalently,

$$\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) - (1+q^k)\phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k) = (-1)^{n+1} q^{\frac{kn(n-1)}{2}} c_{k,n-1}, \tag{35}$$

where $c_{k,0} = 1$, $c_{k,i} = q^k c_{k,i-1}$, and $\phi_n^{(0)}(q,k)$, $\phi_n^{(1)}(q,k)$ are the canonical solutions of (5) *expressed under the determinantal form* (25) *and* (26)*.*

The identity (35) represents a Cassini identity for the (q,k)-Pell sequence, it is called the (q,k)-*Cassini identity* related to the (q,k)-Pell sequence.

On the other side, Expressions (28)–(31) show that the canonical solutions (25) and (26) of Equation (5), are expressed in the terms of nested sums. Then, we have,

$$\begin{aligned}\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) &= \left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n)\left(\prod_{i=0}^{n-1}(1+q^{i+1})\right)\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\Delta_m^{(0)}(n+1) \\ &= (1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n)\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\Delta_m^{(0)}(n+1),\end{aligned}$$

and

$$\begin{aligned}\phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k) &= \left(\prod_{i=1}^{n-1}(1+q^{i+1})\right)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(1)}(n+1).\left(\prod_{i=0}^{n-2}(1+q^{i+1})\right)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n) \\ &= (1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(1)}(n+1)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n).\end{aligned}$$

Therefore, the (q,k)-Cassini identity (35) can be expressed in the terms of nested sums. Indeed, we have the following result.

Proposition 4.4. ((q,k)-Cassini identity for (5): Nested sums approach) *For the (q,k)-Pell sequence we have the following identity,*

$$\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) - (1+q)\phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k) = (-1)^{n+1} q^{\frac{kn(n-1)}{2}} c_{k,n-1},$$

where $c_{k,0} = 1$, $c_{k,i} = q^k c_{k,i-1}$, and

$$\begin{aligned}\phi_n^{(1)}(q,k)\phi_{n+1}^{(0)}(q,k) &= (1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n)\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\Delta_m^{(0)}(n+1), \\ \phi_{n+1}^{(1)}(q,k)\phi_n^{(0)}(q,k) &= (1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(1)}(n+1)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n).\end{aligned}$$

with $\Delta_m^{(0)}(n)$ and $\Delta_m^{(1)}(n)$ are given by (29) *and* (31)*, or equivalently,*

$$\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n)\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\Delta_m^{(0)}(n+1)-\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(1)}(n+1)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n)$$
$$=(-1)^{n+1}\frac{q^{\frac{kn(n-1)}{2}}}{(1+q)(1+q^n)\left(\prod_{i=1}^{n-2}(1+q^{i+1})\right)^2}c_{k,n-1}. \quad (36)$$

It seems to us that the identities in Proposition 4.3 and Proposition 4.4, related to the (q, k)-Cassini identities expressed in the forms (35) and (36), are not existing in the literature.

5 (q, k)-Fibonacci–Pell sequence and the Rogers–Ramanujan identities

5.1 Slater identities and the (q, k)-Fibonacci–Pell sequences

Slater in [24] presented a list of 130 identities of the Rogers–Ramanujan type via Bailey pairs. In [21], Santos obtained extensions for 74 of the series that appear in the 130 Rogers–Ramanujan type identities listed by Slater. He proved some of these identities, using the so-called Andrews Method (see [4]). This method consists in considering an associated function of two variables $f(q, t)$ such that $f(q, t)$ satisfies a first order non-homogeneous equation in q (functional equation in q) and is a generating function of a sequence $\{P_n(q)\}_{n\in\mathbb{N}}$, namely, $f(q, t) = \sum_{n=0}^{\infty} P_n(q)t^n$, where $P_n(q)$ is a polynomial in q, with $\lim_{n\to\infty} P_n(q)$ equal than an infinite product in q.

Santos presented in [21], using the Andrews Method, the combinatorial relation between the function of two variables $f(q, t)$ and the linear difference equation of type (1) and (2). In [9], Craveiro established many results about the combinatorial extensions and interpretations for the Fibonacci, Pell and Jacobsthal numbers, and proved the 37-th and 38-th Slater identities. In this section, we are interested in to study the 16-th and 12-th Slater identities and their parametric extension, namely,

$$\sum_{n=0}^{+\infty}\frac{q^{n^2+2n}}{(q^4;q^4)_n}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\prod_{n=0}^{\infty}(1-q^{5n+5})(1-q^{5n+2})(1-q^{5n+1}), \quad (37)$$

$$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\prod_{n=0}^{\infty}(1-q^{4n+4})(1-q^{4n+2})(1-q^{4n+2})=1+2\sum_{n=1}^{\infty}\frac{(-q;q)_{n-1}q^{\frac{n(n+1)}{2}}}{(q;q)_n}, \quad (38)$$

where $(a;q)_n = (1-a)(1-aq)\cdot(1-aq^{n-1})$ and $(a;q)_\infty = \lim_{n\to+\infty}(a;q)_n$. More precisely, we establish determinantal and combinatorial explicit formulas for the (q, k)-Fibonacci sequence and the (q, k)-Pell sequence, with parameters $q = 1$, and $k = 0$ and $k = 1$, using the previous related results from , Sections 2 and 3.

5.2 The Slater identity 16 and (q, k)-Fibonacci sequence

In the present subsection, we consider the results of previous Section 2, with the aim to study the Slater Identities 16, 20, their parameterized extension, , and their related (q, k)-Fibonacci sequence. Santos in [21] established that the generating function $f(q,t)$ associated with Identity 37, is given by

$$f_{16}(q,t)=\sum_{n=0}^{+\infty}\frac{t^{2n}q^{n^2+2n}}{(t;q^2)_{n+1}(-tq^2;q^2)_n}.$$

In addition, it was proved in [21] that the function $f_{16}(q,t)$ is the generating function of the sequence $\{F_n^{(1)}(q)\}_{n\geq 0}$, namely, we have the following series

$$f_{16}(q,t)=\sum_{n=0}^{+\infty}F_n^{(1)}(t)t^n.$$

Consider the parameterized extension of $f_{16}(q,t)$, defined as follows,

$$f_{20-4k}(q,t)=\sum_{n=0}^{\infty}\frac{t^n q^{n^2+2kn}}{(1-t)(t^2q^4;q^4)_n}, \tag{39}$$

where k is a positive integer parameter. We show easily that when we set $k=0$ and $k=1$, we get the functions of two variables $f_{20}(q,t)$ and $f_{16}(q,t)$, respectively. Moreover, we observe that both functions are associated with (q,k)-Fibonacci sequence given (respectively) by Equations (14) and (15), studied as special cases given in Section 2.

On the other side, a long straightforward computation on Expression (39) allows us to show that the function $f_{20-4k}(q,t)$ satisfies the following functional equation,

$$(1-t)(1+tq^2)f_{20-4k}(q,t)=1+tq^2+tq^{2k+1}f_{20-4k}(tq^2,q). \tag{40}$$

By replacing the representative series $f_{20-4k}(q,t)=\sum_{n=0}^{\infty}F_n^{(k)}(q)t^n$ of $f_{20-4k}(q,t)$ in Expression (40), we obtain

$$(1+tq^2-t-t^2q^2)\sum_{n=0}^{\infty}F_n^{(k)}(q)t^n=1+tq^2+tq^{2k+1}\sum_{n=0}^{\infty}F_n^{(k)}(tq^2)^n,$$

or

$$\sum_{n=0}^{\infty}F_n^{(k)}(q)t^n+\sum_{n=1}^{\infty}q^2F_{n-1}^{(k)}(q)t^n-\sum_{n=1}^{\infty}F_{n-1}^{(k)}(q)t^n-\sum_{n=2}^{\infty}q^2F_{n-2}^{(k)}(q)t^n$$
$$=1+tq^2+\sum_{n=1}^{\infty}F_{n-1}^{(k)}(q)t^nq^{2n+2k-1}.$$

Comparing the coefficients of t^n in both sides of the preceding equality, we get

$$\begin{cases} F_0^{(k)}(q)= & 1, \\ F_1^{(k)}(q)= & 1 + q^{2k+1}, \\ F_n^{(k)}(q)= & (1 - q^2 + q^{2n-1+2k})F_{n-1}^{(k)}(q) + q^2F_{n-2}^{(k)}(q). \end{cases} \tag{41}$$

An immediate observation shows that the sequence $\{F_n^{(k)}(q)\}_{n\geq 0}$ satisfies Equation (4). When $k=0$, Santos in [21] established an explicit formula for the $F_n^{(0)}(q)$ given in (41), as follows,

$$F_n^{(0)}(q)=\sum_{j=-\infty}^{\infty}q^{10j^2+j}U(n,5j)-\sum_{j=-\infty}^{\infty}q^{10j^2+11j+3}U(n,5j+2),$$

where the following notation was adopted:

$$U(m,A)=U(m,A,q)=T_0(m,A,q)+T_0(m,A+1,q), \tag{42}$$

with

$$T_0(m,A)=T_0(m,A,q)=\sum_{j=0}^{m}(-1)^j\begin{bmatrix} m \\ j\end{bmatrix}_{q^2}\begin{bmatrix} 2m-2j \\ m-A-j\end{bmatrix}, \tag{43}$$

where

$$\begin{bmatrix} N \\ M\end{bmatrix}_q=\begin{bmatrix} N \\ M\end{bmatrix}=\frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-M+1})}{(1-q^M)(1-q^{M-1})(1-q)}$$

is the Gauss polynomial. In addition, Santos conjectured the explicit formula for $F_n^{(1)}(q)$, with the aid of the formula

$$c(n,q)=\sum_{j=-\infty}^{\infty}q^{10j^2+3j}U(n,5j)-\sum_{j=-\infty}^{\infty}q^{10j^2+13j+4}U(n,5j+3), \tag{44}$$

and showed that $c(n,q)=F_{n-1}^{(1)}(q)$, such that

$$\begin{cases} F_0^{(1)}(q)= & 1, \\ F_1^{(1)}(q)= & 1+q^3, \\ F_n^{(1)}(q)= & (1-q^2+q^{2n+1})F_{n-1}^{(1)}(q)+q^2F_{n-2}^{(1)}(q), \end{cases} \tag{45}$$

(for more details, see [21]). In [9, Theorem 1.2.1] the identity $c(n,q)=F_{n-1}^{(1)}(q)$, was proved.

Theorem 5.1. [9, Theorem 1.2.1] *When $k=1$, the expression $c(n,q)=F_{n-1}^{(1)}(q)$ as in (44), satisfies the recurrence relation (45) with the same initial conditions.*

The proof of the Theorem 5.1 is established showing that the sequence $\{F_{n-1}^{(1)}(q)\}_{n\geq 0}$ given in (44) satisfies the same relation of recurrence given in (45), with the same initial conditions. That is, we consider the following expression

$$U(n+1,A)-(1-q^2+q^{2n+1})U(n,A)-q^2U(n-1,A), \tag{46}$$

for A and n positive integer numbers. Making $n\to n+1$ in Expression (46) and replacing by Expressions (42) $--$(43) with some immediate cancellations and replaces, we get the result.

In the context of the previous discussion, comparing Expressions (20) and (44) allows us to obtain the following result.

Theorem 5.2. *For every positive integer n, it is verified the identity*

$$\begin{aligned} F_{n-1}^{(1)}(q) &= \sum_{j=-\infty}^{\infty}q^{10j^2+3j}U(n,5j)-\sum_{j=-\infty}^{\infty}q^{10j^2+13j+4}U(n,5j+3) \\ &= q^2\left(\prod_{i=1}^{n-2}(1+q^2+q^{2k+2i-1})\right)\sum_{m=0}^{\lfloor\frac{n-2}{2}\rfloor}\Delta_m^{(1)}(n) \\ &\quad +(1+q^{2k+1})\left(\prod_{i=0}^{n-2}(1+q^2+q^{2k+2i-1})\right)\sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}\Delta_m^{(0)}(n), \end{aligned}$$

where $\Delta_m^{(0)}(n)$ and $\Delta_m^{(1)}(n)$ are given as in (17) *and* (19)*, respectively.*

It follows from Theorem 5.1 an explicit formula for the Fibonacci numbers in terms of trinomial coefficients, since

$$\lim_{q\to 1} F^{(1)}_{n-1}(q) = F_n,$$

where F_n is the n-th Fibonacci number. Indeed, when $q \to 1$ we derive the form,

$$\lim_{q\to 1}\left\{\sum_{j=-\infty}^{\infty} q^{10j^2+3j}U(n,5j) - \sum_{j=-\infty}^{\infty} q^{10j^2+13j+4}U(n,5j+3)\right\} = \sum_{j=-\infty}^{\infty}\left\{\binom{n+1}{5j+1}_2 - \binom{n+1}{5j+3}_2\right\},$$

where $\binom{n}{j}_2 = \sum_{h\geq 0}(-1)^h\left(\begin{array}{c} n \\ h \end{array}\right)\left(\begin{array}{c} 2n-2h \\ n-j-h \end{array}\right)$. Therefore, we show that the Fibonacci numbers can be expressed in the form

$$F_n = \sum_{j=-\infty}^{\infty}\left\{\binom{n+1}{5j+1}_2 - \binom{n+1}{5j+3}_2\right\}.$$

Moreover, replacing $k=1$ in $f_{20-4k}(q,t)$, expressed in the form (39), we derive

$$f_{16}(q,t) = \sum_{n=0}^{\infty} \frac{t^n q^{n^2+2n}}{(t;q^2)_{n+1}(-tq^2;q^2)_n} = \sum_{n=0}^{\infty} Q_n(q)t^n,$$

where $Q_n(q) = F^{(1)}_n(q)$. Then, as a consequence of Theorem 5.1, setting $q = k = 1$, we obtain the following identity for the n-th Fibonacci numbers F_n.

Corollary 5.1. *For every positive integer n, is verified the identity*

$$F_n = \sum_{j=-\infty}^{\infty}\left\{\binom{n+1}{5j+1}_2 - \binom{n+1}{5j+3}_2\right\}.$$

In addition, setting $q = k = 1$, as a result of Theorems 5.2 and 5.1, we get the following corollary.

Corollary 5.2. *For every positive integer n, is verified the identity*

$$F_n = 3^{(n-4)}\left(\sum_{m=0}^{\lfloor\frac{n-4}{2}\rfloor}\Delta^{(1)}_m(n-2) + 6\sum_{m=0}^{\lfloor\frac{n-3}{2}\rfloor}\Delta^{(0)}_m(n-2)\right) = \sum_{j=-\infty}^{\infty}\left\{\tbinom{n+1}{5j+1}_2 - \tbinom{n+1}{5j+3}_2\right\},$$

where F_n is the n-th Fibonacci number and

$$\Delta^{(0)}_m(n) = \left(\frac{-1}{9}\right)^m \sum_{k_1=2(m-1)+1}^{n-2}\left(\sum_{k_2=2(m-2)+1}^{k_1-2}\left(\cdots\left(\sum_{k_m=1}^{k_{m-1}-2} 1\right)\cdots\right)\right) = \frac{(-1)^m}{9^m}\binom{n-1-m}{m},$$

and

$$\Delta^{(1)}_m(n) = \left(\frac{-1}{9}\right)^m \sum_{k_1=2(m-2)+2}^{n-2}\left(\sum_{k_2=2(m-2)+1}^{k_1-2}\left(\cdots\left(\sum_{k_m=1}^{k_{m-1}-2} 1\right)\cdots\right)\right) = \frac{(-1)^m}{9^m}\binom{n-2-m}{m}.$$

It seems to us that the identities given in Theorem 5.2 and Corollary 5.2 are not existing in the literature.

5.3 The Slater Identity 12 and (q, k)-Pell sequence

As mentioned above, in the present subsection, we consider the results of previous Section 3, in the aim to study the Slater Identity 12 and (q, k)-Pell sequence. In [21], is given a list of functions $f(q, t)$ related to the identities proposed by Slater. Among them, the function of two variables associated with the Identity 38, is given by

$$f_{12}(q,t) = \sum_{n=0}^{\infty} \frac{(-t;q)_n t^n q^{\frac{n(n+1)}{2}}}{(t;q)_{n+1}}. \tag{47}$$

A long straightforward computation of the Expression (47), allows us to show that the function $f_{12}(q, t)$ satisfies the following functional equation

$$(1-t)f_{12}(q,t) = 1 + (1+t)tqf_{12}(q,tq). \tag{48}$$

By substituting the expression $f(q,t) = \sum_{n=0}^{\infty} P_q^{(1)}(n)t^n$ in the functional equation (48), we derive that $P_q^{(1)}(n)$ satisfies the same recurrence Equation (2), namely,

$$\begin{cases} P_0^{(1)}(q) = 1, \\ P_1^{(1)}(q) = 1 + q, \\ P_n^{(1)}(q) = (1 + q^n) P_{n-1}^{(1)}(q) + q^{n-1}P_{n-2}^{(1)}(q). \end{cases} \tag{49}$$

In [21], a conjecture for an explicit formula for $P_n^{(1)}(q)$ in terms of q-analogous of binomial and trinomial coefficients was proposed, namely,

$$c(n,q) = P_{n-1}^{(1)}(q) = \sum_{j=-\infty}^{\infty} q^{8j^2} CT(n, 1+8j) - \sum_{j=-\infty}^{\infty} q^{8j^2-8j+2} CT(n, 3-8j), \tag{50}$$

where

$$CT(m,A) = \sum_{j=0}^{m} (-q^{\frac{1}{2}})^j \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix}_{q^{1/2}}$$

with $\begin{bmatrix} N \\ M \end{bmatrix}_q = \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-M+1})}{(1-q^M)(1-q^{M-1})(1-q)}$ being the Gauss polynomial. This conjecture was proved in [9], where the result below was established.

Theorem 5.3. [9, Theorem 2.1.1] *The formula* $c(n, q)$ *given in* (50) *satisfies the recurrence* (49)*, with the same initial conditions, namely,* $c(n, q) = P_{n-1}^{(1)}(q)$*, for every* $n \geq 0$.

In the context of the previous discussion, comparing the Expressions (50) and (32) allows us to obtain the follow result.

Theorem 5.4. *For a positive integer* n*, it is verified the identity*

$$\begin{aligned} P_n^{(1)}(q) &= \sum_{j=-\infty}^{\infty} q^{8j^2} CT(n+1, 1+8j) - \sum_{j=-\infty}^{\infty} q^{8j^2-8j+2} CT(n+1, 3-8j) \\ &= \left(\prod_{i=1}^{n-2}(1+q^{i+1})\right) \sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} \Delta_m^1(n) + (1+q)\left(\prod_{i=0}^{n-2}(1+q^{i+1})\right) \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \Delta_m^0(n). \end{aligned}$$

As a consequence of Theorem 5.3, an explicit formula for the Pell numbers in terms of trinomial coefficients is provided, since $\lim_{q\to 1} P_{n-1}^{(1)}(q) = P_n$, where P_n is the n-th Pell number. Indeed, when $q \to 1$, the Pell numbers can be written in the form

$$P_n = \sum_{j=-\infty}^{\infty} \left\{ \binom{n+1}{1+8j}_2 - \binom{n+1}{3-8j}_2 \right\}.$$

Then, we derive the following identity for Pell numbers,

Corollary 5.3. *For every positive integer n, the n-th Pell number, P_n, verifies the identity*

$$P_n = \sum_{j=-\infty}^{\infty} \left\{ \binom{n+1}{1+8j}_2 - \binom{n+1}{3-8j}_2 \right\}.$$

Also, setting $q = 1$ in Theorem 5.3 and Corollary 3.2, we get the following result,

Corollary 5.4. *For every positive integer n, the n-th Pell number, P_n, verifies the identity*

$$P_n = 2^{n-2} \left(\sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} \Delta_m^{(1)}(n) + 4 \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \Delta_m^{(0)}(n) \right) = \sum_{j=-\infty}^{\infty} \left\{ \binom{n+1}{1+8j}_2 - \binom{n+1}{3-8j}_2 \right\},$$

where

$$\Delta_m^{(0)}(n) = \frac{1}{4^m} \sum_{k_1=2(m-1)+1}^{n-2} \left(\sum_{k_2=2(m-2)+1}^{k_1-2} \left(\cdots \left(\sum_{k_m=1}^{k_{m-1}-2} 1 \right) \cdots \right) \right)$$

and

$$\Delta_m^{(1)}(n) = \frac{1}{4^m} \sum_{k_1=2(m-2)+2}^{n-2} \left(\sum_{k_2=2(m-2)+1}^{k_1-2} \left(\cdots \left(\sum_{k_m=1}^{k_{m-1}-2} 1 \right) \cdots \right) \right).$$

It seems to us that the equalities given in Theorem 5.4 and Corollary 5.4 are not existing in the literature.

6 Conclusion and perspective

In this study, we have presented new results regarding the explicit formulas for the (q, k)-Fibonacci and (q, k)-Pell sequences. Specifically, we have based our construction on two approaches for solving the Equations (4) and (5) considered as a difference equation of the second order with variable coefficients, namely, the determinantal and the nested sums approaches. As consequences, we derived the (q, k)-Cassini identities relativeto (q, k)-Fibonacci and (q, k)-Pell sequences and the combinatorial interpretations concerned to identities of Rogers– Ramanujan type.

To the best of our knowledge, our results are not existing in the literature. Moreover, our approaches can be applied to any classes of (q, k)-sequences described by a linear difference equation of type (3). In addition, the applications as (q, k)-Cassini identities and combinatorial identities of Rogers–Ramanujan type can be established.

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