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On a generalization of dual-generalized complex Fibonacci quaternions

Elif Tan[1] and Umut Öcal[2]

[1] Department of Mathematics, Ankara University
06100 Tandogan Ankara, Turkey
e-mail: etan@ankara.edu.tr

[2] Department of Mathematics, Ankara University
06100 Tandogan Ankara, Turkey
e-mail: ocalum@gmail.com

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Abstract: In this study, we introduce a new class of generalized quaternions whose components are dual-generalized complex Horadam numbers. We investigate some algebraic properties of them.
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1 Introduction

Hypercomplex numbers are defined by Kantor and Solodovnikov [23] as an extension of real numbers. They are finite-dimensional algebra over $\mathbb{R}$ which need not be commutative or associative. They have many applications in geometry, trigonometry, physics, robotics, quantum mechanics, color image processing etc. Especially, associative and commutative hypercomplex algebras are suitable for digital signal processing [3]. Whereas the quaternions, octonions, and sedenions are the best known noncommutative hypercomplex numbers, the two-dimensional hypercomplex numbers are those of commutative.

A general two-dimensional hypercomplex numbers are defined by

$$\mathbb{C}_{\mathfrak{p},\mathfrak{q}} = \left\{a_0 + a_1 J \mid J^2 = \mathfrak{p} + \mathfrak{q}J,\ a_0, a_1, \mathfrak{p}, \mathfrak{q} \in \mathbb{R}, J \notin \mathbb{R}\right\}.$$

For $\mathfrak{q} = 0$, these number systems are denoted by $\mathbb{C}_{\mathfrak{p}}$, and called as the system of generalized complex numbers. It is well-known that the set $\mathbb{C}_{\mathfrak{p}}$ yields the complex numbers, dual numbers, and hyperbolic (perplex, double, split-complex) numbers when $\mathfrak{p}$ is equal to $-1, 0$, and 1, respectively. The geometry of these number systems was investigated by Harkin and Harkin [19]. For an overview of these numbers, we refer to [5, 13, 24, 37].

Gurses et al. [16] introduced the dual-generalized complex numbers by taking any dual number with generalized complex number coefficients instead of real numbers. In particular, the set of dual-generalized complex numbers is defined by

$$\mathbb{DC}_{\mathfrak{p}} = \{a_0 + a_1 J + a_2 \varepsilon + a_3 J\varepsilon \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

where the dual unit ε and generalized complex unit J satisfy the following rules:

$$J^2 = \mathfrak{p}, -\infty < \mathfrak{p} < \infty, \varepsilon^2 = 0, \varepsilon \neq 0, \varepsilon J = J\varepsilon. \tag{1}$$

It is clear to see that this new commutative number system reduces to dual-complex numbers when $\mathfrak{p} = -1$, hyper-dual numbers when $\mathfrak{p} = 0$, and dual-hyperbolic numbers when $\mathfrak{p} = 1$. Thus, by using the dual-generalized complex numbers, one can study dual-complex, hyper-dual, and dual-hyperbolic numbers simultaneously. For details related to dual-generalized complex numbers, we refer to [4, 8, 9, 11, 12, 25–27].

On the other hand, the generalized quaternion algebra [10, 14] is defined by

$$\mathbb{H}_{\lambda,\mu} = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\},$$

where the basis $\{1, i, j, k\}$ satisfies the following multiplication rules:

$$\begin{aligned} i^2 &= -\lambda,\ j^2 = -\mu, k^2 = -\lambda\mu, \\ ij &= -ji = k,\ jk = -kj = \mu i,\ ki = -ik = \lambda j \end{aligned} \tag{2}$$

with $\lambda, \mu \in \mathbb{R}$. For $\lambda = \mu = 1$, it reduces to the real quaternion algebra, and for $\lambda = 1, \mu = -1$, it reduces to the split quaternion algebra. The addition, subtraction and multiplication of two generalized quaternions $q_1 = a_0 + a_1 i + a_2 j + a_3 k$ and $q_2 = b_0 + b_1 i + b_2 j + b_3 k$ are defined by

$$\begin{aligned} q_1 \pm q_2 &= (a_0 \pm b_0) + (a_1 \pm b_1) i + (a_2 \pm b_2) j + (a_3 \pm b_3) k, \\ q_1 q_2 &= a_0 b_0 - a_1 b_1 \lambda - a_2 b_2 \mu - a_3 b_3 \lambda\mu \\ &\quad + (a_0 b_1 + a_1 b_0 + a_2 b_3 \mu - a_3 b_2 \mu) i \\ &\quad + (a_0 b_2 + a_2 b_0 + a_3 b_1 \lambda - a_1 b_3 \lambda) j \\ &\quad + (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1) k. \end{aligned}$$

The norm of a generalized quaternion q_1 is defined by

$$N(q_1) := q_1 \overline{q}_1 = a_0^2 + a_1^2 \lambda + a_2^2 \mu + a_3^2 \lambda\mu,$$

where $\overline{q}_1 = a_0 - a_1 i - a_2 j - a_3 k$ is the conjugate of a generalized quaternion q_1. For details on generalized quaternion algebra, see [7, 18, 22, 29].

Many works related to quaternion sequences over some special quaternion algebras have been extensively studied. In particular, Horadam [20] studied Fibonacci quaternions over the real quaternion algebra $\mathbb{H}_{1,1}$, which is based on the quaternion sequences with Fibonacci number components. Nurkan and Guven [28], defined dual Fibonacci quaternions. Also, Halıcı and Karatas [17] introduced Horadam quaternions over real quaternion algebra $\mathbb{H}_{1,1}$ as

$$Q_{w,n} = w_n + w_{n+1} i + w_{n+2} j + w_{n+3} k,$$

where $\{w_n\}$ is the Horadam sequence [21] and is defined by

$$w_n = p w_{n-1} + q w_{n-2}, \quad n \geq 2$$

with the arbitrary initial values w_0, w_1 and nonzero integers p, q. We note that the Horadam sequence $\{w_n\}$ reduces to the (p, q)-Fibonacci sequence $\{u_n\}$ when $w_0 = 0, w_1 = 1$, and the (p, q)-Lucas sequence $\{v_n\}$ when $w_0 = 2, w_1 = p$. For $p = q = 1$, these sequences reduce to the classical Fibonacci sequence $\{F_n\}$ and Lucas sequence $\{L_n\}$, resepectively. The Binet formula of Horadam sequence $\{w_n\}$ is

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $A := w_1 - w_0\beta, B := w_1 - w_0\alpha$, and α, β are the roots of the characteristic polynomial $x^2 - px - q$, that is; $\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \beta = \frac{p-\sqrt{p^2+4q}}{2}$. Also we have $\alpha\beta = -q, \alpha + \beta = p$, $\Delta := \alpha - \beta = \sqrt{p^2 + 4q}$ with $p^2 + 4q > 0$. Thus the Binet formula of Horadam quaternions [17] is obtained by

$$Q_{w,n} = \frac{A\alpha^*\alpha^n - B\beta^*\beta^n}{\alpha - \beta}, \tag{3}$$

where $\alpha^* = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ and $\beta^* = 1 + \beta i + \beta^2 j + \beta^3 k$. For more on Horadam sequences and Horadam quaternions, we refer to [17, 33–36].

Similar to the Fibonacci quaternions over real quaternion algebra, Akyiğit et al. [2] studied the Fibonacci quaternions over the generalized quaternion algebra $\mathbb{H}_{\lambda,\mu}$, and called it as, Fibonacci generalized quaternions. Senturk et al. [32] studied a generalization of Horadam quaternions over the generalized quaternion algebra $\mathbb{H}_{\lambda,\mu}$. Also, many authors have studied the dual-generalized complex numbers with Fibonacci-like numbers components. In particular, Cihan et al. [6] introduced the dual-hyperbolic Fibonacci and Lucas numbers. Gungor and Azak [15] defined the dual-complex Fibonacci and Lucas numbers. Dual-complex Fibonacci p-numbers were studied by Prasad [30]. Recently, Ait-Amrane et al. [1] have introduced the hyper-dual Horadam quaternions. These numbers can also be seen as hyper-dual numbers with Horadam quaternion coefficients. Senturk et al. [31] have introduced the dual-generalized complex Fibonacci quaternions by taking dual Fibonacci numbers instead of real numbers as coefficients.

Motivated by the above mentioned studies, here we introduce generalized quaternions whose components are dual-generalized complex Horadam numbers. We obtain the generating function and the Binet formula of these new quaternions. Some algebraic properties of these quaternions such as Vajda's identity, Catalan's identity, Cassini's identity, and d'Ocagne's identity are derived with the help of the Binet formula. Our results can be seen as a generalization of many previous works in the literature such as [1, 6, 15, 16].

2 Main results

In this section, first we define the dual-generalized complex Horadam numbers, then by using these numbers we introduce the dual-generalized complex Horadam quaternions over the generalized quaternion algebra $\mathbb{H}_{\lambda,\mu}$, and call it as dual-generalized complex Horadam generalized quaternions.

Definition 2.1. *The n-th dual-generalized complex Horadam number is defined as*

$$\widetilde{w}_n = w_n + w_{n+1}J + w_{n+2}\varepsilon + w_{n+3}J\varepsilon,$$

where w_n is the n-th Horadam number, ε is dual unit, and J is generalized complex unit that satisfies the multiplication rules in (1).

Definition 2.2. *The n-th dual-generalized complex Horadam generalized quaternion is defined as*

$$\widetilde{Q}_{w,n} = \widetilde{w}_n + \widetilde{w}_{n+1}i + \widetilde{w}_{n+2}j + \widetilde{w}_{n+3}k,$$

where $\widetilde{w}_n$ is the n-th dual-generalized complex Horadam number and i, j, k satisfies the generalized quaternion multiplication rules in (2).

It is clear to see that when $w_0 = 0, w_1 = 1, p = q = 1$, and $\lambda = \mu = 1$, the dual-generalized complex Horadam generalized quaternion sequence $\{\widetilde{Q}_{w,n}\}$ reduces to the DGC Fibonacci quaternions in [31]. Thus depend on the value of $\mathfrak{p}$, we have the following special cases:

1. For $\mathfrak{p} = -1$, we get dual-complex Fibonacci and Lucas quaternions [15].
2. For $\mathfrak{p} = 1$, we get dual-hyperbolic Fibonacci and Lucas quaternions [6].
3. For $\mathfrak{p} = 0$, we get hyper-dual Horadam quaternions [1].

The n-th dual-generalized complex Horadam generalized quaternion can also be expressed as

$$\widetilde{Q}_{w,n} = Q_{w,n} + Q_{w,n+1}J + Q_{w,n+2}\varepsilon + Q_{w,n+3}J\varepsilon,$$

where $Q_{w,n}$ is the n-th Horadam quaternion. The addition, subtraction, and multiplication of two dual-generalized complex Horadam generalized quaternions $\widetilde{Q}_{w,n}$ and $\widetilde{Q}_{w,m}$ are defined as

$$\begin{aligned}\widetilde{Q}_{w,n} \pm \widetilde{Q}_{w,m} &= (Q_{w,n} \pm Q_{w,m}) + (Q_{w,n+1} \pm Q_{w,m+1})J \\ &\quad + (Q_{w,n+2} \pm Q_{w,m+2})\varepsilon + (Q_{w,n+3} \pm Q_{w,m+3})J\varepsilon,\end{aligned}$$

$$\begin{aligned}\widetilde{Q}_{w,n}\widetilde{Q}_{w,m} &= (Q_{w,n}Q_{w,m} + \mathfrak{p}Q_{w,n+1}Q_{w,m+1}) + (Q_{w,n}Q_{w,m+1} + Q_{w,n+1}Q_{w,m})J \\ &\quad + (Q_{w,n}Q_{w,m+2} + Q_{w,n+2}Q_{w,m} + \mathfrak{p}Q_{w,n+1}Q_{w,m+3} + \mathfrak{p}Q_{w,n+3}Q_{w,m+1})\varepsilon \\ &\quad + (Q_{w,n}Q_{w,m+3} + Q_{w,n+1}Q_{w,m+2} + Q_{w,n+2}Q_{w,m+1} + Q_{w,n+3}Q_{w,m})J\varepsilon,\end{aligned}$$

respectively. The multiplication of two dual-generalized complex Horadam generalized quaternions can be written in terms of dual-generalized complex Horadam numbers as:

$$\begin{aligned}\widetilde{Q}_{w,n}\widetilde{Q}_{w,m} &= \widetilde{w}_n\widetilde{w}_m - \widetilde{w}_{n+1}\widetilde{w}_{m+1}\lambda - \widetilde{w}_{n+2}\widetilde{w}_{m+2}\mu - \widetilde{w}_{n+3}\widetilde{w}_{m+3}\lambda\mu \\ &\quad + (\widetilde{w}_n\widetilde{w}_{m+1} + \widetilde{w}_{n+1}\widetilde{w}_m + \widetilde{w}_{n+2}\widetilde{w}_{m+3}\mu - \widetilde{w}_{n+3}\widetilde{w}_{m+2}\mu)i \\ &\quad + (\widetilde{w}_n\widetilde{w}_{m+2} + \widetilde{w}_{n+2}\widetilde{w}_m + \widetilde{w}_{n+3}\widetilde{w}_{m+1}\lambda - \widetilde{w}_{n+1}\widetilde{w}_{m+3}\lambda)j \\ &\quad + (\widetilde{w}_n\widetilde{w}_{m+3} + \widetilde{w}_{n+3}\widetilde{w}_m + \widetilde{w}_{n+1}\widetilde{w}_{m+2} - \widetilde{w}_{n+2}\widetilde{w}_{m+1})k.\end{aligned}$$

The multiplication scheme for the basis elements can be given in the following table.

Table 1. Please add table caption here

	1	i	j	k	J	ε	$J\varepsilon$
1	1	i	j	k	J	ε	$J\varepsilon$
i	i	$-\lambda$	k	$-\lambda j$	Ji	εi	$J\varepsilon i$
j	j	$-k$	$-\mu$	μi	Jj	εj	$J\varepsilon j$
k	k	λj	$-\mu i$	$-\lambda\mu$	Jk	εk	$J\varepsilon k$
J	J	Ji	Jj	Jk	$\mathfrak{p}$	$J\varepsilon$	$\mathfrak{p}\varepsilon$
ε	ε	εi	εj	εk	$J\varepsilon$	0	0
$J\varepsilon$	$J\varepsilon$	$J\varepsilon i$	$J\varepsilon j$	$J\varepsilon k$	$\mathfrak{p}\varepsilon$	0	0

Theorem 2.1. *The dual-generalized complex Horadam generalized quaternions satisfy the following relation:*

$$\widetilde{Q}_{w,n} = p\widetilde{Q}_{w,n-1} + q\widetilde{Q}_{w,n-2},\ n \geq 2.$$

Proof. From the definitions of dual-generalized complex Horadam quaternions and Horadam quaternions, we have

$$\begin{aligned}
p\widetilde{Q}_{w,n-1} + q\widetilde{Q}_{w,n-2} &= p\left(Q_{w,n-1} + Q_{w,n}J + Q_{w,n+1}\varepsilon + Q_{w,n+2}J\varepsilon\right) \\
&\quad + q\left(Q_{w,n-2} + Q_{w,n-1}J + Q_{w,n}\varepsilon + Q_{w,n+1}J\varepsilon\right) \\
&= \left(pQ_{w,n-1} + qQ_{w,n-2}\right) + \left(pQ_{w,n} + qQ_{w,n-1}\right)J \\
&\quad + \left(pQ_{w,n+1} + qQ_{w,n}\right)\varepsilon + \left(pQ_{w,n+2} + qQ_{w,n+1}\right)J\varepsilon \\
&= Q_{w,n} + Q_{w,n+1}J + Q_{w,n+2}\varepsilon + Q_{w,n+3}J\varepsilon.
\end{aligned}$$

□

Theorem 2.2. *The generating function for dual-generalized complex Horadam generalized quaternions is*

$$G(x) = \frac{\widetilde{Q}_{w,0} + \left(\widetilde{Q}_{w,1} - p\widetilde{Q}_{w,0}\right)x}{1 - px - qx^2}.$$

Proof. Let

$$G(x) := \sum_{n=0}^{\infty} \widetilde{Q}_{w,n}x^n = \widetilde{Q}_{w,0} + \widetilde{Q}_{w,1}x + \sum_{n=2}^{\infty} \widetilde{Q}_{w,n}x^n.$$

From Theorem 2.1, we have

$$\begin{aligned}
&\left(1 - px - qx^2\right)G(x) \\
&= \widetilde{Q}_{w,0} + \widetilde{Q}_{w,1}x + \sum_{n=2}^{\infty} \widetilde{Q}_{w,n}x^n - p\widetilde{Q}_{w,0}x - p\sum_{n=2}^{\infty} \widetilde{Q}_{w,n-1}x^n - q\sum_{n=2}^{\infty} \widetilde{Q}_{w,n-2}x^n \\
&= \widetilde{Q}_{w,0} + \widetilde{Q}_{w,1}x - p\widetilde{Q}_{w,0}x + \sum_{n=2}^{\infty} \left(\widetilde{Q}_{w,n} - p\widetilde{Q}_{w,n-1} - q\widetilde{Q}_{w,n-2}\right)x^n \\
&= \widetilde{Q}_{w,0} + \left(\widetilde{Q}_{w,1} - p\widetilde{Q}_{w,0}\right)x.
\end{aligned}$$

Thus, we get the desired result. □

Theorem 2.3. *The Binet formula of dual-generalized complex Horadam generalized quaternions is*

$$\widetilde{Q}_{w,n}=\frac{A\alpha^{*}\underline{\alpha}\alpha^{n}-B\beta^{*}\underline{\beta}\beta^{n}}{\alpha-\beta},$$

where $\alpha^{*}=1+\alpha i+\alpha^{2}j+\alpha^{3}k$ *and* $\beta^{*}=1+\beta i+\beta^{2}j+\beta^{3}k$ *and* $\underline{\alpha}=1+\alpha J+\alpha^{2}\varepsilon+\alpha^{3}J\varepsilon$ *and* $\underline{\beta}=1+\beta J+\beta^{2}\varepsilon+\beta^{3}J\varepsilon$.

Proof. From the Binet formula of Horadam quaternions in (3), we have

$$\begin{aligned}
\widetilde{Q}_{w,n} &= Q_{w,n}+Q_{w,n+1}J+Q_{w,n+2}\varepsilon+Q_{w,n+3}J\varepsilon\\
&= \left(\frac{A\alpha^{*}\alpha^{n}-B\beta^{*}\beta^{n}}{\alpha-\beta}\right)+\left(\frac{A\alpha^{*}\alpha^{n+1}-B\beta^{*}\beta^{n+1}}{\alpha-\beta}\right)J\\
&\quad+\left(\frac{A\alpha^{*}\alpha^{n+2}-B\beta^{*}\beta^{n+2}}{\alpha-\beta}\right)\varepsilon+\left(\frac{A\alpha^{*}\alpha^{n+3}-B\beta^{*}\beta^{n+3}}{\alpha-\beta}\right)J\varepsilon\\
&= \frac{A\alpha^{*}\alpha^{n}}{\alpha-\beta}\left(1+\alpha J+\alpha^{2}\varepsilon+\alpha^{3}J\varepsilon\right)-\frac{B\beta^{*}\beta^{n}}{\alpha-\beta}\left(1+\beta J+\beta^{2}\varepsilon+\beta^{3}J\varepsilon\right)\\
&= \frac{A\alpha^{*}\underline{\alpha}\alpha^{n}-B\beta^{*}\underline{\beta}\beta^{n}}{\alpha-\beta}.
\end{aligned}$$

□

From Theorem 2.3, we obtain the Binet formulas of (p,q)-Fibonacci and Lucas cases:

$$\widetilde{Q}_{u,n}=\frac{\alpha^{*}\underline{\alpha}\alpha^{n}-\beta^{*}\underline{\beta}\beta^{n}}{\alpha-\beta}\quad\text{and}\quad\widetilde{Q}_{v,n}=\alpha^{*}\underline{\alpha}\alpha^{n}+\beta^{*}\underline{\beta}\beta^{n},\tag{4}$$

respectively. By considering (4), we can easily obtain the following relation:

$$\widetilde{Q}_{v,n}=\widetilde{Q}_{u,n+1}+q\widetilde{Q}_{u,n-1}.$$

We need the following lemma to obtain several properties of dual-generalized complex Horadam generalized quaternions.

Lemma 2.1. *Let* $\theta_{\lambda,\mu}:=1-q\lambda+q^{2}\mu-q^{3}\lambda\mu$, $\omega_{\lambda,\mu}:=(1-q\mu)\,i+(p-p\lambda)\,j+(1+p^{2}+q)\,k$. *Then we have the followings:*

(i) $\alpha^{*}\beta^{*}=Q_{v,0}-\theta_{\lambda,\mu}-\Delta q\left(Q_{u,0}-\omega_{\lambda,\mu}\right)$

(ii) $\beta^{*}\alpha^{*}=Q_{v,0}-\theta_{\lambda,\mu}+\Delta q\left(Q_{u,0}-\omega_{\lambda,\mu}\right)$

(iii) $\underline{\alpha}\underline{\beta}=\widetilde{v}_{0}-\left(1+\mathfrak{p}q\left(1+v_{2}\varepsilon\right)+pqJ\varepsilon\right).$

Proof. (i) By using the multiplication rules in (2), we have

$$\begin{aligned}
\alpha^{*}\beta^{*} &= \left(1+\alpha i+\alpha^{2}j+\alpha^{3}k\right)\left(1+\beta i+\beta^{2}j+\beta^{3}k\right)\\
&= 1-(\alpha\beta)\lambda-(\alpha\beta)^{2}\mu-(\alpha\beta)^{3}\lambda\mu+\left(\beta+\alpha+\alpha^{2}\beta^{3}\mu-\alpha^{3}\beta^{2}\mu\right)i\\
&\quad+\left(\beta^{2}+\alpha^{2}+\alpha^{3}\beta\lambda-\alpha\beta^{3}\lambda\right)j+\left(\beta^{3}+\alpha^{3}+\alpha\beta^{2}-\alpha^{2}\beta\right)k\\
&= 1+q\lambda-q^{2}\mu+q^{3}\lambda\mu+\left(p-q^{2}\Delta\mu\right)i\\
&\quad+\left(p^{2}+2q-qp\Delta\lambda\right)j+\left(p^{3}+3qp+q\Delta\right)k\\
&= 2+pi+\left(p^{2}+2q\right)j+\left(p^{3}+3pq\right)k-\left(1-q\lambda+q^{2}\mu-q^{3}\lambda\mu\right)\\
&\quad-q\Delta\left(q\mu i+p\lambda j-k\right)\\
&\quad=Q_{v,0}-\theta_{\lambda,\mu}-q\Delta\left(Q_{u,0}-\omega_{\lambda,\mu}\right).
\end{aligned}$$

(ii) It can be proven in a similar manner to the first identity.

(iii) By using the multiplication rules in (1), we have

$$\begin{aligned}
\underline{\alpha\beta} &= \left(1+\alpha J+\alpha^2\varepsilon+\alpha^3 J\varepsilon\right)\left(1+\beta J+\beta^2\varepsilon+\beta^3 J\varepsilon\right) \\
&= \left(1+\mathfrak{p}\left(\alpha\beta\right)\right)+\left(\alpha+\beta\right)J+\left(\alpha^2+\beta^2+\mathfrak{p}\left(\alpha\beta\right)\left(\alpha^2+\beta^2\right)\right)\varepsilon \\
&\quad +\left(\alpha^3+\beta^3+\left(\alpha\beta\right)\left(\alpha+\beta\right)\right)J\varepsilon \\
&= 1-\mathfrak{p}q+v_1 J+\left(v_2\left(1-\mathfrak{p}q\right)\right)\varepsilon+\left(v_3-pq\right)J\varepsilon \\
&= 2+v_1 J+v_2\varepsilon+v_3 J\varepsilon-1-\mathfrak{p}q-\mathfrak{p}qv_2\varepsilon-pqJ\varepsilon \\
&= \widetilde{v}_0-\left(1+\mathfrak{p}q\left(1+v_2\varepsilon\right)+pqJ\varepsilon\right). \qquad \square
\end{aligned}$$

Note that for $\lambda=\mu=1$ and $\mathfrak{p}=0$, the identities in Lemma 2.1 reduce to the identities for hyper-dual Horadam quaternions in [1, Lemma 1].

It is clear that we have the following results from Lemma 2.1:

$$\alpha^*\beta^*+\beta^*\alpha^* = 2\left(Q_{v,0}-\theta_{\lambda,\mu}\right), \tag{5}$$

$$\alpha^*\beta^*-\beta^*\alpha^* = -2\Delta q\left(Q_{u,0}-\omega_{\lambda,\mu}\right). \tag{6}$$

By using the Binet formula of dual-generalized complex Horadam generalized quaternions and using the Lemma 2.1, we obtain the following identity.

Theorem 2.4. *(Vajda's identity) For nonnegative integers $n, r,$ and s, we have*

$$\begin{aligned}
&\widetilde{Q}_{w,n+r}\widetilde{Q}_{w,n+s}-\widetilde{Q}_{w,n}\widetilde{Q}_{w,n+r+s} \\
&= AB\left(-q\right)^n\left(\widetilde{v}_0-\left(1+\mathfrak{p}q\left(1+v_2\varepsilon\right)+pqJ\varepsilon\right)\right)u_r\left(\left(Q_{v,0}-\theta_{\lambda,\mu}\right)u_s+q\left(Q_{u,0}-\omega_{\lambda,\mu}\right)v_s\right).
\end{aligned}$$

Proof. From the Binet formula of dual-generalized complex Horadam generalized quaternions, we have

$$\begin{aligned}
&\Delta^2\left(\widetilde{Q}_{w,n+r}\widetilde{Q}_{w,n+s}-\widetilde{Q}_{w,n}\widetilde{Q}_{w,n+r+s}\right) \\
&= \left(A\alpha^*\underline{\alpha}\alpha^{n+r}-B\beta^*\underline{\beta}\beta^{n+r}\right)\left(A\alpha^*\underline{\alpha}\alpha^{n+s}-B\beta^*\underline{\beta}\beta^{n+s}\right) \\
&\quad -\left(A\alpha^*\underline{\alpha}\alpha^{n}-B\beta^*\underline{\beta}\beta^{n}\right)\left(A\alpha^*\underline{\alpha}\alpha^{n+r+s}-B\beta^*\underline{\beta}\beta^{n+r+s}\right) \\
&= A^2\left(\alpha^*\underline{\alpha}\right)^2\alpha^{2n+r+s}-AB\alpha^*\beta^*\underline{\alpha\beta}\alpha^{n+r}\beta^{n+s}-AB\beta^*\alpha^*\underline{\beta\alpha}\alpha^{n+s}\beta^{n+r}+B^2\left(\beta^*\underline{\beta}\right)^2\beta^{2n+r+s} \\
&\quad -A^2\left(\alpha^*\underline{\alpha}\right)^2\alpha^{2n+r+s}+AB\alpha^*\beta^*\underline{\alpha\beta}\alpha^{n}\beta^{n+r+s}+AB\beta^*\alpha^*\underline{\beta\alpha}\beta^{n}\alpha^{n+r+s}-B^2\left(\beta^*\underline{\beta}\right)^2\beta^{2n+r+s} \\
&= AB\left(\alpha\beta\right)^n\underline{\alpha\beta}\left(\alpha^*\beta^*\left(-\alpha^r\beta^s+\beta^{r+s}\right)+\beta^*\alpha^*\left(-\alpha^s\beta^r+\alpha^{r+s}\right)\right).
\end{aligned}$$

By using Lemma 2.1, we have

$$\begin{aligned}
&\widetilde{Q}_{w,n+r}\widetilde{Q}_{w,n+s}-\widetilde{Q}_{w,n}\widetilde{Q}_{w,n+r+s} \\
&\quad = \frac{AB}{\Delta^2}\left(-q\right)^n\underline{\alpha\beta}\left(-\alpha^*\beta^*\beta^s\left(\alpha^r-\beta^r\right)+\beta^*\alpha^*\alpha^s\left(\alpha^r-\beta^r\right)\right) \\
&\quad = \frac{AB}{\Delta}\left(-q\right)^n\underline{\alpha\beta}u_r\left(\beta^*\alpha^*\alpha^s-\alpha^*\beta^*\beta^s\right) \\
&\quad = \frac{AB}{\Delta}\left(-q\right)^n\underline{\alpha\beta}u_r\left(Q_{v,0}-\theta_{\lambda,\mu}+\Delta q\left(Q_{u,0}-\omega_{\lambda,\mu}\right)\right)\alpha^s \\
&\quad\quad -\frac{AB}{\Delta}\left(-q\right)^n\underline{\alpha\beta}u_r\left(Q_{v,0}-\theta_{\lambda,\mu}-\Delta q\left(Q_{u,0}-\omega_{\lambda,\mu}\right)\right)\beta^s \\
&\quad = AB\left(-q\right)^n\left(\widetilde{v}_0-\left(1+\mathfrak{p}q\left(1+v_2\varepsilon\right)+pqJ\varepsilon\right)\right)u_r\left(\left(Q_{v,0}-\theta_{\lambda,\mu}\right)u_s+q\left(Q_{u,0}-\omega_{\lambda,\mu}\right)v_s\right). \qquad \square
\end{aligned}$$

If we set $r, s \to m$ and $n \to n-m$ in Theorem 2.4, we get the following corollary which corresponds to Catalan's identity for dual-generalized complex Horadam generalized quaternions.

Corollary 2.1. *For nonnegative integers n and m with $n \geq m$, we have*

$$\begin{aligned}&\widetilde{Q}_{w,n-m}\widetilde{Q}_{w,n+m}-\widetilde{Q}_{w,n}^{2}\\&=-AB\left(-q\right)^{n-m}\left(\widetilde{v}_{0}-\left(1+\mathfrak{p}q\left(1+v_{2}\varepsilon\right)+pqJ\varepsilon\right)\right)u_{m}\left(\left(Q_{v,0}-\theta_{\lambda,\mu}\right)u_{m}+q\left(Q_{u,0}-\omega_{\lambda,\mu}\right)v_{m}\right).\end{aligned}$$

If we set $r = s = 1$ and $n \to n-1$ in Theorem 2.4, we get the following corollary which corresponds to Cassini's identity for dual-generalized complex Horadam generalized quaternions.

Corollary 2.2. *For positive integer n, we have*

$$\begin{aligned}&\widetilde{Q}_{w,n-1}\widetilde{Q}_{w,n+1}-\widetilde{Q}_{w,n}^{2}\\&=-AB\left(-q\right)^{n-1}\left(\widetilde{v}_{0}-\left(1+\mathfrak{p}q\left(1+v_{2}\varepsilon\right)+pqJ\varepsilon\right)\right)\left(Q_{v,0}-\theta_{\lambda,\mu}+pq\left(Q_{u,0}-\omega_{\lambda,\mu}\right)\right).\end{aligned}$$

If we set $r = 1$ and $s \to m-n$ in Theorem 2.4, we get d'Ocagne's identity for dual-generalized complex Horadam generalized quaternions.

Corollary 2.3. *For nonnegative integers n and m with $m \geq n$, we have*

$$\begin{aligned}&\widetilde{Q}_{w,n+1}\widetilde{Q}_{w,m}-\widetilde{Q}_{w,n}\widetilde{Q}_{w,m+1}\\&=AB\left(-q\right)^{n}\left(\widetilde{v}_{0}-\left(1+\mathfrak{p}q\left(1+v_{2}\varepsilon\right)+pqJ\varepsilon\right)\right)\left(\left(Q_{v,0}-\theta_{\lambda,\mu}\right)u_{m-n}+q\left(Q_{u,0}-\omega_{\lambda,\mu}\right)v_{m-n}\right).\end{aligned}$$

Next, we give some relations which are obtained by using Lemma 2.1 and the Binet formula of dual-generalized complex Horadam generalized quaternions and the Binet formulas in (4). To avoid repetition we only give the proof of the identity (i).

Theorem 2.5. *For nonnegative integers n and m such that $m \geq n$, we have*

$$\begin{aligned}(i)\ &\widetilde{Q}_{v,n}\widetilde{Q}_{u,m}-\widetilde{Q}_{v,m}\widetilde{Q}_{u,n}\\&\quad=2\left(-q\right)^{n}\left(\widetilde{v}_{0}-\left(1+\mathfrak{p}q\left(1+v_{2}\varepsilon\right)+pqJ\varepsilon\right)\right)u_{m-n}\left(Q_{v,0}-\theta_{\lambda,\mu}\right).\\(ii)\ &\widetilde{Q}_{u,n}\widetilde{Q}_{w,m}-\widetilde{Q}_{w,m}\widetilde{Q}_{u,n}\\&\quad=2\left(-q\right)^{n+1}\left(\widetilde{v}_{0}-\left(1+\mathfrak{p}q\left(1+v_{2}\varepsilon\right)+pqJ\varepsilon\right)\right)w_{m-n}\left(Q_{u,0}-\omega_{\lambda,\mu}\right).\end{aligned}$$

Proof. (i) From the Binet formula of dual-generalized complex Horadam generalized quaternions, we have

$$\begin{aligned}&\Delta\left(\widetilde{Q}_{v,n}\widetilde{Q}_{u,m}-\widetilde{Q}_{v,m}\widetilde{Q}_{u,n}\right)\\&\quad=\left(\alpha^{*}\underline{\alpha}\alpha^{n}+\beta^{*}\underline{\beta}\beta^{n}\right)\left(\alpha^{*}\underline{\alpha}\alpha^{m}-\beta^{*}\underline{\beta}\beta^{m}\right)-\left(\alpha^{*}\underline{\alpha}\alpha^{m}+\beta^{*}\underline{\beta}\beta^{m}\right)\left(\alpha^{*}\underline{\alpha}\alpha^{n}-\beta^{*}\underline{\beta}\beta^{n}\right)\\&\quad=\left(\alpha^{*}\right)^{2}\underline{\alpha}^{2}\alpha^{n+m}-\alpha^{*}\beta^{*}\underline{\alpha}\underline{\beta}\alpha^{n}\beta^{m}+\beta^{*}\alpha^{*}\underline{\alpha}\underline{\beta}\alpha^{m}\beta^{n}-\left(\beta^{*}\right)^{2}\underline{\beta}^{2}\beta^{n+m}\\&\qquad-\left(\alpha^{*}\right)^{2}\underline{\alpha}^{2}\alpha^{n+m}+\alpha^{*}\beta^{*}\underline{\alpha}\underline{\beta}\alpha^{m}\beta^{n}-\beta^{*}\alpha^{*}\underline{\alpha}\underline{\beta}\alpha^{n}\beta^{m}+\left(\beta^{*}\right)^{2}\underline{\beta}^{2}\beta^{n+m}\\&\quad=\alpha^{*}\beta^{*}\left(\alpha\beta\right)^{n}\underline{\alpha}\underline{\beta}\left(\alpha^{m-n}-\beta^{m-n}\right)+\beta^{*}\alpha^{*}\left(\alpha\beta\right)^{n}\underline{\alpha}\underline{\beta}\left(\alpha^{m-n}-\beta^{m-n}\right)\\&\quad=\left(\alpha\beta\right)^{n}\underline{\alpha}\underline{\beta}\left(\alpha^{m-n}-\beta^{m-n}\right)\left(\alpha^{*}\beta^{*}+\beta^{*}\alpha^{*}\right).\end{aligned}$$

From (5), we have

$$\begin{aligned}&\Delta\left(\widetilde{Q}_{v,n}\widetilde{Q}_{u,m}-\widetilde{Q}_{v,m}\widetilde{Q}_{u,n}\right)\\&\quad=2\left(-q\right)^{n}\left(\widetilde{v}_{0}-\left(1+\mathfrak{p}q\left(1+v_{2}\varepsilon\right)+pqJ\varepsilon\right)\right)\Delta u_{m-n}\left(Q_{v,0}-\theta_{\lambda,\mu}\right).\end{aligned}$$ □

Now, we give some summation formulas for dual-generalized complex Horadam generalized quaternions.

Theorem 2.6. *For $n \geq 2$, we have*

$$\sum_{r=1}^{n-1} \widetilde{Q}_{w,r} = \frac{\widetilde{Q}_{w,n} - \widetilde{Q}_{w,1} + q\left(\widetilde{Q}_{w,n-1} - \widetilde{Q}_{w,0}\right)}{p+q-1}.$$

Proof. From the Binet formula for dual-generalized complex Horadam generalized quaternions, we have

$$\begin{aligned}
\sum_{r=1}^{n-1} \widetilde{Q}_{w,r} &= \sum_{r=1}^{n-1} \frac{A\alpha^{*}\underline{\alpha}\alpha^{r} - B\beta^{*}\underline{\beta}\beta^{r}}{\alpha-\beta} \\
&= \frac{A\alpha^{*}\underline{\alpha}}{\alpha-\beta}\sum_{r=1}^{n-1}\alpha^{r} - \frac{B\beta^{*}\underline{\beta}}{\alpha-\beta}\sum_{r=1}^{n-1}\beta^{r} \\
&= \frac{A\alpha^{*}\underline{\alpha}}{\alpha-\beta}\left(\frac{\alpha^{n}-\alpha}{\alpha-1}\right) - \frac{B\beta^{*}\underline{\beta}}{\alpha-\beta}\left(\frac{\beta^{n}-\beta}{\beta-1}\right) \\
&= \frac{1}{(\alpha-\beta)(1-p-q)}\Big(-\left(A\alpha^{*}\underline{\alpha}\alpha^{n} - B\beta^{*}\underline{\beta}\beta^{n}\right) - q\left(A\alpha^{*}\underline{\alpha}\alpha^{n-1} - B\beta^{*}\underline{\beta}\beta^{n-1}\right) \\
&\qquad\qquad + q\left(A\alpha^{*}\underline{\alpha} - B\beta^{*}\underline{\beta}\right) + \left(A\alpha^{*}\underline{\alpha}\alpha - B\beta^{*}\underline{\beta}\beta\right)\Big) \\
&= \frac{-\widetilde{Q}_{w,n} - q\widetilde{Q}_{w,n-1} + q\widetilde{Q}_{w,0} + \widetilde{Q}_{w,1}}{1-p-q}.
\end{aligned}$$

□

Theorem 2.7. *For nonnegative integers n and r, we have*

$$\sum_{m=0}^{n} \binom{n}{m} q^{n-m} p^{m} \widetilde{Q}_{w,m+r} = \widetilde{Q}_{w,2n+r}.$$

Proof. From the Binet formula for dual-generalized complex Horadam generalized quaternions, we have

$$\begin{aligned}
\sum_{m=0}^{n} \binom{n}{m} q^{n-m} p^{m} \widetilde{Q}_{w,m+r} &= \sum_{m=0}^{n} \binom{n}{m} q^{n-m} p^{m} \left(\frac{A\alpha^{*}\underline{\alpha}\alpha^{m+r} - B\beta^{*}\underline{\beta}\beta^{m+r}}{\alpha-\beta}\right) \\
&= \frac{A\alpha^{*}\underline{\alpha}\alpha^{r}}{\alpha-\beta}\sum_{m=0}^{n}\binom{n}{m} q^{n-m}(p\alpha)^{m} - \frac{B\beta^{*}\underline{\beta}\beta^{r}}{\alpha-\beta}\sum_{m=0}^{n}\binom{n}{m} q^{n-m}(p\beta)^{m} \\
&= \frac{A\alpha^{*}\underline{\alpha}\alpha^{r}}{\alpha-\beta}(q+p\alpha)^{n} - \frac{B\beta^{*}\underline{\beta}\beta^{r}}{\alpha-\beta}(q+p\beta)^{n} \\
&= \frac{A\alpha^{*}\underline{\alpha}\alpha^{2n+r} - B\beta^{*}\underline{\beta}\beta^{2n+r}}{\alpha-\beta} = \widetilde{Q}_{w,2n+r}.
\end{aligned}$$

□

Finally, we give a matrix representation for dual-generalized complex Horadam quaternions.

Theorem 2.8. *For $n \geq 0$, we have*

$$\begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}^{n} \begin{bmatrix} \widetilde{Q}_{w,2} & \widetilde{Q}_{w,1} \\ \widetilde{Q}_{w,1} & \widetilde{Q}_{w,0} \end{bmatrix} = \begin{bmatrix} \widetilde{Q}_{w,n+2} & \widetilde{Q}_{w,n+1} \\ \widetilde{Q}_{w,n+1} & \widetilde{Q}_{w,n} \end{bmatrix}.$$

Proof. It can be proven easily by the mathematical induction on n. □

If we take the determinant of both sides of the above matrix equality, we obtain the Cassini's identity for the sequence $\left\{\widetilde{Q}_{w,n}\right\}$ in a simple way as:

$$\widetilde{Q}_{w,n+1}\widetilde{Q}_{w,n-1}-\widetilde{Q}_{w,n}^{2}=(-q)^{n-1}\left(\widetilde{Q}_{w,2}\widetilde{Q}_{w,0}-\widetilde{Q}_{w,1}^{2}\right).$$

3 Conclusion

In this paper, we define generalized quaternions with dual-generalized complex Horadam numbers. The main advantage of introducing the dual-generalized complex Horadam generalized quaternions is that many dual-generalized complex numbers with celebrated numbers such as Fibonacci, Lucas, Pell, Pell–Lucas, Jacosthal, Jacobsthal–Lucas numbers can be deduced as particular cases of these quaternions. Also, one can obtain real quaternions, split quaternions, and degenere-split quaternions. We give generating function and Binet formula for these quaternions. With the help of the Binet formula of dual-generalized Horadam generalized quaternions, we derive many properties of these quaternions such as summation formulas, binomial sum identities, Vajda's identity, Catalan's identity, Cassini identity, and d'Ocagne's identity. The algebra of generalized quaternions is noncommutative, whereas the algebra of dual-generalized complex numbers is commutative. For interested readers, the results of this paper could be applied for hyperbolic generalized complex Horadam numbers and complex-generalized complex Horadam numbers.

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