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Objects generated by an arbitrary natural number. Part 4: New aspects

Krassimir Atanassov[1,2]

[1] Department of Bioinformatics and Mathematical Modelling, Institute of Biophysics and Biomedical Engineering, Bulgarian Academy of Sciences
"Acad. G. Bonchev" Str., Bl. 105, Sofia-1113, Bulgaria
e-mail: krat@bas.bg

Intelligent Systems Laboratory, "Prof. Dr Asen Zlatarov" University
1 "Prof. Yakimov" Blvd., Burgas-8000, Bulgaria

To Tony for his $F_1 + F_6 + F_8 + F_{10}$ birthday!

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Abstract: The set $\underline{SET}(n)$, generated by an arbitrary natural number n, was defined in [3]. There, and in [5,6], some arithmetic functions and arithmetic operators of a modal and topological types are defined over the elements of $\underline{SET}(n)$. Here, over the elements of $\underline{SET}(n)$ new arithmetic functions are defined and some of their basic properties are studied. Two standard modal topological structures over $\underline{SET}(n)$ are described. Perspectives for future research are discussed.
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1 Introduction

In the middle of 2022, the author for a first time introduced the concept of a Modal Topological Structure (MTS) in [4]. In this paper, he discussed examples of such structures related to the concept of an Intuitionistic Fuzzy Set (IFS), introduced by him in 1983 (see, e.g., [2]). During the last year, a lot of other examples with IFSs were constructed by the author, but simultaneously with them, he found the first examples of MTSs in number theory (see [5, 6]).

The present paper is a continuation of [3, 5, 6]. Here, we give only the basic definitions from them and after this, in Section 2, we will introduce new arithmetic operations, related to already existing ones. They will be the foundation for new MTS, introduced in Section 3, while in Section 4 new mathematical objects, related to the results from Section 3, are discussed.

Let everywhere

$$n = \prod_{i=1}^{k} p_i^{\alpha_i} \geq 2,$$

where k, $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$ are natural numbers and $p_1, p_2, \ldots, p_k$ are different prime numbers. In [1, 3, 5, 6], the following notations related to n that we will use below, are introduced:

$$\underline{set}(n) = \{p_1, p_2, \ldots, p_k\},$$
$$\underline{mult}(n) = \prod_{i=1}^{k} p_i,$$
$$\omega(n) = k,$$

$$\underline{SET}(n) = \{m \mid m = \prod_{i=1}^{k} p_i^{\beta_i} \ \&\ \delta(n) \leq \beta_i \leq \Delta(n)\},$$

where *

$$\delta(n) = \min(\alpha_1, \ldots, \alpha_k), \qquad \boxcircle\, n = (\underline{mult}(n))^{\delta(n)},$$
$$\Delta(n) = \max(\alpha_1, \ldots, \alpha_k), \qquad \boxast\, n = (\underline{mult}(n))^{\Delta(n)},$$

for each $m \in \underline{SET}(n)$, i.e., for $m = \prod_{i=1}^{k} p_i^{\beta_i}$, where $\delta(m) \leq \beta_i \leq \Delta(m)$:

$$\neg m = \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\beta_i},$$
$$\Box\, m = (\underline{mult}(n))^{\delta(m)},$$
$$\Diamond m = (\underline{mult}(n))^{\Delta(m)},$$
$$\nabla(n) = \frac{\delta(n) + \Delta(n)}{2},$$
$$\mathcal{C}(m) = \prod_{i=1}^{k} p_i^{\max(\lceil \nabla(n) \rceil, \beta_i)},$$
$$\mathcal{I}(m) = \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor, \beta_i)},$$

where $\lfloor x \rfloor$ is the floor function, or the integer part of the real number x and

* Other authors (see, e.g. [13]) denote the functions δ and Δ by h and H, respectively.

$$\lceil x \rceil = \begin{cases} x, & \text{if } x \text{ is integer} \\ \lfloor x \rfloor + 1, & \text{otherwise} \end{cases}.$$

For the natural numbers $l, m \in \underline{SET}(n)$ with canonical forms:

$$l = \prod_{i=1}^{k} p_i^{\beta_i}, \qquad m = \prod_{i=1}^{k} p_i^{\gamma_i},$$

let us define the operations:

$$l \times m = \prod_{i=1}^{k} p_i^{\min(\beta_i+\gamma_i,\Delta(n))}.$$

$$(l, m) = \prod_{i=1}^{k} p_i^{\min(\beta_i,\gamma_i)},$$

$$[l, m] = \prod_{i=1}^{k} p_i^{\max(\beta_i,\gamma_i)},$$

because

$$\delta(n) \leq \min(\beta_i, \gamma_i) \leq \max(\beta_i, \gamma_i) \leq \Delta(n).$$

In some sense, operators $\mathcal{C}$ and $\mathcal{I}$ are analogues of the topological operators "closure" and "interior", see, e.g., [8, 12].

While in [3, 5] it was shown that the new objects have properties specific to algebra and for modal logic[†], respectively, here we will discuss their topological properties.

2 New arithmetic functions, defined in $\underline{SET}(n)$

Let us define for the natural numbers $m \in \underline{SET}(n)$ and $s \in [\delta(n), \Delta(n)]$:

$$P_s(l) = \prod_{i=1}^{k} p_i^{\max(s,\beta_i)}, \qquad Q_s(l) = \prod_{i=1}^{k} p_i^{\min(s,\beta_i)}.$$

Having in mind that by the definition

$$\delta(n) \leq \min(s, \beta_i) \leq \max(s, \beta_i) \leq \Delta(n)$$

we see that $P_s(l), Q_s(l) \in \underline{SET}(n)$.

Theorem 1. For each $l \in \underline{SET}(n)$ and $s \in [\delta(n), \Delta(n)]$:

$$\neg P_s(\neg l) = Q_{\Delta(n)+\delta(n)-s}(l), \tag{1}$$

$$\neg Q_s(\neg l) = P_{\Delta(n)+\delta(n)-s}(l). \tag{2}$$

[†] As we discussed in [5], operators $\Box$ and $\Diamond$ are in some sense analogues of the modal operators "necessity" and "possibility", respectively, see, e.g., [7, 9, 10].

Proof. Let $l \in \underline{SET}(n)$ and $s \in [\delta(n), \Delta(n)]$. Then for (1) we obtain:

$$\begin{aligned}
\neg P_s(\neg l) &= \neg P_s\left(\neg \prod_{i=1}^{k} p_i^{\beta_i}\right) \\
&= \neg P_s\left(\prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\beta_i}\right) \\
&= \neg \prod_{i=1}^{k} p_i^{\max(s,\Delta(n)+\delta(n)-\beta_i)} \\
&= \prod_{i=1}^{k} p_i^{\Delta(n)+\delta(n)-\max(s,\Delta(n)+\delta(n)-\beta_i)} \\
&= \prod_{i=1}^{k} p_i^{\min(\Delta(n)+\delta(n)-s,\beta_i)} \\
&= Q_{\Delta(n)+\delta(n)-s}(l).
\end{aligned}$$

The proof of (2) is similar. □

Theorem 2. For the natural numbers $m \in \underline{SET}(n)$ and $s, t \in [\delta(n), \Delta(n)]$:

$$P_s(P_t(l)) = P_{\max(s,t)}(l), \tag{3}$$

$$P_s(Q_t(l)) = Q_{\max(s,t)}P_s(l), \tag{4}$$

$$Q_s(Q_t(l)) = Q_{\min(s,t)}(l), \tag{5}$$

$$Q_s(P_t(l)) = P_{\min(s,t)}Q_s(l). \tag{6}$$

Proof. Let the natural numbers $l \in \underline{SET}(n)$ and $s, t \in [\delta(n), \Delta(n)]$. Then for (3) we obtain:

$$\begin{aligned}
P_s(P_t(l)) &= P_s\left(\prod_{i=1}^{k} p_i^{\max(t,\beta_i)}\right) \\
&= \prod_{i=1}^{k} p_i^{\max(s,\max(t,\beta_i))}) \\
&= \prod_{i=1}^{k} p_i^{\max(\max(s,t),\beta_i))}) \\
&= P_{\max(s,t)}(l).
\end{aligned}$$

For (4) we obtain:

$$\begin{aligned}
P_s(Q_t(l)) &= P_s(\prod_{i=1}^{k} p_i^{\min(t,\beta_i)}) \\
&= \prod_{i=1}^{k} p_i^{\max(s,\min(t,\beta_i))} \\
&= \prod_{i=1}^{k} p_i^{\min(\max(s,t),\max(s,\beta_i))} \\
&= Q_{\max(s,t)}\left(\prod_{i=1}^{k} p_i^{\max(s,\beta_i)}\right) \\
&= Q_{\max(s,t)}(P_s(l)).
\end{aligned}$$

The proofs of (5) and (6) are similar. □

Finally, we must mention that for the natural numbers $l \in \underline{SET}(n)$ and $s \in [\delta(n), \Delta(n)]$ the inequalities:

$$Q_s(l) \leq l \leq P_s(l) \tag{7}$$

and the equalities

$$\begin{aligned} P_{\delta(n)}(l) &= l, \\ P_{\Delta(n)}(l) &= \boxast n, \\ Q_{\delta(n)}(l) &= \boxcircle n, \\ Q_{\Delta(n)}(l) &= l, \end{aligned}$$

hold.

Therefore, the two new operators have properties that are similar to these of the two topological operators, discussed in [3, 5, 6]. Moreover, we can see immediately that

$$\mathcal{C}(l) = \prod_{i=1}^{k} p_i^{\max(\lceil \nabla(n) \rceil, \beta_i)} = P_{\lceil \nabla(n) \rceil}(l),$$

$$\mathcal{I}(l) = \prod_{i=1}^{k} p_i^{\min(\lfloor \nabla(n) \rfloor, \beta_i)} = Q_{\lfloor \nabla(n) \rfloor}(l),$$

i.e., the new topological operators are extensions of the previous ones.

In the next Section, we will modify the results from [6] in the sense of the new operators.

3 Two standard modal topological structures over $\underline{SET}(n)$

Below, we use the definitions for a Modal Topological Structures (MTSs) from [4, 6] and will prove two theorems.

Theorem 3. Let n, s be fixed natural numbers and let $s \in [\delta(n), \Delta(n)]$. Then $\langle \underline{SET}(n), P_s, (.), \diamondsuit, \boxast n \rangle$ is a cl-MTS.

Proof. Let $l, m \in \underline{SET}(n)$. We check sequentially the conditions C1–C3 and C9, while the checks of conditions C5–C8 coincide with these from [6], Theorem 3 and the validity of condition C4 follows from (3).

$$\begin{aligned} \text{C1:} \quad P_s((l, m)) &= P_s(\prod_{i=1}^{k} p_i^{\min(\beta_i, \gamma_i)}) \\ &= \prod_{i=1}^{k} p_i^{\max(s, \min(\beta_i, \gamma_i))} \\ &= \prod_{i=1}^{k} p_i^{\min(\max(s, \beta_i), \max(\max(s, \gamma_i))} \\ &= \left(\prod_{i=1}^{k} p_i^{\max(s, \beta_i)}, \prod_{i=1}^{k} p_i^{\max(s, \gamma_i)} \right) \\ &= (P_s(l), P_s(m)); \end{aligned}$$

$$
\begin{aligned}
\text{C2:} \qquad P_s(l) &= \prod_{i=1}^{k} p_i^{\max(s,\beta_i)} \\
&\geq \prod_{i=1}^{k} p_i^{\beta_i} = l; \\
\text{C3:} \quad P_s(\boxplus n) &= P_s((\underline{mult}(n))^{\Delta(n)}) \\
&= (\underline{mult}(n))^{\max(s,\Delta(n))} \\
&= (\underline{mult}(n))^{\Delta(n)} = \boxplus n;
\end{aligned}
$$

$$
\begin{aligned}
\text{C9:} \quad \Diamond P_s(l) &= \Diamond \prod_{i=1}^{k} p_i^{\max(s,\beta_i)} \\
&= \prod_{i=1}^{k} p_i^{\max\limits_{1\leq i\leq k}(\max(s,\beta_i))} \\
&= \prod_{i=1}^{k} p_i^{\max(s,\max\limits_{1\leq i\leq k}\beta_i)} \\
&= \prod_{i=1}^{k} p_i^{\max(s,\Delta(l))} \\
&= P_s\left(\prod_{i=1}^{k} p_i^{\Delta(l))}\right) = \mathcal{P}(\Diamond l).
\end{aligned}
$$

Therefore, Theorem 3 is valid. $\square$

Theorem 4. Let n, s be fixed natural numbers and let $s \in [\delta(n), \Delta(n)]$. Then $\langle \underline{SET}(n), Q_s, [.], \square, \boxdot n\rangle$ is a in-MTS.

Proof. Let $l, m \in \underline{SET}(n)$. We check sequentially the conditions I1–I3 and I9, while the checks of conditions I5–I8 coincide with these from [6], Theorem 4 and the validity of condition C4 follows from (5).

$$
\begin{aligned}
\text{I1:} \quad Q_s([l,m]) &= Q_s(\prod_{i=1}^{k} p_i^{\max(\beta_i,\gamma_i)}) \\
&= \prod_{i=1}^{k} p_i^{\min(s,\max(\beta_i,\gamma_i))} \\
&= \prod_{i=1}^{k} p_i^{\max(\min(s,\beta_i),\min(s,\gamma_i))} \\
&= \left[\prod_{i=1}^{k} p_i^{\min(s,\beta_i)}, \prod_{i=1}^{k} p_i^{\min(s,\gamma_i)}\right] \\
&= [Q_s(l), Q_s(m)];
\end{aligned}
$$

$$
\begin{aligned}
\text{I2:} \qquad Q_s(l) &= \prod_{i=1}^{k} p_i^{\min(s,\beta_i)} \\
&\leq \prod_{i=1}^{k} p_i^{\beta_i} = l; \\
\text{I3:} \quad Q_s(\boxdot(l)) &= Q_s((\underline{mult}(n))^{\delta(l)}) \\
&= (\underline{mult}(n))^{\min(s,\delta(l))} \\
&= (\underline{mult}(n))^{\delta(l)} = \boxdot(l);
\end{aligned}
$$

$$
\begin{aligned}
\text{I9:} \quad \Box\, Q_s(l) &= \Box \prod_{i=1}^{k} p_i^{\min(s,\beta_i)} \\
&= \prod_{i=1}^{k} p_i^{\min\limits_{1\leq i\leq k}(\min(s,\beta_i))} \\
&= \prod_{i=1}^{k} p_i^{\min(s,\min\limits_{1\leq i\leq k}\beta_i)} \\
&= \prod_{i=1}^{k} p_i^{\min(s,\delta(l))} \\
&= Q_s\left(\prod_{i=1}^{k} p_i^{\delta(l))}\right) = Q_s(\Box\, l).
\end{aligned}
$$

Therefore, Theorem 4 is valid. $\Box$

4 New aspects of the research over $\underline{SET}(n)$

Obviously, for each natural number n, when $\Delta(n) - \delta(n) = 0$,

$$\underline{SET}(n) = \{n\}$$

and all research in [3, 5, 6] do not have sense; when $\Delta(n) - \delta(n)$ equals 1 or 2, we can already construct a topological structure, but it will be very trivial and in practice, operators $\mathcal{P}_s$ and $\mathcal{Q}_s$ will be trivial ones.

Therefore, it will be interesting to discuss the case when $\Delta(n) - \delta(n) \geq 3$. Now, over set $\underline{SET}(n)$ we can define at least two *different* cl-MTS $\langle \underline{SET}(n), P_s, (.), \Diamond, \boxtimes\, n\rangle$ and $\langle \underline{SET}(n), P_t, (.), \Diamond, \boxtimes\, n\rangle$, and two *different* in-MTS $\langle \underline{SET}(n), Q_s, [.], \Box, \boxdot\, n\rangle$ and $\langle \underline{SET}(n), Q_t, [.], \Box, \boxdot\, n\rangle$, where $s, t \in [\delta(n), \Delta(n)]$ and $s < t$. Therefore, we can define

$$
\begin{aligned}
\mathcal{SET}(n) = \quad & \{\langle \underline{SET}(n), P_s, (.), \Diamond, \boxtimes\, n\rangle | s \in [\lfloor \tfrac{\Delta(n)-\delta(n)}{2} \rfloor + 1, \Delta(n) - \delta(n) - 1]\} \\
& \cup \{\langle \underline{SET}(n), Q_s, [.], \Box, \boxdot\, n\rangle | s \in [\delta(n) + 1, \lfloor \tfrac{\Delta(n)-\delta(n)}{2} \rfloor]\}.
\end{aligned}
$$

Since the components "(.)", "$\Diamond$", "$\boxtimes\, n$", "[.]", "$\Box$", "$\boxdot\, n$" depend of "P_s" and "Q_s", respectively, for brevity, the above 5-tuples can be denoted as $\langle \underline{SET}(n), P_s\rangle$ and $\langle \underline{SET}(n), Q_s\rangle$.

Let V and W are some of the symbols P and Q (equal or different) and let $v, w \in [\delta(n) + 1, \Delta(n) - 1]$. Now, we can define for two elements of set $\mathcal{SET}(n)$ the relation of strong order as follows

$$\langle \underline{SET}(n), V_v \rangle \ll \langle \underline{SET}(n), W_w \rangle \text{ if and only if } v < w. \tag{8}$$

Theorem 5. For each fixed natural number n, $\langle \mathcal{SET}(n), \ll \rangle$ is strong ordered.

Proof. Let n be a fixed natural number. First, we see that from (8) they follow that

- for $r, s \in [\delta(n) + 1, \lfloor \frac{\Delta(n)-\delta(n)}{2} \rfloor]$ and $r < s$:

$$\langle \underline{SET}(n), Q_r, [.], \square\,, \boxcircle\, n \rangle \ll \langle \underline{SET}(n), Q_s, [.], \square\,, \boxcircle\, n \rangle;$$

- for $r, s \in [\lfloor \frac{\Delta(n)-\delta(n)}{2} \rfloor, \Delta(n) - 1]$ and $r < s$:

$$\langle \underline{SET}(n), P_r, (.), \Diamond, \boxast\, n \rangle \ll \langle \underline{SET}(n), P_s, (.), \Diamond, \boxast\, n \rangle;$$

- for $r \in [\delta(n) + 1, \lfloor \frac{\Delta(n)-\delta(n)}{2} \rfloor]$ and $s \in [\lfloor \frac{\Delta(n)-\delta(n)}{2} \rfloor, \Delta(n) - 1]$ (obviously, $r < s$):

$$\langle \underline{SET}(n), Q_r, [.], \square\,, \boxcircle\, n \rangle \ll \langle \underline{SET}(n), P_s, (.), \Diamond, \boxast\, n \rangle.$$

Second, the set $\mathcal{SET}(n)$ is a finite one and hence the interval $[\delta(n), \Delta(n)]$, as well.

The minimal element of $\mathcal{SET}(n)$ is $\langle \underline{SET}(n), Q_{\delta(n)+1}, [.], \square\,, \boxcircle\, n \rangle$, its maximal element is $\langle \underline{SET}(n), P_{\Delta(n)-1}, (.), \Diamond, \boxast\, n \rangle$, and all its elements are strong ordered. $\square$

Now, we will cite a part of the Conclusion of [4]. *"If we, using the terminology from* [11, 14]*, call* "maps" *the two types of MTSs, [...], then all these maps, based on a fixed universe will generate an* "atlas". In all research over MTSs by the moment, there is not an example of an atlas. The idea from the present Section is the first particular illustration of the idea for maps and atlas. $\mathcal{SET}(n)$ is an atlas, its elements are maps, and that is very interesting, this atlas is paginated, because the strong order of the elements of $\mathcal{SET}(n)$ can be used as a pagination, starting with $\delta(n)$ and finishing with $\Delta(n)$.

5 Conclusion

In the present paper, new examples for a MTS are given – here, over set $\underline{SET}(n)$. The idea for a set of MTSs $\mathcal{SET}(n)$ and a relation over its elements is discussed.

In the next part of the research, a new set of $\underline{SET}(n)$ for different natural numbers n will be constructed and its properties will be studied.

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