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The 2-successive partial Bell polynomials

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Abstract: In this paper, we discuss a new class of partial Bell polynomials. The first section gives an overview of partial Bell polynomials and their related 2-successive Stirling numbers. In the second section, we introduce the concept of 2-successive partial Bell polynomials. We give an explicit formula for computing these polynomials and establish their generating function. In addition, we derive several recurrence relations that govern the behaviour of these polynomials. Furthermore, we study specific cases to illustrate the applicability and versatility of this new class of polynomials.

Keywords: 2-successive associated Stirling numbers, Exponential partial Bell polynomials, Generating function.
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1 Introduction

Stirling numbers have proven to be invaluable in various branches of mathematics such as algebra, geometry, and combinatorics. They play a crucial role in counting partitions and permutations of sets, and their applications extend to the study of degenerate versions of special numbers and polynomials. Researchers have been particularly interested in exploring generalizations and special cases of Stirling numbers, including degenerate Stirling numbers and r-Stirling numbers, which have been extensively investigated (see [8]). In line with this research, Belbachir and Tebtoub [1] introduced a novel class of Stirling numbers known as the 2-successive associated Stirling numbers, denoted by $\left\{ {n \atop k} \right\}^{[2]}$, counting the number of partitions of the set $[n] := \{1, 2, \ldots, n\}$ into k non-empty blocks, so that each block contains at least two consecutive numbers. Moreover, the last element n must either form a part with its predecessor or belong to another part satisfying the previous property. The ordinary generating function of these numbers is to be

$$\sum_{n\geq 2k} \left\{ {n \atop k} \right\}^{[2]} x^n = \frac{x^{2k}}{(1-x)(1-2x)\cdots(1-kx)}.$$

They satisfy the following recurrence relation, for $n \geq 2k \geq 0$

$$\left\{ {n \atop k} \right\}^{[2]} = k \left\{ {n-1 \atop k} \right\}^{[2]} + \left\{ {n-2 \atop k-1} \right\}^{[2]},$$

with

$$\left\{ {0 \atop 0} \right\}^{[2]} = 1, \ \left\{ {n \atop 1} \right\}^{[2]} = 1 \ (n \geq 2), \ \left\{ {n \atop n-1} \right\}^{[2]} = 0 \ \text{ and } \ \left\{ {n \atop 0} \right\}^{[2]} = 0 \ (n \geq 1).$$

The following Table 1 presents the first values for those numbers.

Table 1. The first values of the 2-successive associated Stirling numbers

$n \backslash k$	0	1	2	3	4	5
0	1					
1	0					
2	0	1				
3	0	1				
4	0	1	1			
5	0	1	3			
6	0	1	7	1		
7	0	1	15	6		
8	0	1	31	25	1	
9	0	1	63	90	10	
10	0	1	127	301	65	1

On the other side, an important tool employed in this manuscript is the partial exponential Bell polynomial. These polynomials have been extensively investigated by various researchers (see [2–4]). These polynomials are a class of special polynomials in combinatorial analysis that find widespread applications in mathematics. They are defined as follows:

$$\sum_{n\geq k} B_{n,k}(\varphi) \frac{t^n}{n!} = \frac{1}{k!} \left(\varphi\left(x\right)\right)^k, \quad \varphi\left(x\right) = \sum_{l\geq 1} a_l \frac{x^l}{l!},$$

with

$$B_{n,k}(\varphi) = B_{n,k}(a_j) := B_{n,k}(a_1, a_2, \ldots),$$

and are given explicitly by

$$B_{n,k}(\varphi) = \sum_{\pi(n,k)} \frac{n!}{k_1! \cdots k_n!} \left(\frac{a_1}{1!}\right)^{k_1} \left(\frac{a_2}{2!}\right)^{k_2} \cdots ,$$

when

$$\pi(n,k) = \left\{ K := (k_1, k_2, \ldots) : k_j \in \mathbb{N}, \sum_{i\geq 1} k_i = k \text{ and } \sum_{i\geq 1} ik_i = n \right\}.$$

The exponential partial Bell polynomials reduce to some special combinatorial sequences when the variables a_j are appropriately chosen. We notice the following exceptions:

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= B_{n,k}(0!, 1!, 2!, \ldots) &&: \text{ unsigned Stirling numbers of the first kind,} \\
\begin{Bmatrix} n \\ k \end{Bmatrix} &= B_{n,k}(1, 1, 1, \ldots) &&: \text{ Stirling numbers of the second kind,} \\
\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor &= B_{n,k}(1!, 2!, 3!, \ldots) &&: \text{ Lah numbers,} \\
\binom{n}{k} k^{n-k} &= B_{n,k}(1, 2, 3, \ldots) &&: \text{ idempotent numbers.}
\end{aligned}$$

The above polynomials have been considered as important combinatorial tools and have been applied in many different contexts, including: the development of complementary inverse relations for binomial polynomials [9–11], the properties of complete and degenerate partial Bell polynomials [7], the establishment of new properties of polynomials such as the degenerate Fubini polynomials [6], and many other topics. This wide application of Bell polynomials inspired the further development of this mathematical tool.

Motivated by this work and the work of Belbachir and Tebtoub [1], in this manuscript we propose a new family of special numbers called 2-successive partial Bell polynomials denoted by $B_{n,k}^{[2]}(a_l)$. We prove properties of these numbers, derive some identities and discuss some special cases. This family includes well-known numbers and polynomials such as Stirling numbers, Bell numbers and polynomials, and so on. We study their properties using generating functions and combinatorial interpretations.

2 The 2-successive partial Bell polynomials

We begin in this section by identifying the 2-successive exponential partial Bell polynomials.

Definition 2.1. *Let the* 2*-successive associated partial Bell polynomial*

$$B_{n,k}^{[2]}(a_2, a_3, a_4, \ldots)$$

be defined by the number of coloured partitions of the set $[n]$ *into* k *non-empty parts, such that each part contains at least* 2 *consecutive numbers, and any block of cardinality* i *may be coloured with* a_i *colors.*

Theorem 2.1. *For $n \geq 2k \geq 2$, the 2-successive partial Bell polynomials $B_{n,k}^{[2]}(a_l)$ can be written as*

$$B_{n,k}^{[2]}(a_2, a_3, \ldots) = \frac{(n-k)!}{k!} \sum_{n_1+\cdots+n_k=n-2k} \frac{a_{n_1+2} \cdots a_{n_k+2}}{n_1! \cdots n_k!}.$$

Proof. To partition a set $[n]$ into k blocks $B_1, \ldots, B_k$ according to Definition 2.1, for fixed elements $i_1 \in B_1, \ldots, i_k \in B_k$, we pick the predecessor of i_j $(j = 1, \ldots, k)$, must be in B_j, i.e.,

$$\{i_1, i_1 + 1\} \subset B_1, \ldots, \{i_k, i_k + 1\} \subset B_k.$$

The elements $i_1, \ldots, i_k$ are all different and different of n and can be chosen in $\binom{n-k}{k}$ ways.

From the $n - 2k$ remaining elements, we choose n_1 elements to be in $B_1, \ldots, n_k$ elements to be in B_k. Then the total number of these partitions is $\binom{n-2k}{n_1,\ldots,n_k}$ and using colors, the number must be

$$\binom{n-2k}{n_1, \ldots, n_k} a_{n_1+2} \cdots a_{n_k+2}.$$

So, the total number of these partitions is

$$B_{n,k}^{[2]}(a_2, a_3, \ldots) = \binom{n-k}{k} \sum_{n_1+\cdots+n_k=n-2k} \binom{n-2k}{n_1, \ldots, n_k} a_{n_1+2} \cdots a_{n_k+2}. \qquad \square$$

From a simple technological approach to this theorem, we may get the following corollary.

Corollary 2.1. *The 2-successive partial Bell numbers have the generating function*

$$\sum_{n \geq k} B_{n+k,k}^{[2]}(a_2, a_3, \ldots) \frac{x^n}{n!} = \frac{x^k}{k!} \left(\varphi''(x)\right)^k.$$

Proof. From Theorem 2.1, we obtain

$$\begin{aligned}
\sum_{n \geq k} B_{n+k,k}^{[2]}(a_2, a_3, \ldots) \frac{x^n}{n!} &= \frac{1}{k!} \sum_{n \geq k} \left(\sum_{n_1+\cdots+n_k=n-k} \frac{a_{n_1+2} \ldots a_{n_k+2}}{n_1! \cdots n_k!} \right) x^n \\
&= \frac{x^k}{k!} \sum_{j \geq 0} x^j \sum_{n_1+\cdots+n_k=j} \frac{a_{n_1+2} \ldots a_{n_k+2}}{n_1! \cdots n_k!} \\
&= \frac{x^k}{k!} \sum_{n_1,\ldots,n_k \geq 0} \frac{a_{n_1+2} x^{n_1}}{n_1!} \frac{a_{n_2+2} x^{n_2}}{n_2!} \cdots \frac{a_{n_k+2} x^{n_k}}{n_k!} \\
&= \frac{x^k}{k!} \left(\sum_{j \geq 0} \frac{a_{j+2} x^j}{j!} \right)^k \\
&= \frac{x^k}{k!} \left(\varphi''(x)\right)^k. \qquad \square
\end{aligned}$$

In the following, we discuss numerous special cases of $B_{n,k}^{[2]}(a_j)$ using previously established numbers. In this context, we present a proposition that proves a connection between the 2-successive partial Bell polynomials and the classical partial Bell polynomials.

Proposition 2.1. *For $n \geq 2k$ and $k \geq 1$, we have*

$$B_{n+k,k}^{[2]}(a_2, a_3, \ldots) = B_{n,k}(j a_{j+1}).$$

Proof. By Corollary 2.1, we obtain

$$\begin{aligned}\sum_{n\geq k} B^{[2]}_{n+k,k}(a_2,a_3,\ldots)\frac{x^n}{n!} &= \frac{x^k}{k!}\left(\sum_{j\geq 0} a_{j+2}\frac{x^j}{j!}\right)^k \\ &= \frac{1}{k!}\left(\sum_{j\geq 1} ja_{j+1}\frac{x^j}{j!}\right)^k \\ &= \sum_{n\geq k} B_{n,k}(ja_{j+1})\frac{x^n}{n!}.\end{aligned}$$

Therefore, through identification, we can deduce the desired identity. □

For $a_j = 1$, $j \geq 2$, we get

$$B^{[2]}_{n+k,k}(1,1,1,1,\ldots) = \binom{n}{k} k^{n-k}.$$

For $a_j = \frac{1}{j-1}$, $j \geq 2$, we get

$$B^{[2]}_{n,k}\left(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\right) = \left\{ {n-k \atop k} \right\}.$$

For $a_j = \frac{(j-2)!}{j-1}$, $j \geq 2$, we get

$$B^{[2]}_{n,k}\left(\frac{0!}{1},\frac{1!}{2},\frac{2!}{3},\ldots\right) = \left[{n-k \atop k} \right].$$

For $a_j = (j-2)!$, $j \geq 2$, we get

$$B^{[2]}_{n,k}(0!,1!,2!,\ldots) = \left\lfloor {n-k \atop k} \right\rfloor.$$

Proposition 2.2. *The* 2*-successive partial Bell numbers $B^{[2]}_{n,k}(a_j)$ verify the following recurrence relation*

$$B^{[2]}_{n,k}(a_j) = \sum_{i=0}^{n}\binom{n}{i} a_2^i B^{[2]}_{n-2i,k-i}(a_{j+1}).$$

Proof. To partition a set $[n]$ into k blocks with respect to Definition 2.1, first let us choose let i elements and their predecessors from n to formed i blocks of the same cardinality 2, so the number of these colored blocks is $\binom{n}{i}a_2^i$. Then, the remaining $n-2i$ elements can be partitioned into $k-i$ colored partitions with the condition that each block contains at least 2 consecutive numbers, so there are $B^{[2]}_{n-2i,k-i}(a_j)$ ways to do it. Then, the total number of all partitions is $\sum\limits_{i=0}^{n}\binom{n}{i} a_2^i B^{[2]}_{n-2i,k-i} B^{[2]}_{n,k}(a_{j+1})$. □

Corollary 2.2. *Let $n,k,j \in \mathbb{N}$ such that $n \geq 2k$ and $j \geq 1$. We have*

$$B_{n-k,k}(ja_{j+1}) = \sum_{i=0}^{k}\binom{n-2i}{i} a_2^i B_{n-k-2i,k-i}(ja_{j+2}).$$

For $a_j = \frac{1}{j-1}$, $j \geq 2$, we get

$$\left\{ {n-k \atop k} \right\} = \sum_{i=0}^{k} \binom{n-2i}{i} \left\{ {n-k-2i \atop k} \right\}.$$

For $a_j = \frac{(j-2)!}{j-1}$, $j \geq 2$, we get

$$\left[{n-k \atop k} \right] = \sum_{i=0}^{k} \binom{n-2i}{i} \left[{n-k-2i \atop k} \right].$$

For $a_j = (j-2)!$, $j \geq 2$, we get:

$$\left\lfloor {n-k \atop k} \right\rfloor = \sum_{i=0}^{k} \binom{n-2i}{i} \left\lfloor {n-k-2i \atop k} \right\rfloor.$$

The next proposition is inspired by the work of Djurdje Cvijović [5], which gives an explicit closed-form formula for the partial Bell polynomials $B_{n,k}$ and shows that they admit a new recurrence relation.

Proposition 2.3. *For $n \geq 2k$ and $k \geq 1$, we have*

$$B^{[2]}_{n-k,k}(a_j) = \frac{1}{a_2} \cdot \frac{1}{n-2k} \sum_{\alpha=1}^{n-2k} [(\alpha+1)(k+1) - n - k + 1] \frac{a_{\alpha+2}}{\alpha!} B^{[2]}_{n-k-\alpha,k}(a_j).$$

Proof. First note that

$$\left(\sum_{n=1}^{\infty} f_n x^n \right)^k = \sum_{n=k}^{\infty} g_n(k) x^n.$$

For a fixed positive integer k, we have:

$$g_k(k) = f_1^k, \tag{1}$$

$$g_n(k) = \frac{1}{(n-k)f_1} \sum_{\alpha=1}^{n-k} [(\alpha+1)(k+1) - (n+1)] f_{\alpha+1} g_{n-\alpha}(k) \quad (n \geq k+1). \tag{2}$$

Conversely, we can deduce from the definition of $B^{[2]}_{n,k}$

$$\sum_{n \geq 2k} B^{[2]}_{n,k}(a_j) \frac{x^n}{(n-2k)! \binom{n-k}{k}} = \sum_{n \geq k} \binom{n}{k}^{-1} B^{[2]}_{n+k,k}(a_j) \frac{x^{n+k}}{(n-k)!}$$
$$= \left(\sum_{j \geq 1} a_{j+1} \frac{x^{j+1}}{(j-1)!} \right)^k.$$

Then, by choosing the following values of f_n and $g_n(k)$ in the equation (1)

$$f_n = \sum_{n \geq 1} a_{n+1} \frac{x^{n+1}}{(n-1)!}$$

and

$$g_n(k) = \frac{B^{[2]}_{n+k,k}}{\binom{n}{k}(n-k)!}.$$

According to the equation (2), we get the result. □

3 Conclusion

In this paper we have thoroughly investigated and established significant connections, closed-form formulas and combinatorial identities related to partial Bell polynomials and 2-successive partial Bell polynomials. These results are of great importance and applicability in combinatorial number theory and various other areas of mathematics. The results presented in this paper contribute to a deeper understanding of these polynomials and provide valuable tools for further research and applications in related fields.

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