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# Enumeration of cyclic vertices and components over the congruence $a^{11} \equiv b \pmod{n}$

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**Abstract:** For each positive integer $n$, we assign a digraph $\Gamma(n, 11)$ whose set of vertices is $Z_n = \{0, 1, 2, \ldots, n-1\}$ and there exists exactly one directed edge from the vertex $a$ to the vertex $b$ iff $a^{11} \equiv b \pmod{n}$. Using the ideas of modular arithmetic, cyclic vertices are presented and established for $n = 3^k$ in the digraph $\Gamma(n, 11)$. Also, the number of cycles and the number of components in the digraph $\Gamma(n, 11)$ is presented for $n = 3^k, 7^k$ with the help of Carmichael's lambda function. It is proved that for $k \geq 1$ , the number of components in the digraph $\Gamma(3^k, 11)$ is $(2k+1)$ and for $k > 2$ the digraph $\Gamma(3^k, 11)$ has $(k-1)$ non-isomorphic cycles of length greater than 1, whereas the number of components of the digraph $\Gamma(7^k, 11)$ is $(8k-3)$.

**Keywords:** Digraph, Fixed point, Power digraph, Carmichael $\lambda$-function, Cycles, Components.
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# 1 Introduction

For each pair of integers $n$ $(\geq 1)$ and $k$ $(\geq 1)$, a power digraph modulo $n$ denoted by $\Gamma(n,k)$ is a digraph with vertex set $Z_n = \{0,1,2,\ldots,n-1\}$ and the ordered pair $(a,b)$ is a directed edge of $\Gamma(n,k)$ from $a$ to $b$ if and only if $a^k \equiv b \pmod{n}$, where $a,b \in Z_n$(for detail see [15]). It has evolved a powerful connection between number theory, group theory and graph theory. The concept of power digraph was introduced in the year 1967 by Bryant [2] in which for each element $x$ of a finite group, a point $P(x)$ was assigned and joined with $P(x^2)$ by a directed line arising a graph called the graph of the group. Thereafter, Blanton et al. [1], Somer and Křížek [7, 13], Szalay [16], Rogers [11] have considered and investigated the properties of a variety of power digraphs corresponding to the quadratic congruence $a^2 \equiv b \pmod{n}$. In 1996, Lucheta et al. [8] studied the power digraph using the congruence $a^k \equiv b \pmod{p}$, where $k \geq 2$ is a positive integer and $p$ is a prime. Skowronek- Kaziw et al. [12], Rahmati [10] defined the power digraphs using the congruences $a^3 \equiv b \pmod{n}$ and $a^5 \equiv b \pmod{n}$ respectively and established some results. Mateen et al. [9] discussed the power digraph with the help of the congruence $a^7 \equiv b \pmod{n}$ and gave explicit formula for fixed points and the condition for which the digraph has exactly 7 components.

It is important to mention that the problem of enumeration of cyclic vertices, cycles and components of power digraph $\Gamma(n,k)$ is still open. In this paper, we try to enumerate the cyclic vertices and components with respect to the congruence $a^{11} \equiv b \pmod{n}$ for $n = 3^k$. We organize our paper as follows:

In Section 2, we provide some definition from graph theory and number theory. In Section 3, we have discussed some properties of Carmichael lambda function. Finally, in Section 4, we have tried to formulate an explicit formula to enumerate the cyclic vertices and components of the digraphs $\Gamma(3^k,11)$ and $\Gamma(7^k,11)$, where $k$ is any positive number. Throughout the paper all notations are usual. For example, the greatest common divisor of two integers $m$ and $n$ is denoted by $\gcd(m,n)$, the order of $a$ modulo $n$ is denoted by $\mathrm{ord}_n(a)$ etc.

# 2 Preliminaries

For a positive integer $n$, $Z_n = \{0,1,2,\ldots,n-1\}$ denotes the complete set of residues modulo $n$. We consider a directed graph $\Gamma(n,11)$ whose vertex set is $Z_n$ and any two vertices $a,b \in Z_n$ are connected by exactly one directed edge from $a$ to $b$ iff

$$a^{11} \equiv b \pmod{n} \tag{2.1}$$

The distinct vertices $a_1,a_2,a_3,\ldots,a_t$ in $Z_n$ will form a cycle of length $t$ if

$$\begin{aligned} a_1^{11} &\equiv a_2 \pmod{n}\\ a_2^{11} &\equiv a_3 \pmod{n}\\ a_3^{11} &\equiv a_4 \pmod{n}\\ &\vdots\\ a_t^{11} &\equiv a_1 \pmod{n} \end{aligned}$$

We call a cycle of length $t$ as a $t$-cycle and a cycle of length $1$ is named as a fixed point. A vertex is isolated if it is not connected to any other vertex in $\Gamma(n, 11)$.

The indegree of a vertex $a \in Z_n$, denoted by $\mathrm{indeg}(a)$ is the number of directed edges coming into the vertex $a$ and the outdegree of a vertex $a$, denoted by $\mathrm{outdeg}(a)$ is the number of directed edges leaving the vertex $a$. Since the residue of a number modulo $n$ is unique, so $\mathrm{outdeg}(a) = 1$ and $\mathrm{indeg}(a) \geq 0$ for each vertex $a \in Z_n$. Also, for an isolated fixed point $a \in Z_n$, $\mathrm{outdeg}(a) = \mathrm{indeg}(a) = 1$.

A component of a digraph is a subdigraph which is a maximal connected subgraph of the associated nondirected graph (for details see [5]). As the outdegree of each vertex of the digraph $\Gamma(n, 11)$ is equal to $1$, so the number of components of $\Gamma(n, 11)$ is equal to the number of all cycles. The cycles may or may not be isolated. Moreover, by using the properties of the congruence relation (2.1) and the definition of cycle, it can be established that the digraph $\Gamma(k^{11^t} - k, 11)$ has a $t$-cycle containing the vertex $k$, for an arbitrary integer $t \geq 1$.

For $n > 1$, let $\Gamma_1(n, 11)$ and $\Gamma_2(n, 11)$ be two subdigraphs of the digraph $\Gamma(n, 11)$, where $\Gamma_1(n, 11)$ is the subdigraph induced on the set of vertices $a \in Z_n$ such that $\gcd(a, n) = 1$ and $\Gamma_2(n, 11)$ is the subdigraph induced on the set of vertices $a \in Z_n$ such that $\gcd(a, n) \neq 1$. Clearly, the vertex set of $\Gamma_1(n, 11)$ is the unit group $Z_n^*$ with order $\phi(n)$, where $\phi$ denotes Euler's totient function. Also, the vertices $1$ and $(n - 1)$ are the vertices of $\Gamma_1(n, 11)$ and $0$ is always a vertex of $\Gamma_2(n, 11)$. It can be easily observed that $\Gamma_1(n, 11) \cup \Gamma_2(n, 11) = \Gamma(n, 11)$ and $\Gamma_1(n, 11) \cap \Gamma_2(n, 11) = \varnothing$.

A tree is a connected acyclic graph. A tree in which one vertex has been designated as the root is a rooted tree. The edges of a rooted tree can be assigned a natural orientation, either away from or towards the root, in which case the structure becomes a directed rooted tree. When a directed rooted tree has an orientation away from the root, it is called an arborescence or out-tree when it has an orientation towards the root, it is called an anti-arborescence or in-tree.

# 3 Properties of the Carmichael $\lambda$-function

**Definition 3.1** ([3]). *The Carmichael lambda function of a positive integer $n$, denoted by $\lambda(n)$ is defined as the smallest positive integer $m$ such that $a^m \equiv 1 \pmod{n}$ for every integer $a$ relatively prime to $n$.*

**Lemma 3.2** ([3]). *Let $n$ be a positive integer, and $\phi$ denote Euler's totient function. Then*

$$\begin{aligned}
\lambda(1) &= 1 = \phi(1) \\
\lambda(2) &= 1 = \phi(1) \\
\lambda(4) &= 2 = \phi(4) \\
\lambda(2^k) &= 2^{k-2} = \frac{1}{2}\phi(2^k) \quad \textit{for } k \geq 3 \\
\lambda(p^k) &= (p-1)p^{k-1} = \phi(p^k) \quad \textit{for any odd prime } p \textit{ and } k \geq 1 \\
\lambda(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) &= \mathrm{lcm}[\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \ldots, \lambda(p_r^{k_r})],
\end{aligned}$$

*where $p_1, p_2, \ldots, p_r$ are distinct primes for $k_i \geq 1, i = 1, 2, \ldots, r$.*

It follows from Lemma 3.2, that $\lambda(n) \mid \phi(n), \forall n$ and that $\lambda(n) = \phi(n)$ if and only if $n \in \{1, 2, 4, q^k, 2q^k\}$ where $q$ is an odd prime and $k \geq 1$.

The following theorem generalizes the well-known Euler's Theorem which says that $a^{\phi(n)} \equiv 1 \pmod{n}$ if and only if $\gcd(a, n) = 1$. It shows that $\lambda(n)$ is the least possible order modulo $n$.

**Theorem 3.3** (Carmichael's Theorem, see [3,6])**.** *Let $a, n \in N$. Then*

$$a^{\lambda(n)} \equiv 1 \pmod{n} \quad \textit{if and only if } \gcd(a, n) = 1.$$

Moreover, there exists an integer $g$ such that $\mathrm{ord}_n(g) = \lambda(n)$, where $\mathrm{ord}_n(g)$ denotes the multiplicative order of $g$ modulo $n$.

Assume now that $\lambda(n)$ has the following prime power factorization:

$$\lambda(n) = \Pi_{j=1}^{r} q_j^{l_j},$$

where $q_1 < q_2 < \cdots < q_r$ are primes and $l_j > 0$. It is evident that from the definition of $\lambda$ that $q_1 = 2$ , if $n > 2$.

**Theorem 3.4** ([4])**.** *(i) Let $n > 2$. Then there exists a cycle of length $t$ in the digraph $\Gamma(n, 11)$ if and only if $t = \mathrm{ord}_d(11)$, for some even positive divisor $d$ of $\lambda(n)$.*

*(ii) If there exists a $t$-cycle in $\Gamma(n, 11)$ then there exists a $t$-cycle in $\Gamma_1(n, 11)$.*

Suppose $A_t(\Gamma(n, k))$ denotes the number of $t$-cycles in $\Gamma(n, k)$ where $t$ is a positive integer.

**Theorem 3.5** ([14])**.** *Let $n = \Pi_{i=1}^{r} p_i^{\alpha_i}$, where $p_i, i = 1, 2, \ldots, r$ are distinct primes. Then*

$$A_t\big(\Gamma(n, k)\big) = \frac{1}{t}\Big[\prod_{i=1}^{r}\big(\delta_i \gcd\big(\lambda(p_i^{\alpha_i}), k^t - 1\big) + 1\big) - \sum_{d \mid t, d \neq t} dA_d\big(\Gamma(n, k)\big)\Big],$$

*where $\delta_i = 2$ if $2 \mid k^t - 1$ and $8 \mid p_i^{\alpha_i}$, and $\delta_i = 1$ otherwise.*

# 4 Enumeration of cycles and components

In this section, we try to formulate an explicit formula to enumerate the cyclic vertices and components of the digraph $\Gamma(3^k, 11)$, where $k$ is a positive number. According to Theorem 3.4, we need to calculate $\mathrm{ord}_d(11)$, for every even positive divisor $d$ of $\lambda(n)$. The following two lemmas showed that $\mathrm{ord}_{2\cdot 3^r}(11) = 2 \cdot 3^{r-1}$ for $1 \leq r \leq k - 1$.

**Lemma 4.1.** *Suppose $x^n \equiv y \pmod{\alpha p^\beta}$. Then $x^{np} \equiv y^p \pmod{\alpha p^{\beta+1}}$.*

*Proof.* Since $x^n \equiv y \pmod{\alpha p^\beta}$, we have $x^n = y + \alpha p^\beta k$ for some integer $k$. A binomial expansion yields

$$x^{np} = (y + \alpha p^\beta k)^p = y^p + \sum_{i=1}^{p} \binom{p}{i} y^{p-i} (\alpha p^\beta)^i.$$

We have $\binom{p}{1} y^{p-1} (\alpha p^\beta) = \alpha p^{\beta+1} y^{p-1} \equiv 0 \pmod{\alpha p^{\beta+1}}$. For $i \geq 2$, we have $\binom{p}{i} y^{p-i} \alpha^i p^{i\beta} \equiv 0 \pmod{\alpha p^{\beta+1}}$. Thus, $x^{np} \equiv (y + \alpha p^\beta k)^p \equiv y^p \pmod{\alpha p^{\beta+1}}$. □

**Lemma 4.2.** *Suppose $k \geq 2$ and $1 \leq r \leq k-1$. Then $\mathrm{ord}_{2\cdot 3^r}(11) = 2\cdot 3^{r-1}$. Also, $\mathrm{ord}_{3^{r+1}}(11) = 2 \cdot 3^r$ for $0 \leq r \leq k - 1$.*

*Proof.* It is easy to check that $\mathrm{ord}_6(11) = 2$. So, we assume $r \geq 2$. We need to show that $11^{2\cdot 3^{r-1}} \equiv 1 \pmod{2\cdot 3^r}$ and $11^d \not\equiv 1 \pmod{2\cdot 3^r}$ for every positive proper divisor $d$ of $2\cdot 3^{r-1}$. Since every proper divisor of $2\cdot 3^{r-1}$ divides either $2\cdot 3^{r-2}$ or $3^{r-1}$, it suffices to show that $11^d \not\equiv 1 \pmod{2\cdot 3^r}$ for $d \in \{2\cdot 3^{r-2}, 3^{r-1}\}$.

We use induction to show that $11^{2\cdot 3^{r-1}} \equiv 1 \pmod{2\cdot 3^r}, 11^{3^{r-1}} \equiv -1 \pmod{2\cdot 3^r}$, and $11^{2\cdot 3^{r-2}} \equiv 1+4\cdot 3^{r-1} \pmod{2\cdot 3^r}$. A calculation verifies that $11^{2\cdot 3^1} \equiv 1 \pmod{2\cdot 3^2}, 11^{3^1} \equiv -1 \pmod{2\cdot 3^2}$, and $11^{2\cdot 3^0} \equiv 1+4\cdot 3^1 \pmod{2\cdot 3^2}$. This establishes the base case.

Suppose that for $r \geq 2$, we have $11^{2\cdot 3^{r-1}} \equiv 1 \pmod{2\cdot 3^r}, 11^{3^{r-1}} \equiv -1 \pmod{2\cdot 3^r}$, and $11^{2\cdot 3^{r-2}} \equiv 1+4\cdot 3^{r-1} \pmod{2\cdot 3^r}$. We need to show that $11^{2\cdot 3^r} \equiv 1 \pmod{2\cdot 3^{r+1}}, 11^{3^r} \equiv -1 \pmod{2\cdot 3^{r+1}}$, and $11^{2\cdot 3^{r-1}} \equiv 1+4\cdot 3^r \pmod{2\cdot 3^{r+1}}$. By Lemma 4.1, we have $11^{2\cdot 3^r} \equiv 1^3 \equiv 1 \pmod{2\cdot 3^{r+1}}, 11^{3^r} \equiv (-1)^3 \equiv 1 \pmod{2\cdot 3^{r+1}}$, and $11^{2\cdot 3^{r-1}} \equiv (1+4\cdot 3^{r-1})^3 \pmod{2\cdot 3^{r+1}}$. Since $3^i \equiv 3 \pmod{2\cdot 3}$ for $i \geq 1$, we have $3^i \equiv 3^{r+1} \pmod{2\cdot 3^{r+1}}$ for $i \geq r+1$. A binomial expansion yields

$$\begin{aligned} 11^{2\cdot 3^{r-1}} &\equiv (1+4\cdot 3^{r-1})^3 \equiv 1+3\cdot 4\cdot 3^{r-1}+3\cdot 4^2\cdot 3^{2r-2}+4^3\cdot 3^{3r-3} \\ &\equiv 1+4\cdot 3^r+3\cdot 4^2\cdot 3^{r+1}+4^3\cdot 3^{r+1} \\ &\equiv 1+4\cdot 3^r \pmod{2\cdot 3^{r+1}}. \end{aligned}$$

A similar argument shows that $\mathrm{ord}_{3^{r+1}}(11) = 2\cdot 3^r$. □

**Theorem 4.3.** *For $k \geq 1, \Gamma(3^k, 11)$ has $2k+1$ cycles. There are three 1-cycles and two cycles of length $2\cdot 3^r$ for $0 \leq r \leq k-2$.*

*Proof.* The proof is by induction. By Theorem 3.4, there is a $t$-cycle in $\Gamma(3^k, 11)$ if and only if there exists an even divisor $d$ of $\lambda(3^k) = 2\cdot 3^{k-1}$ such that $\mathrm{ord}_d(11) = t$. The only even divisors of $2\cdot 3^{k-1}$ are $2\cdot 3^r$ for $0 \leq r \leq k-1$. We have $\mathrm{ord}_2(11) = 1$. By Lemma 4.2, $\mathrm{ord}_{2\cdot 3^r}(11) = 2\cdot 3^{r-1}$ for $1 \leq r \leq k-1$. Thus, $A_1(\Gamma(3^k, 11)) \geq 1$, $A_{2\cdot 3^r}(\Gamma(3^k, 11)) \geq 1$ for $0 \leq r \leq k-2$, and $A_d(\Gamma(3^k, 11)) = 0$ for all other positive divisors $d$ of $2\cdot 3^{k-1}$. By Theorem 3.5, we have

$$A_t(\Gamma(\widehat{n}, \widehat{k})) = \frac{1}{t}\Big[\prod_{i=1}^{\widehat{r}}\big(\delta_i \gcd(\lambda(p_i^{\alpha_i}), \widehat{k}^t - 1) + 1\big) - \sum_{d|t,\ d\neq t} dA_d\big(\Gamma(\widehat{n}, \widehat{k})\big)\Big].$$

Since $\widehat{n} = \prod_{i=1}^{\widehat{r}} p_1^{\alpha_i} = 3^k$, we have $\widehat{r} = 1, p_1 = 3$, and $\alpha_1 = k$. Also, $\widehat{k} = 11$ and $\delta_1 = 1$. Thus

$$A_t\big(\Gamma(3^k, 11)\big) = \frac{1}{t}\Big[\gcd\big(\lambda(3^k), 11^t - 1\big) + 1 - \sum_{d|t,\ d\neq t} dA_d\big(\Gamma(3^k, 11)\big)\Big].$$

We have

$$\begin{aligned} A_1\big(\Gamma(3^k, 11)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(3^k), 11^1 - 1\big) + 1 - \sum_{d|1,\ d\neq 1} dA_d\big(\Gamma(3^k, 11)\big)\Big] \\ &= \Big[\gcd\big(2\cdot 3^{k-1}, 10\big) + 1 - 0\Big] = 2+1-0 = 3 \end{aligned}$$

and

$$\begin{aligned} A_2\big(\Gamma(3^k, 11)\big) &= \frac{1}{2}\Big[\gcd\big(\lambda(3^k), 11^2 - 1\big) + 1 - \sum_{d|2,\ d\neq 2} dA_d\big(\Gamma(3^k, 11)\big)\Big] \\ &= \frac{1}{2}\Big[\gcd\big(2\cdot 3^{k-1}, 120\big) + 1 - 1\cdot A_1\big(\Gamma(3^k, 11)\big)\Big] = \frac{1}{2}[6+1-3] = 2. \end{aligned}$$

Suppose $r \geq 1$ and $A_{2\cdot 3^i}\big(\Gamma(3^k, 11)\big) = 2$ for all $0 \leq i < r$. We need to show $A_{2\cdot 3^r}\big(\Gamma(3^k, 11)\big) = 2$. Since $A_{3^i}\big(\Gamma(3^k, 11)\big) = 0$ for all $1 \leq i \leq r$, we have

$$A_{2\cdot 3^r}\big(\Gamma(3^k,11)\big) = \frac{1}{2\cdot 3^r}\Big[\gcd\big(2\cdot 3^{k-1}, 11^{2\cdot 3^r}-1\big)+1-1\cdot A_1\big(\Gamma(3^k,11)\big)-\sum_{i=0}^{r-1} 2\cdot 3^i\cdot A_{2\cdot 3^i}\big(\Gamma(3^k,11)\big)\Big]$$

By Lemma 4.2, $\operatorname{ord}_{2\cdot 3^{r+1}}(11) = 2\cdot 3^r$ and $\operatorname{ord}_{2\cdot 3^{r+2}}(11) = 2\cdot 3^{r+1}$. Thus, $11^{2\cdot 3^r} \equiv 1 \pmod{2\cdot 3^{r+1}}$ and $11^{2\cdot 3^{r+1}} \equiv 1 \pmod{2\cdot 3^{r+2}}$, but $11^{2\cdot 3^r} \not\equiv 1 \pmod{2\cdot 3^{r+2}}$. Hence, $2\cdot 3^{r+1} \mid (11^{2\cdot 3^r}-1)$, but $2\cdot 3^{r+2} \nmid (11^{2\cdot 3^r}-1)$. Therefore, $\gcd\big(2\cdot 3^{k-1}, 11^{2\cdot 3^r}-1\big) = 2\cdot 3^{r+1}$. We have

$$\begin{aligned} A_{2\cdot 3^r}\big(\Gamma(3^k,11)\big) &= \frac{1}{2\cdot 3^r}\Big[2\cdot 3^{r+1}+1-1\cdot 3-\sum_{i=0}^{r-1} 2\cdot 3^i\cdot 2\Big] \\ &= \frac{1}{2\cdot 3^r}\Big[2\cdot 3^{r+1}+1-3-\Big(\frac{2^2\cdot 3^r-2^2}{3-1}\Big)\Big] = 2. \end{aligned}$$

Thus, $A_1\big(\Gamma(3^k,11)\big) = 3$ and $A_{2\cdot 3^r}\big(\Gamma(3^k,11)\big) = 2, 0 \leq r \leq k-2$. So, by using addition principle of counting, total number of cycles in $\Gamma(3^k,11) = 3+\underbrace{2+2+\cdots+2}_{(k-1)\text{ terms}} = 2k+1$. Hence, there are $(2k+1)$ number of cycles in the digraph $\Gamma(3^k,11), k \geq 1$. $\square$

Fig. 1 displays the digraph $\Gamma(3^k, 11)$ for $k = 5$. It illustrates that the number of cycles of $\Gamma(3^k, 11)$ is $2\cdot 5+1 = 11$.

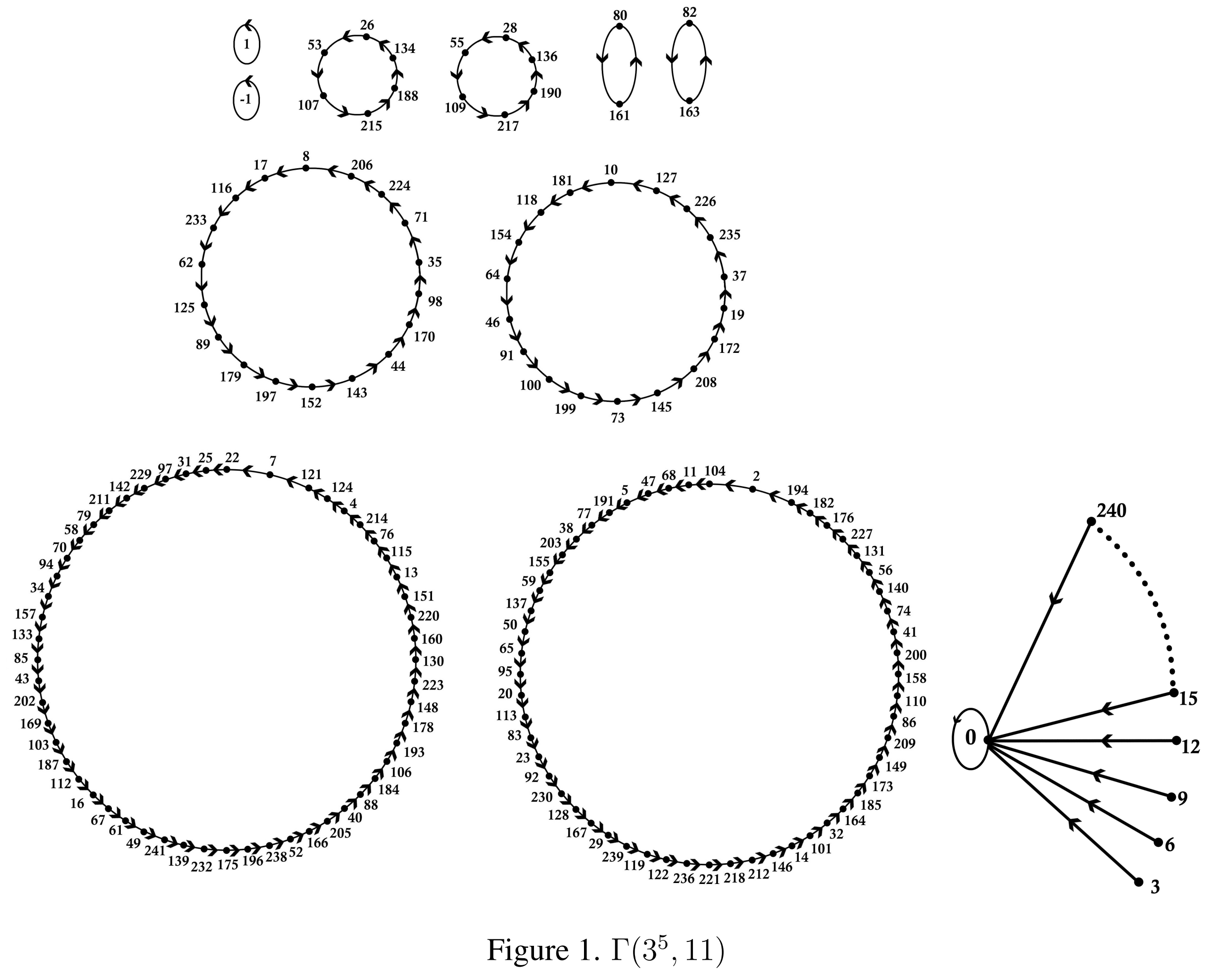



Figure 1. $\Gamma(3^5, 11)$

**Lemma 4.4.** *Suppose $k \geq 2$ and $1 \leq r \leq k-1$. Then vertices $1+3^r$ and $-1+3^r$ lie on different cycles.*

*Proof.* A binomial expansion yields

$$(1+3^r)^{120} = 1 + 120 \cdot 3^r + \sum_{i=2}^{120} \binom{120}{i} 3^{ir} \equiv 1 \pmod{3^{r+1}}.$$

Thus,

$$(1+3^r)^{11^2} \equiv 1 + 3^r \not\equiv -1 + 3^r \pmod{3^{r+1}}.$$

Furthermore, we have

$$(1+3^r)^{11} = 1 + 11 \cdot 3^r + \sum_{i=2}^{11} \binom{11}{i} 3^{ir} \equiv 1 + 2 \cdot 3^r \not\equiv -1 + 3^r \pmod{3^{r+1}}.$$

Hence, $(1+3^r)^{11^s} \not\equiv -1+3^r \pmod{3^k}$ for any positive integer $s$. This complete the proof. $\square$

**Theorem 4.5.** *Suppose $k \geq 2, m = \lfloor k/2 - 1 \rfloor$, and $0 < r \leq m$. Then the sequences of vertices given by $\left((1+3^{k-(r+1)})^{11^s} : 0 \leq s \leq 2 \cdot 3^r\right)$ and $\left((-1+3^{k-(r+1)})^{11^s} : 0 \leq s \leq 2 \cdot 3^r\right)$ are the two cycles of length $2 \cdot 3^r$ in $\Gamma(3^k, 11)$. Also, the three 1-cycles are $0, 1$, and $-1$.*

*Proof.* It is clear $0, 1$, and $-1$ are the three 1-cycles. A binomial expansion yields

$$\left(1+3^{k-(r+1)}\right)^n = 1 + n \cdot 3^{k-(r+1)} + \sum_{i=2}^{n} \binom{n}{i} 3^{i(k-r-1)}.$$

Since $i \geq 2$ and $r \leq k/2 - 1$, we have $i(k-r-1) \geq 2k - 2(r+1) \geq k$. Thus, $\left(1+3^{k-(r+1)}\right)^n \equiv 1 + n \cdot 3^{k-(r+1)} \pmod{3^k}$. Let $n = 11^s$. Then $\left(1+3^{k-(r+1)}\right)^{11^s} \equiv 1 + 11^s \cdot 3^{k-(r+1)} \pmod{3^k}$. By Lemma 4.2, $\mathrm{ord}_{3^{r+1}}(11) = 2 \cdot 3^r$. Thus, $11^{2 \cdot 3^r} \equiv 1 \pmod{3^{r+1}}$ and $11^d \not\equiv 1 \pmod{3^{r+1}}$ for every positive proper divisor $d$ of $2 \cdot 3^r$. Hence, $\left(1+3^{k-(r+1)}\right)^{11^{2 \cdot 3^r}} \equiv 1 + 3^{k-(r+1)} \pmod{3^k}$ and $\left(1+3^{k-(r+1)}\right)^{11^d} \not\equiv 1 + 3^{k-(r+1)} \pmod{3^k}$ for every positive proper divisor $d$ of $2 \cdot 3^r$. Therefore, the sequence of vertices given by $((1+3^{k-(r+1)})^{11^s} : 0 \leq s \leq 2 \cdot 3^r)$ is a cycle of length $2 \cdot 3^r$ in $\Gamma(3^k, 11)$. A similar argument shows that the sequence of vertices given by $\left((-1+3^{k-(r+1)})^{11^s} : 0 \leq s \leq 2 \cdot 3^r\right)$ is a cycle of length $2 \cdot 3^r$ in $\Gamma(3^k, 11)$. $\square$

**Conjecture 4.6.** *Suppose $k \geq 2, m = \lfloor k/2 - 1 \rfloor$, and $m < r \leq k-2$. Then the sequences of vertices given by $\left((1+3^{k-(r+1)})^{11^s} : 0 \leq s \leq 2 \cdot 3^r\right)$ and $\left((-1+3^{k-(r+1)})^{11^s} : 0 \leq s \leq 2 \cdot 3^r\right)$ are the two cycles of length $2 \cdot 3^r$ in $\Gamma(3^k, 11)$.*

Note that, for $k > 1$, the digraph $\Gamma(3^k, 11)$ is classified into two subdigraphs $\Gamma_1(3^k, 11)$ and $\Gamma_2(3^k, 11)$ , where $\Gamma_1(3^k, 11)$ is the subdigraph induced on the set of vertices $a \in Z_{3^k}$ such that $\gcd(a, 3^k) = 1$ and $\Gamma_2(3^k, 11)$ is the subdigraph induced on the set of vertices $a \in Z_{3^k}$ such that $\gcd(a, 3^k) \neq 1$. Clearly, the vertices $1$ and $3^k - 1$ (or $-1$) are the vertices of $\Gamma_1(3^k, 11)$ and the vertex $0$ is always a vertex of $\Gamma_2(3^k, 11)$. Clearly, $0, 1$ and $-1$ are the fixed points of the digraph $\Gamma(3^k, 11)$. Also, $\Gamma_1(3^k, 11) \cup \Gamma_2(3^k, 11) = \Gamma(3^k, 11)$ and $\Gamma_1(3^k, 11) \cap \Gamma_2(3^k, 11) = \varnothing$.

**Theorem 4.7.** *Suppose $k \geq 2$. The digraph $\Gamma(3^k, 11)$ is a directed rooted in-tree with root $0$. The in-degree of the root $0$ is $3^{k-\lceil k/11 \rceil} - 1$. Let $k_1 = \lceil (\lceil k/11 \rceil)/11 \rceil$ and $k_2 = \lfloor (k-1)/11 \rfloor$. Then the number of branch points (non-leaf vertices adjacent to the root $0$) in $\Gamma_2(3^k, 11)$ is given below:*

- *If $k \leq 11$, then $\Gamma_2(3^k, 11)$ has no branch points.*
- *If $k > 11$, then the number of branch points in $\Gamma_2(3^k, 11)$ is*

$$2 \cdot 3^{k-11k_2-1} \cdot \big(3^{11(k_2-k_1+1)} - 1\big)/(3^{11} - 1).$$

*Furthermore, if $k > 11$, the number of leaves adjacent to the root $0$ is*

$$3^{k-\lceil k/11 \rceil} - 1 - 2 \cdot 3^{k-11k_2-1} \cdot \big(3^{11(k_2-k_1+1)} - 1\big)/(3^{11} - 1).$$

*Proof.* Since every vertex $v$ in $\Gamma_2(3^k, 11)$ is divisible by 3, there exist integers $i$ and $t$ such that $v = 3^i t, i \geq 1, 1 \leq t \leq 3^{k-i}$ and $\gcd(3, t) = 1$. Let $\ell = \lceil \log_{11} k \rceil$. Then $11^\ell \geq k$. Thus $v^{11^\ell} \equiv 3^{i11^\ell} t^{11^\ell} \equiv 0 \pmod{3^k}$. Hence, $\Gamma(3^k, 11)$ is a directed rooted in-tree with root $0$.

Suppose $3^i t$ is a vertex of $\Gamma_2(3^k, 11)$ such that $\lceil k/11 \rceil \leq i \leq k-1$ and $t$ is an integer satisfying $1 \leq t < 3^{k-i}$ and $\gcd(3, t) = 1$. Since $i < k$, $3^i t$ is not the root $0$. Also, since $i \geq \lceil k/11 \rceil$, we have $i \geq k/11$. Thus, $(3^i t)^{11} \equiv 3^{11i} t^{11} \equiv 0 \pmod{3^k}$. Hence, $3^i t$ is adjacent to the root $0$. On the other hand, if $i < \lceil k/11 \rceil$, then $11i < 11(\lceil k/11 \rceil - 1) < k$. Thus, $3^i t$ is not adjacent to the root $0$. Hence, for fixed $\lceil k/11 \rceil \leq i \leq k-1$, the number of distinct vertices of the form $3^i t$ adjacent to the root $0$ is $\phi(3^{k-i}) = 2 \cdot 3^{k-i-1}$. Therfore, the number of vertices adjacent to the root $0$ is

$$\sum_{i=\lceil k/11 \rceil}^{k-1} \phi(3^{k-i}) = \sum_{i=\lceil k/11 \rceil}^{k-1} 2 \cdot 3^{k-i-1} = 3^{k-\lceil k/11 \rceil} - 1.$$

Suppose vertex $3^i t$ is adjacent to the root $0$ where $\lceil k/11 \rceil \leq i \leq k-1, 1 \leq t < 3^{k-i}$ and $\gcd(3, t) = 1$. If $11 \mid i$, then $i = 11j$ for some integer $j$ such that $\lceil k/11 \rceil \leq 11j \leq k-1$. Then $k_1 \leq j \leq k_2$. Since $\gcd(11, \phi(3^{k-11j})) = \gcd(11, 2 \cdot 3^{k-11j-1}) = 1$, the mapping $x \mapsto x^{11} \pmod{3^{k-11j}}$ is an automorphism of $U(3^{k-11j})$. So, there exists an integer $s$ in $U(3^{k-11j})$ such that $s^{11} \equiv t \pmod{3^{k-11j}}$. Thus, $(3^j s)^{11} \equiv 3^i t \pmod{3^k}$. Hence, $3^i t$ is a branch point of $\Gamma_2(3^k, 11)$. On the other hand, if $11 \nmid i$, then there is no vertex $3^j s$ such that $(3^j s)^{11} \equiv 3^i t \pmod{3^k}$. Thus, $3^i t$ is a leaf.

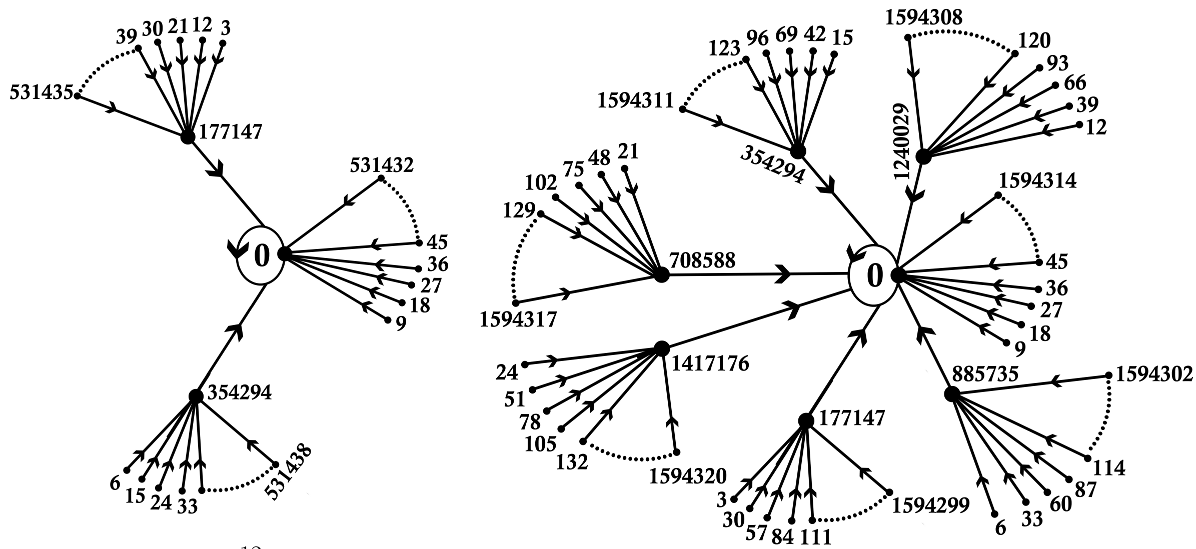
If $k \leq 11$, then every vertex of the form $3^{11j} t$, where $j$ is a positive integer, is the root $0$. Thus, $\Gamma_2(3^k, 11)$ has no branch points. On the other hand, if $k > 11$, we observe that the list of vertices of the form $3^{11j} t$, where $k_1 \leq j \leq k_2, 1 \leq t \leq 3^{k-11j}$ and $\gcd(3, t) = 1$, is a list of the distinct branch points of $\Gamma_2(3^k, 11)$. Hence, the number of branch points of $\Gamma_2(3^k, 11)$ is

$$\sum_{j=k_1}^{k_2} \phi(3^{k-11j}) = \sum_{j=k_1}^{k_2} 2 \cdot 3^{k-11j-1} = 2 \cdot 3^{k-11k_2-1} \cdot \big(3^{11(k_2-k_1+1)} - 1\big)/(3^{11} - 1).$$

Also, if $k > 11$, the number of leaves adjacent to the root $0$ is

$$3^{k-\lceil k/11 \rceil} - 1 - 2 \cdot 3^{k-11k_2-1} \cdot \big(3^{11(k_2-k_1+1)} - 1\big)/(3^{11} - 1). \qquad \square$$

Figures 2(a) and 2(b) display the digraph $\Gamma_2(3^k, 11)$ for $k = 12$ and $k = 13$. They illustrate that the vertices of $\Gamma_2(3^k, 11)$ form a directed rooted in-tree with root at 0 and have $2 \cdot 3^{k-12}$ branch points in the case when $11 < k \leq 22$.




(a) $\Gamma_2(3^{12}, 11)$

(b) $\Gamma_2(3^{13}, 11)$

Figure 2. Digraph $\Gamma_2(3^k, 11)$ for $k = 12$ and $k = 13$

**Note.** We have observed that the digraph $\Gamma(3^k, 11)$ has a subdigraph $\Gamma_1(3^k, 11)$ that consist of two 1-cycle and two cycles of length $2 \cdot 3^r$ for each $0 \leq r \leq k-2$, and a subdigraph $\Gamma_2(3^k, 11)$ that is a directed rooted in-tree with root 0. For an odd prime $q < 11$, we tried to determine for which values of $q$ does the digraph $\Gamma(q^k, 11)$ have a structure similar to that of $\Gamma(3^k, 11)$. We were able to show that the digraph $\Gamma(7^k, 11)$ has three 1-cycles, two 2-cycles, four cycles of length $3 \cdot 7^r$, $0 \leq r \leq k-2$, four cycles of length $6 \cdot 7^r, 0 \leq r \leq k-2$ and a directed rooted in-tree with root 0. First we show that $\operatorname{ord}_{2\cdot 7^r}(11) = 3 \cdot 7^{r-1}$ and $\operatorname{ord}_{6\cdot 7^r}(11) = 6 \cdot 7^{r-1}$ for $1 \leq r \leq k-1$.

**Lemma 4.8.** *Suppose $k \geq 2$ and $1 \leq r \leq k-1$. Then $\operatorname{ord}_{2\cdot 7^r}(11) = 3 \cdot 7^{r-1}$ and $\operatorname{ord}_{6\cdot 7^r}(11) = 6 \cdot 7^{r-1}$.*

*Proof.* We show that $\operatorname{ord}_{6\cdot 7^r}(11) = 6 \cdot 7^{r-1}$. The proof that $\operatorname{ord}_{2\cdot 7^r}(11) = 3 \cdot 7^{r-1}$ is similar, and we leave the details of the proof to the reader. It is easy to check that $\operatorname{ord}_{6\cdot 7^1}(11) = 6 \cdot 7^0$. So, we assume $r \geq 2$. We need to show that $11^{6\cdot 7^{r-1}} \equiv 1 \pmod{6 \cdot 7^r}$ and $11^d \not\equiv 1 \pmod{6 \cdot 7^r}$ for every positive proper divisor $d$ of $6 \cdot 7^{r-1}$.

We use induction to show that $11^{6\cdot 7^{r-1}} \equiv 1 \pmod{6 \cdot 7^r}$ and $11^{6\cdot 7^{r-2}} \equiv 1 - 12 \cdot 7^{r-1} \pmod{6 \cdot 7^r}$ for $r \geq 2$. A calculation verifies that $11^{6\cdot 7^1} \equiv 1 \pmod{6 \cdot 7^2}$ and $11^{6\cdot 7^0} \equiv 1 - 12 \cdot 7^1 \pmod{6 \cdot 7^2}$. Suppose we have $11^{6\cdot 7^{r-1}} \equiv 1 \pmod{6\cdot 7^r}$ and $11^{6\cdot 7^{r-2}} \equiv 1 - 12\cdot 7^{r-1} \pmod{6 \cdot 7^r}$ for $r \geq 2$. By Lemma 4.1, we have $11^{6\cdot 7^r} \equiv 1^7 \equiv 1 \pmod{6 \cdot 7^{r+1}}$ and $11^{6\cdot 7^{r-1}} \equiv (1 - 12 \cdot 7^{r-1})^7 \pmod{6 \cdot 7^{r+1}}$. Since $7^i \equiv 7 \pmod{6 \cdot 7}$ for $i \geq 1$, we have $7^i \equiv 7^{r+1} \pmod{6 \cdot 7^{r+1}}$ for $i \geq r+1$. A binomial expansion yields:

$$
\begin{aligned}
(1-12\cdot 7^{r-1})^7 &\equiv 1-12\cdot 7^r+3\cdot 12\cdot 7^{2r-1}+\sum_{i=3}^{7}\binom{7}{i}(-1)^i\cdot 2^{2i}\cdot 3^i\cdot 7^{ir-i}\\
&\equiv 1-12\cdot 7^r+6\cdot 6\cdot 7^{r+1}+\sum_{i=3}^{7}\binom{7}{i}(-1)^i\cdot 2^{2i-1}\cdot 3^{i-1}\cdot 6\cdot 7^{r+1}\\
&\equiv 1-12\cdot 7^r \pmod{6\cdot 7^{r+1}}.
\end{aligned}
$$

Thus, $11^{6\cdot 7^{r-1}}\equiv 1-12\cdot 7^r \pmod{6\cdot 7^{r+1}}$.

Since every proper divisor of $6\cdot 7^{r-1}$ divides either $3\cdot 7^{r-1}, 2\cdot 7^{r-1}$, or $6\cdot 7^{r-2}$, it suffices to show that $11^d\not\equiv 1 \pmod{6\cdot 7^r}$ for $d\in\{3\cdot 7^{r-1}, 2\cdot 7^{r-1}, 6\cdot 7^{r-2}\}$. We have already verified that $11^{6\cdot 7^{r-2}}\equiv 1-12\cdot 7^{r-1}\not\equiv 1 \pmod{6\cdot 7^r}$. Since $11\equiv -1 \pmod 6$, we have $11^{3\cdot 7^{r-1}}\equiv(-1)^{3\cdot 7^{r-1}}\equiv -1\not\equiv 1 \pmod 6$. Thus, $11^{3\cdot 7^{r-1}}\not\equiv 1 \pmod{6\cdot 7^r}$. By Fermat's Little Theorem, $11^7\equiv 11 \pmod 7$. Thus $11^{7^i}\equiv 11 \pmod 7$ for $i\geq 1$. Hence, $11^{2\cdot 7^{r-1}}\equiv 11^2\not\equiv 1 \pmod 7$. Therefore, $11^{2\cdot 7^{r-1}}\not\equiv 1 \pmod{6\cdot 7^r}$. □

**Remark 4.9.** *One may use Lemma 4.1 to show* $11^{3\cdot 7^{r-1}}\equiv 1+4\cdot 7^r \pmod{6\cdot 7^r}$.

**Theorem 4.10.** *For* $k\geq 1, \Gamma(7^k, 11)$ *has* $8k-3$ *cycles. There are three* 1*-cycles, two* 2*-cycles, four cycles of length* $3\cdot 7^r$ *for* $0\leq r\leq k-2$*, and four cycles of length* $6\cdot 7^r$ *for* $0\leq r\leq k-2$.

*Proof.* The proof is by induction. By Theorem 3.4, there is a $t$-cycle in $\Gamma(7^k, 11)$ if and only if there exists an even divisor $d$ of $\lambda(7^k)=6\cdot 7^{k-1}$ such that $\mathrm{ord}_d(11)=t$. The only even divisors of $6\cdot 7^{k-1}$ are $2\cdot 7^r$ and $6\cdot 7^r$ for $0\leq r\leq k-1$. We have $\mathrm{ord}_2(11)=1$ and $\mathrm{ord}_6(11)=2$. By Lemma 4.8, $\mathrm{ord}_{2\cdot 7^r}(11)=3\cdot 7^{r-1}$ and $\mathrm{ord}_{6\cdot 7^r}(11)=6\cdot 7^{r-1}$ for $1\leq r\leq k-1$. Thus $A_1(\Gamma(7^k, 11))\geq 1$, $A_2(\Gamma(7^k, 11))\geq 1$, $A_{3\cdot 7^r}(\Gamma(7^k, 11))\geq 1$ and $A_{6\cdot 7^r}(\Gamma(7^k, 11))\geq 1$ for $0\leq r\leq k-2$. Also, $A_{7^r}(\Gamma(7^k, 11))=0$ and $A_{2\cdot 7^r}(\Gamma(7^k, 11))=0$ for $1\leq r\leq k-1$, $A_{3\cdot 7^{k-1}}(\Gamma(3^k, 11))=0$, and $A_{6\cdot 7^{k-1}}(\Gamma(3^k, 11))=0$. By Theorem 3.5, we have

$$
A_t\big(\Gamma(\widehat{n},\widehat{k})\big)=\frac{1}{t}\Big[\prod_{i=1}^{\widehat{r}}\big(\delta_i\gcd\big(\lambda(p_i^{\alpha_i}),\widehat{k}^t-1\big)+1\big)-\sum_{d|t,d\neq t}dA_d\big(\Gamma(\widehat{n},\widehat{k})\big)\Big].
$$

Since $\widehat{n}=\prod_{i=1}^{\widehat{r}}p_i^{\alpha_i}=7^k$, we have $\widehat{r}=1, p_1=7$, and $\alpha_1=k$. Also, $\widehat{k}=11$ and $\delta=1$. Thus

$$
\begin{aligned}
A_1\big(\Gamma(7^k, 11)\big) &= \frac{1}{1}\Big[\gcd\big(\lambda(7^k), 11^1-1\big)+1-\sum_{d|1,d\neq 1}A_d\big(\Gamma(7^k, 11)\big)\Big]\\
&= \Big[\gcd(6\cdot 7^{k-1}, 2\cdot 5)+1-0\Big]=2+1-0=3,
\end{aligned}
$$

$$
\begin{aligned}
A_2\big(\Gamma(7^k, 11)\big) &= \frac{1}{2}\Big[\gcd\big(\lambda(7^k), 11^2-1\big)+1-\sum_{d|2,d\neq 2}A_d\big(\Gamma(7^k, 11)\big)\Big]\\
&= \frac{1}{2}\Big[\gcd(6\cdot 7^{k-1}, 2^3\cdot 3\cdot 5)+1-1\cdot 3\Big]=\frac{1}{2}[6+1-3]=2,
\end{aligned}
$$

$$\begin{aligned} A_3\big(\Gamma(7^k,11)\big) &= \frac{1}{3}\Big[\gcd\big(\lambda(7^k), 11^3-1\big)+1-\sum_{d|3, d\neq 3} A_d\big(\Gamma(7^k,11)\big)\Big] \\ &= \frac{1}{3}\Big[\gcd(6\cdot 7^{k-1}, 2\cdot 5\cdot 7\cdot 19)+1-1\cdot 3\Big] = \frac{1}{3}[14+1-3] = 4, \end{aligned}$$

$$\begin{aligned} A_6\big(\Gamma(7^k,11)\big) &= \frac{1}{6}\Big[\gcd\big(\lambda(7^k), 11^6-1\big)+1-\sum_{d|6, d\neq 6} A_d\big(\Gamma(7^k,11)\big)\Big] \\ &= \frac{1}{6}\Big[\gcd(6\cdot 7^{k-1}, 2^3\cdot 3^2\cdot 5\cdot 7\cdot 19\cdot 37)+1-1\cdot 3-2\cdot 2-3\cdot 4\Big] \\ &= \frac{1}{6}[42+1-3-4-12] = 4. \end{aligned}$$

Suppose $r \geq 1$ and $A_{3\cdot 7^i}(\Gamma(7^k,11)) = 4$ for all $0 \leq i < r$. We need to show $A_{3\cdot 7^r}(\Gamma(7^k,11) = 4$. Since $A_{7^i}(\Gamma(7^k,11)) = 0$ for all $1 \leq i \leq r$, we have

$$A_{3\cdot 7^r}\big(\Gamma(7^k,11)\big) = \frac{1}{3\cdot 7^r}\Big[\gcd\big(6\cdot 7^{k-1}, 11^{3\cdot 7^r}-1\big)+1-1\cdot A_1\big(\Gamma(7^k,11)\big)-\sum_{i=0}^{r-1} 3\cdot 7^i\cdot A_{3\cdot 7^i}\big(\Gamma(7^k,11)\big)\Big].$$

By Lemma 4.8, $\mathrm{ord}_{2\cdot 7^{r+1}}(11) = 3\cdot 7^r$ and $\mathrm{ord}_{2\cdot 7^{r+2}}(11) = 3\cdot 7^{r+1}$. Thus, $11^{3\cdot 7^r} \equiv 1 \pmod{2\cdot 7^{r+1}}$ and $11^{3\cdot 7^{r+1}} \equiv 1 \pmod{2\cdot 7^{r+2}}$, but $11^{3\cdot 7^r} \not\equiv 1 \pmod{2\cdot 7^{r+2}}$. Hence, $2\cdot 7^{r+1} \mid (11^{3\cdot 7^r}-1)$, but $2\cdot 7^{r+2} \nmid (11^{3\cdot 7^r}-1)$. Since $11^{3\cdot 7^r} \equiv (-1)^{3\cdot 7^{r-1}} \equiv -1 \pmod 3$, $3 \nmid (11^{3\cdot 7^r}-1)$. Therefore, $\gcd(6\cdot 7^{k-1}, 11^{3\cdot 7^r}-1) = 2\cdot 7^{r+1}$. We have

$$\begin{aligned} A_{3\cdot 7^r}(\Gamma(7^k,11)) &= \frac{1}{3\cdot 7^r}\Big[2\cdot 7^{r+1}+1-1\cdot 3-\sum_{i=0}^{r-1} 3\cdot 7^i\cdot 4\Big] \\ &= \frac{1}{3\cdot 7^r}\Big[2\cdot 7^{r+1}+1-3-\Big(\frac{3\cdot 4\cdot 7^r-3\cdot 4}{7-1}\Big)\Big] = 4. \end{aligned}$$

By Lemma 4.8, $\mathrm{ord}_{6\cdot 7^{r+1}}(11) = 6\cdot 7^r$ and $\mathrm{ord}_{6\cdot 7^{r+2}}(11) = 6\cdot 7^{r+1}$. Thus $11^{6\cdot 7^r} \equiv 1 \pmod{6\cdot 7^{r+1}}$ and $11^{6\cdot 7^{r+1}} \equiv 1 \pmod{6\cdot 7^{r+2}}$, but $11^{6\cdot 7^r} \not\equiv 1 \pmod{6\cdot 7^{r+2}}$. Hence. $6\cdot 7^{r+1} \mid (11^{6\cdot 7^r}-1)$, but $6\cdot 7^{r+2} \nmid (11^{6\cdot 7^r}-1)$. Therefore, $\gcd(6\cdot 7^{k-1}, 11^{6\cdot 7^r}-1) = 6\cdot 7^{r+1}$.

Suppose, $r \geq 1$ and $A_{6\cdot 7^i}(\Gamma(7^k,11)) = 4$ for all $0 \leq i < r$. We need to show $A_{6\cdot 7^r}(\Gamma(7^k,11)) = 4$. Since $A_{7^i}(\Gamma(7^k,11)) = 0$ and $A_{2\cdot 7^i}(\Gamma(7^k,11)) = 0$ for all $1 \leq i \leq r$, we have

$$\begin{aligned} A_{6\cdot 7^r}\big(\Gamma(7^k,11)\big) &= \frac{1}{6\cdot 7^r}\Big[\gcd\big(6\cdot 7^{k-1}, 11^{6\cdot 7^r}-1\big)+1-1\cdot A_1\big(\Gamma(7^k,11)\big)-2\cdot A_2\big(\Gamma(7^k,11)\big) \\ &\quad -\sum_{i=0}^{r} 3\cdot 7^i\cdot A_{3\cdot 7^i}\big(\Gamma(7^k,11)\big)-\sum_{i=0}^{r-1} 6\cdot 7^i\cdot A_{3\cdot 7^i}\big(\Gamma(7^k,11)\big)\Big] \\ &= \frac{1}{6\cdot 7^r}\Big[6\cdot 7^{r+1}+1-1\cdot 3-2\cdot 2-\sum_{i=0}^{r} 3\cdot 7^i\cdot 4-\sum_{i=0}^{r-1} 6\cdot 7^i\cdot 4\Big] \\ &= \frac{1}{6\cdot 7^r}\Big[6\cdot 7^{r+1}+1-3-4-(2\cdot 7^{r+1}-2)-(4\cdot 7^r-4)\Big] = 4. \end{aligned}$$

Thus, the number of cycles in $\Gamma(7^k,11)$ is $3+2+4(k-1)+4(k-1) = 8k-3$. □

**Theorem 4.11.** *Suppose $k \geq 2$. The digraph $\Gamma(7^k, 11)$ is s directed rooted in-tree with root $0$. The in-degree of the root $0$ is $7^{k-\lceil k/11 \rceil} - 1$. Let $k_1 = \lceil (\lceil k/11 \rceil)/11 \rceil$ and $k_2 = \lfloor (k-1)/11 \rfloor$. Then the number of branch points (non-leaf vertices adjacent to the root $0$) in $\Gamma_2(7^k, 11)$ is given below:*

- *If $k \leq 11$, then $\Gamma_2(7^k, 11)$ has no branch points.*
- *If $k > 11$, then the number of branch points in $\Gamma_2(7^k, 11)$ is*

$$6 \cdot 7^{k-11k_2-1} \cdot \big(7^{11(k_2-k_1+1)} - 1\big)/(7^1 1 - 1).$$

*Furthermore, if $k > 11$, the number of leaves adjacent to the root $0$ is*

$$7^{k-\lceil k/11 \rceil} - 1 - 6 \cdot 7^{k-11k_2-1} \cdot \big(7^{11(k_2-k_1+1)} - 1\big)/(7^{11} - 1).$$

*Proof.* The proof is similar to the proof of Theorem 4.7. □

# 5 Conclusion

In this paper, we have studied the cyclic vertices and components of the digraph $\Gamma(q^k, p)$ with respect to congruence $a^p \equiv b \pmod{q^k}$, with $p = 11$ and $q = 3, 7$. We have also enumerated cycles and components of the digraph $\Gamma(q^k, p)$ for $q = 3, 7$. We proved that for the pair $q = 3$ and $p = 11$, the digraph $\Gamma(q^k, p)$ has a subdigraph $\Gamma_1(q^k, p)$ that consists of $(q-1)$ cycles of length $1$ and $(q-1)$ cycles of length $(q-1) \cdot q^r$ for each integer $0 \leq r \leq k-2$, and a subdigraph $\Gamma_2(q^k, p)$ that is a directed rooted in-tree with root $0$. For $p = 11$, we tried to investigate the other primes $q$, for which the graph has the above structure. We found that for $q = 7$, along with some other components the digraph $\Gamma(q^k, p)$ has $4$ cycles of length $6 \cdot 7^r, 0 \leq r \leq k-2$. Therefore, for $p = 11$ we found that there is no odd prime $q \neq 3$ with $q < 11$ that could ensure a structure similar to $\Gamma(3^k, 11)$. However, the existence of other pairs of odd primes $(p, q), q < p$ for which the digraph $\Gamma(q^k, p)$ has similar structure with $\Gamma(3^k, 11)$ could be a part of future investigations.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