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Coding theory on the generalized balancing sequence

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Abstract: In this paper, we introduce the generalized balancing sequence and its matrix. Then by using the generalized balancing matrix, we give a coding and decoding method.
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1 Introduction

Balancing numbers were introduced by Behera and Panda in 1999 [2]. For $n \in \mathbb{N}$, a balancing number is defined as follows

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

for some $r \in \mathbb{N}$. For any positive number n, the balancing numbers $\{B_n\}_{n=0}^{\infty}$ satisfy the recurrence relation

$$B_{n+1} = 6B_n - B_{n-1},\ n \geq 1,$$

with initial conditions $B_0 = 0, B_1 = 1$ (see [9]).

In 2015, Ray studied balancing and Lucas-balancing sums by matrix methods [17]. In [6], Frontczak obtained an interesing general hybrid convolution identity involving balancing and Lucas balancing numbers. Also, he studied sums of balancing and Lucas-balancing numbers with binomial coefficients (see [7]).

For any integers $n \geq 1$ and $k \geq 0$, let $\Phi(n)$ and $\sigma_k(n)$ be the Euler Phi function and the sum of the k-th powers of the divisors of n, respectively. In [4], solutions to some Diophantine equations about these functions of balancing and Lucas-balancing numbers are discussed. Also, balancing polynomials which were also applied to coding theory were introduced by Frontczak in 2019 [8]. There are many works devoted to study of the generalized balancing numbers, for example see [11, 13].

In [19], Stakhov introduced the Fibonacci code which is used in source coding, as well as in cryptography. Many authors have studied the generalized Fibonacci sequence and coding theory (see [1, 5, 10, 12, 15, 18]). In [14] an encoding and decoding techinque based on the matrices T^n was introduced, where

$$T^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}.$$

Prasad [16] introduced a matrix, whose elements are balancing polynomials, and developed a new coding and decoding method following from it. Here, by using the generalized balancing sequence matrix, we give a coding and decoding method.

In Section 2, we define the generalized balancing sequence and its matrix. Also, we get the n-th power of its matrix and inverse, respectively. These are employed as the encoding and decoding matrices. Sections 3 and 4 are devoted to obtain some codes by using the generalized balancing matrix.

2 The generalized balancing sequence

In this section, we define the generalized balancing sequence. Then we give some useful results which be used later.

Definition 2.1. *For $m \geq 3$, the generalized balancing sequence $\{B_{m,n}\}_{n=0}^{\infty}$ is defined by*

$$B_{m,n} = 6B_{m,n-1} - B_{m,n-2} - \cdots - B_{m,n-m}, \quad n \geq m,$$

with initial conditions $B_{m,0} = B_{m,1} = \cdots = B_{m,m-2} = 0$ and $B_{m,m-1} = 1$.

For example, when $m = 3$, we have $\{B_{3,n}\}_{n=0}^{\infty} = \{0, 0, 1, 6, 35, 203, 1177, \ldots\}$.

Definition 2.2. *For $m \geq 3$, let Q_m be an $m \times m$ generalized balancing matrix that is defined as follows*

$$Q_m = \begin{bmatrix} 6 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

For example

$$Q_3 = \begin{bmatrix} 6 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let

$$W_{m,n} = \begin{bmatrix} B_{m,n+m-1} & -(B_{m,n} + \cdots + B_{m,n+m-2}) & -(B_{m,n+1} + \cdots + B_{m,n+m-2}) & \cdots & -B_{m,n+m-2} \\ B_{m,n+m-2} & -(B_{m,n-1} + \cdots + B_{m,n+m-3}) & -(B_{m,n} + \cdots + B_{m,n+m-3}) & \cdots & -B_{m,n+m-3} \\ B_{m,n+m-3} & -(B_{m,n-2} + \cdots + B_{m,n+m-4}) & -(B_{m,n-1} + \cdots + B_{m,n+m-4}) & \cdots & -B_{m,n+m-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m,n} & -(B_{m,n-m+1} + \cdots + B_{m,n-1}) & -(B_{m,n-m+2} + \cdots + B_{m,n-1}) & \cdots & -B_{m,n-1} \end{bmatrix},$$

where $B_{m,n}$ is the element of the generalized balancing sequence.

Theorem 2.1. *For $n \geqslant 2$ and $m = 3$, we have*

$$Q_3^n = \begin{bmatrix} B_{3,n+2} & -(B_{3,n+1} + B_{3,n}) & -B_{3,n+1} \\ B_{3,n+1} & -(B_{3,n} + B_{3,n-1}) & -B_{3,n} \\ B_{3,n} & -(B_{3,n-1} + B_{3,n-2}) & -B_{3,n-1} \end{bmatrix} = W_{3,n},$$

where

$$Q_3 = \begin{bmatrix} 6 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Proof. By using the induction method on n, setting $n = 2$ and by Definition 2.2, we have

$$Q_3^2 = \begin{bmatrix} 6 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 35 & -7 & -6 \\ 6 & -1 & -1 \\ 1 & 0 & 0 \end{bmatrix} = W_{3,2}.$$

Now, suppose that the statement holds for $n = k$. Therefore, for $n = k + 1$ we have

$$\begin{aligned} (Q_3)^{k+1} &= \begin{bmatrix} 6 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} B_{3,k+2} & -(B_{3,k+1} + B_{3,k}) & B_{3,k+1} \\ B_{3,k+1} & -(B_{3,k} + B_{3,k-1}) & B_{3,k} \\ B_{3,k} & -(B_{3,k-1} + B_{3,k-2}) & B_{3,k-1} \end{bmatrix} \\ &= \begin{bmatrix} B_{3,k+3} & -(B_{3,k+2} + B_{3,k+1}) & B_{3,k+2} \\ B_{3,k+2} & -(B_{3,k+1} + B_{3,k}) & B_{3,k+1} \\ B_{3,k+1} & -(B_{3,k} + B_{3,k+1}) & B_{3,k} \end{bmatrix} = W_{3,k+1}. \end{aligned}$$ □

Now, we are ready to generalize the idea of the n-th power of the matrix Q_3 to the n-th power of the matrix $Q_m (m > 3)$. Here, we calculate the n-th power of this matrix, denoted by Q_m^n.

Lemma 2.1. *For $m \geq 3$, we have $Q_m^{m-1} = W_{m,m-1}$.*

Proof. Let v_i^t and w_i^t be the i-th rows of Q_m^t and $W_{m,t}$, by the Definitions 2.1 and 2.2, we have

$$Q_m^1 = Q_m = \begin{bmatrix} 6 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and

$$\begin{aligned} W_{m,1} &= \begin{bmatrix} B_{m,m} & -\sum_{i=0}^{m-2} B_{m,1+i} & -\sum_{i=1}^{m-2} B_{m,1+i} & \cdots & -B_{m,m-1} \\ B_{m,m-1} & -\sum_{i=-1}^{m-3} B_{m,1+i} & -\sum_{i=0}^{m-3} B_{m,1+i} & \cdots & -B_{m,m-2} \\ B_{m,m-2} & -\sum_{i=-2}^{m-4} B_{m,1+i} & -\sum_{i=-1}^{m-4} B_{m,1+i} & \cdots & -B_{m,m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m,1} & -\sum_{i=-m+1}^{-1} B_{m,1+i} & -\sum_{i=-m+2}^{-1} B_{m,1+i} & \cdots & -B_{m,0} \end{bmatrix} \\ &= \begin{bmatrix} B_{m,m} & -B_{m,m-1} & -B_{m,m-1} & \cdots & -B_{m,m-1} \\ B_{m,m-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \end{aligned}$$

Then $v_1^1 = w_1^1$ and $v_2^1 = w_2^1$. Also

$$\begin{aligned} W_{m,2} &= \begin{bmatrix} B_{m,m} & -\sum_{i=0}^{m-2} B_{m,2+i} & -\sum_{i=1}^{m-2} B_{m,2+i} & \cdots & -B_{m,m-1} \\ B_{m,m-1} & -\sum_{i=-1}^{m-3} B_{m,2+i} & -\sum_{i=0}^{m-3} B_{m,2+i} & \cdots & -B_{m,m-2} \\ B_{m,m-2} & -\sum_{i=-2}^{m-4} B_{m,2+i} & -\sum_{i=-1}^{m-4} B_{m,2+i} & \cdots & -B_{m,m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m,1} & -\sum_{i=-m+1}^{-1} B_{m,2+i} & -\sum_{i=-m+2}^{-1} B_{m,2+i} & \cdots & -B_{m,0} \end{bmatrix} \\ &= \begin{bmatrix} B_{m,m+1} & -(B_{m,m-1} + B_{m,m}) & -(B_{m,m-1} + B_{m,m}) & \cdots & -B_{m,m} \\ B_{m,m-1} & -B_{m,m-1} & -B_{m,m-1} & \cdots & -B_{m,m-1} \\ B_{m,m-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \end{aligned}$$

and

$$Q_m^2 = Q_m \times Q_m = \begin{bmatrix} 6 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} B_{m,m} & -B_{m,m-1} & -B_{m,m-1} & \cdots & -B_{m,m-1} & -B_{m,m-1} \\ B_{m,m-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6B_{m,m} - B_{m,m-1} & -6B_{m,m-1} - 1 & \cdots & -6B_{m,m-1} - 1 & -6B_{m,m-1} \\ B_{m,m} & -B_{m,m-1} & \cdots & -B_{m,m-1} & -B_{m,m-1} \\ B_{m,m-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} B_{m,m+1} & -(B_{m,m-1} + B_{m,m}) & -(B_{m,m-1} + B_{m,m}) & \cdots & -B_{m,m} \\ B_{m,m-1} & -B_{m,m-1} & -B_{m,m-1} & \cdots & -B_{m,m-1} \\ B_{m,m-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $v_1^2 = w_1^2$, $v_2^2 = v_1^1 = w_1^1 = w_2^2$ and $v_3^2 = v_2^1 = w_2^1 = w_3^2$. By continuing the above process, $m-3$ times, the lemma is proved. □

Theorem 2.2. *For $m \geq 3$ and $n \geq m-1$, we have $Q_m^n = W_{m,n}$.*

Proof. By Lemma 2.1, we have

$$Q_m^{m-1} = W_{m,m-1}.$$

Then by induction on n, we get

$$(Q_m)^{n+1}$$

$$= \begin{bmatrix} 6 & -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} B_{m,n+m-1} & -(B_{m,n} + \cdots + B_{m,n+m-2}) & -(B_{m,n+1} + \cdots + B_{m,n+m-2}) & \cdots & -B_{m,n+m-2} \\ B_{m,n+m-2} & -(B_{m,n-1} + \cdots + B_{m,n+m-3}) & -(B_{m,n} + \cdots + B_{m,n+m-3}) & \cdots & -B_{m,n+m-3} \\ B_{m,n+m-3} & -(B_{m,n-2} + \cdots + B_{m,n+m-4}) & -(B_{m,n-1} + \cdots + B_{m,n+m-4}) & \cdots & -B_{m,n+m-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m,n} & -(B_{m,n-m+1} + \cdots + B_{m,n-1}) & -(B_{m,n-m+2} + \cdots + B_{m,n-1}) & \cdots & -B_{m,n-1} \end{bmatrix}$$

$$= \begin{bmatrix} B_{m,n+m} & -(B_{m,n+1}+\cdots+B_{m,n+m-1}) & -(B_{m,n+2}+\cdots+B_{m,n+m-1}) & \cdots & -B_{m,n+m-1} \\ B_{m,n+m-1} & -(B_{m,n}+\cdots+B_{m,n+m-2}) & -(B_{m,n+1}+\cdots+B_{m,n+m-2}) & \cdots & -B_{m,n+m-2} \\ B_{m,n+m-2} & -(B_{m,n-1}+\cdots+B_{m,n+m-3}) & -(B_{m,n}+\cdots+B_{m,n+m-3}) & \cdots & -B_{m,n+m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m,n+1} & -(B_{m,n-m+2}+\cdots+B_{m,n}) & -(B_{m,n-m+3}+\cdots+B_{m,n}) & \cdots & -B_{m,n} \end{bmatrix}$$
$$= W_{m,n+1}. \qquad \square$$

Example 2.1. *We have*

$$Q_4^3 = \begin{bmatrix} B_{4,6} & -(B_{4,5}+B_{4,4}+B_{4,3}) & -(B_{4,5}+B_{4,4}) & -B_{4,5} \\ B_{4,5} & -(B_{4,4}+B_{4,3}+B_{4,2}) & -(B_{4,4}+B_{4,3}) & -B_{4,4} \\ B_{4,4} & -(B_{4,3}+B_{4,2}+B_{4,1}) & -(B_{4,3}+B_{4,2}) & -B_{4,3} \\ B_{4,3} & -(B_{4,2}+B_{4,1}+B_{4,0}) & -(B_{4,2}+B_{4,1}) & -B_{4,2} \end{bmatrix} = \begin{bmatrix} 203 & -42 & -41 & -35 \\ 35 & -7 & -7 & -6 \\ 6 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we can get the following corollary from Theorem 2.2.

Corollary 2.1. *The determinant of $W_{m,n}$ is equal to $(-1)^{nm}$.*

Proof. By Theorem 2.2, we have $\det W_{m,n} = \det Q_m^n = (-1)^{mn}$. $\square$

Lemma 2.2. *Let $g_{B(x)}$ be the generating function of the generalized balancing sequence. Then*

$$g_{B(x)} = \frac{x^{m-1}}{1-6x+x^2+\cdots+x^m}. \tag{1}$$

Proof. Let $g_{B(x)}$ be the generating function of the generalized balancing sequence. We have

$$\begin{aligned} g_{B(x)} &= \sum_{n=1}^{\infty} B_{m,n}x^n \\ &= B_{m,1}x + B_{m,2}x^2 + \cdots + B_{m,m-1}x^{m-1} + \sum_{n=m}^{\infty} B_{m,n}x^n \\ &= x^{m-1} + \sum_{n=m}^{\infty}(6B_{m,n-1} - B_{m,n-2} - \cdots - B_{m,n-m})x^n \\ &= x^{m-1} + \sum_{n=m}^{\infty} 6B_{m,n-1}x^n - \sum_{n=m}^{\infty} B_{m,n-2}x^n - \cdots - \sum_{n=m}^{\infty} B_{m,n-m}x^n \\ &= x^{m-1} + 6x\sum_{n=1}^{\infty} B_{m,n}x^n - x^2\sum_{n=1}^{\infty} B_{m,n}x^n - \cdots - x^m\sum_{n=1}^{\infty} B_{m,n}x^n \\ &= x^{m-1} + 6xg_{B(x)} - x^2g_{B(x)} - x^3g_{B(x)} - \cdots - x^m g_{B(x)}. \end{aligned}$$

Thus,

$$g_{B(x)} = \frac{x^{m-1}}{1-6x+x^2+\cdots+x^m}. \qquad \square$$

Theorem 2.3. *The generalized balancing sequence $\{B_{m,n}\}_{n=0}^{\infty}$ has the following exponential representation*

$$g_{B(x)} = x^{m-1}\exp\sum_{i=1}^{\infty}\frac{x^i}{i}(6-x-\cdots-x^{m-1})^i, \tag{2}$$

where $m \geq 3$.

Proof. By using (1), we have

$$\ln \frac{g_{B(x)}}{x^{m-1}} = -\ln(1 - 6x + x^2 + \cdots + x^m).$$

Since

$$-\ln(1 - 6x + x^2 + \cdots + x^m) = -[-x(6 - x - \cdots - x^{m-1}) - \frac{1}{2}x^2(6 - x - \cdots - x^{m-1})^2 - \cdots - \frac{1}{n}x^n(6 - x - \cdots - x^{m-1})^n - \cdots],$$

we get the result. □

Let $K(k_1, k_2, \ldots, k_v)$ be a $v \times v$ companion matrix as follows

$$K(k_1, k_2, \ldots, k_v) = \begin{bmatrix} k_1 & k_2 & \cdots & k_{v-1} & k_v \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 2.4 (Chen and Louck, [3]). *The* (i, j) *entry* $k^n_{i,j}(k_1, k_2, \ldots, k_v)$ *in the matrix* $K^n(k_1, k_2, \ldots, k_v)$ *is given by the following formula*

$$k^n_{i,j}(k_1, k_2, \ldots, k_v) = \sum_{(t_1,t_2,\ldots,t_v)} \frac{t_j + t_{j+1} + \cdots + t_v}{t_1 + t_2 + \ldots + t_v} \times \binom{t_1 + t_2 + \cdots + t_v}{t_1, t_2, \ldots, t_v} k_1^{t_1} \ldots k_v^{t_v}, \tag{3}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + vt_v = n - i + j$,

$$\binom{t_1 + t_2 + \cdots + t_v}{t_1, t_2, \ldots, t_v} = \frac{(t_1 + t_2 + \cdots + t_v)!}{t_1!t_2! \ldots t_v!}$$

is a multinomial coefficient, and the coefficients in (1) *are defined to be* 1 *if* $n = i - j$.

According to Theorem 2.4, we can obtain following result.

Corollary 2.2. *For the generalized balancing sequence* $\{B_{m,n}\}_{n=0}^{\infty}$, *we have*

(i)
$$B_{m,n} = \sum_{(t_1,t_2,\cdots,t_m)} \binom{t_1 + t_2 + \cdots + t_m}{t_1, t_2, \ldots, t_m} 6^{t_1}(-1)^{t_2+\cdots+t_m},$$

where the summation is over nonnegative integers satsfying $t_1 + 2t_2 + \cdots + mt_m = n - m - 1$.

(ii)
$$B_{m,n} = -\sum_{(t_1,t_2,\cdots,tm)} \frac{t_m}{t_1 + t_2 + \cdots + t_m} \binom{t_1 + t_2 + \cdots + t_m}{t_1,\ t_2, \ldots,\ t_m} 6^{t_1}(-1)^{t_2+\cdots+t_m},$$

where the summation is over nonnegative integers satsfying $t_1 + 2t_2 + \cdots + mt_m = n + 1$.

Proof. (i) By Theorem 2.2, for $m \geq 3$ and $n \geq m - 1$, we have $W_{m,n} = Q^n_m$. On the other hand, the $(m, 1)$ entry in the matrix $W_{m,n}$ is $B_{m,n}$. Then for $i = m$ and $j = 1$, by Theorem 2.4, we obtain

$$B_{m,n} = \sum_{(t_1,t_2,\dots,t_m)} \frac{t_1+t_2+\cdots+t_m}{t_1+t_2+\cdots+t_m} \times \binom{t_1+t_2+\cdots+t_m}{t_1,t_2,\dots,t_m} 6^{t_1}(-1)^{t_2+\cdots+t_m}$$
$$= \sum_{(t_1,t_2,\dots,t_m)} \binom{t_1+t_2+\cdots+t_m}{t_1,t_2,\dots,t_m} 6^{t_1}(-1)^{t_2+\cdots+t_m}.$$

For the proof of (ii), we know that the $(m-1,m)$ entry in the matrix W_m^n is $-B_{m,n}$. Then for $i=m-1$ and $j=m$, by Theorem 2.4, we obtain

$$-B_{m,n} = \sum_{(t_1,t_2,\dots,tm)} \frac{t_m}{t_1+t_2+\cdots+t_m} \binom{t_1+t_2+\cdots+t_m}{t_1,\ t_2,\dots,\ t_m} 6^{t_1}(-1)^{t_2+\cdots+t_m}.$$

This completes the proof of corollary. □

3 Blocking method on the generalized balancing matrix

Here, we construct a blocking method by using the generalized balancing matrix. For this, we consider P as a message matrix of order $3m$ and explain the symbols of this coding method. For $1 \le i \le m^2$, we present E_i and C_i as follows

$$E_i := \begin{bmatrix} e_1^i & e_2^i & e_3^i \\ e_4^i & e_5^i & e_6^i \\ e_7^i & e_8^i & e_9^i \end{bmatrix}, \qquad C_i := \begin{bmatrix} c_1^i & c_2^i & c_3^i \\ c_4^i & c_5^i & c_6^i \\ c_7^i & c_8^i & c_9^i \end{bmatrix}.$$

For the blocking matrix P, we pursue the following five steps.

- Step 1. The matrix P is divided into submatrices $C_i(1 \le i \le m^2)$ of 3×3 from left to right. Note that if the number of entries are less, then the rest of the entries are set to zero.
- Step 2. We define c and n such that c is the number of C_i's and we get n by the following relation:

$$n = \begin{cases} 2, & \text{if } c \le 4, \\ \lfloor \frac{c}{3} \rfloor, & \text{if } c > 4. \end{cases}$$

- Step 3. By using Table 1, we obtain c_s^i, $1 \le s \le 9$.
- Step 4. For $1 \le i \le m^2$, we get determinant C_i and denote by d_i.
- Step 5. Construct the block matrix $W := [d_i, c_s^i]_{s\in\{1,2,4,5,6,7,8,9\}}$.

Now, we get the decoding matrix by pursuing the following four steps.

- Step 1. We calculate Q_3^n and put following relations:

$$Q_3^n = \begin{bmatrix} B_{3,n+2} & -(B_{3,n+1}+B_{3,n}) & -B_{3,n+1} \\ B_{3,n+1} & -(B_{3,n}+B_{3,n-1}) & -B_{3,n} \\ B_{3,n} & -(B_{3,n-1}+B_{3,n-2}) & -B_{3,n-1} \end{bmatrix} \to \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}.$$

• <u>Step 2.</u> For $1 \leq i \leq m^2$, let

$$\begin{aligned} c_4^i a_1 + c_5^i a_4 + c_6^i a_7 &\longmapsto e_4^i, \\ c_4^i a_2 + c_5^i a_5 + c_6^i a_8 &\longmapsto e_5^i, \\ c_4^i a_3 + c_5^i a_6 + c_6^i a_9 &\longmapsto e_6^i, \\ c_7^i a_1 + c_8^i a_4 + c_9^i a_7 &\longmapsto e_7^i, \\ c_7^i a_2 + c_8^i a_5 + c_9^i a_8 &\longmapsto e_8^i, \\ c_7^i a_3 + c_8^i a_6 + c_9^i a_9 &\longmapsto e_9^i. \end{aligned}$$

• <u>Step 3.</u> For $1 \leq i \leq m^2$, we get d_i

$$\begin{aligned} d_i = &(c_1^i a_1 + c_2^i a_4 + y_i a_7)(e_5^i e_9^i - e_6^i e_8^i) - (c_1^i a_2 + c_2^i a_5 + y_i a_8)(e_4^i e_9^i - e_6^i e_7^i) \\ &+ (c_1^i a_3 + c_2^i a_6 + y_i a_9)(e_4^i e_8^i - e_5^i e_7^i). \end{aligned}$$

• <u>Step 4.</u> We put $y_i = c_3^i$ and construct C_i. Thus, we obtain the message P.

Example 3.1. *Let a message P be*

"mathematics is beautiful and the mother of sciences".

We have

$$P = \begin{bmatrix} m & a & t & h & e & m & a & t & i \\ c & s & 0 & i & s & 0 & b & e & a \\ u & t & i & f & u & l & 0 & a & n \\ d & 0 & t & h & e & 0 & m & o & t \\ h & e & r & 0 & o & f & 0 & s & c \\ i & e & n & c & e & s & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and submatrices C_i, $1 \leqslant i \leqslant 9$, are as follows

$$C_1 = \begin{bmatrix} m & a & t \\ c & s & 0 \\ u & t & i \end{bmatrix}, \quad C_2 = \begin{bmatrix} h & e & m \\ i & s & 0 \\ f & u & l \end{bmatrix}, \quad C_3 = \begin{bmatrix} a & t & i \\ b & e & a \\ 0 & a & n \end{bmatrix},$$
$$C_4 = \begin{bmatrix} d & 0 & t \\ h & e & r \\ i & e & n \end{bmatrix}, \quad C_5 = \begin{bmatrix} h & e & 0 \\ 0 & o & f \\ c & e & s \end{bmatrix}, \quad C_6 = \begin{bmatrix} m & o & t \\ 0 & s & c \\ . & 0 & 0 \end{bmatrix},$$
$$C_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By noting that $1 \leq i \leq 9$, we get $c = 9$ and $n = 3$. So, by using Definition 1.1 and Table 1, we have

$$C_1 = \begin{bmatrix} m & a & t \\ c & s & 0 \\ u & t & i \end{bmatrix} = \begin{bmatrix} 16 & 4 & 3 \\ 6 & 22 & 2 \\ 24 & 23 & 12 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} h & e & m \\ i & s & 0 \\ f & u & l \end{bmatrix} = \begin{bmatrix} 11 & 8 & 16 \\ 12 & 22 & 2 \\ 9 & 24 & 15 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} a & t & i \\ b & e & a \\ 0 & a & n \end{bmatrix} = \begin{bmatrix} 4 & 23 & 12 \\ 5 & 8 & 4 \\ 2 & 4 & 17 \end{bmatrix}, \qquad C_4 = \begin{bmatrix} d & 0 & t \\ h & e & r \\ i & e & n \end{bmatrix} = \begin{bmatrix} 7 & 2 & 23 \\ 11 & 8 & 21 \\ 12 & 8 & 17 \end{bmatrix},$$

$$C_5 = \begin{bmatrix} h & e & 0 \\ 0 & o & f \\ c & e & s \end{bmatrix} = \begin{bmatrix} 11 & 8 & 2 \\ 2 & 18 & 9 \\ 6 & 8 & 22 \end{bmatrix}, \qquad C_6 = \begin{bmatrix} m & o & t \\ 0 & s & c \\ . & 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 18 & 23 \\ 2 & 22 & 6 \\ 3 & 2 & 2 \end{bmatrix},$$

$$C_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}, \qquad C_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix},$$

$$C_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

We calculate $\det(C_i) := d_i$*, for* $i = 1, 2, \dots, 9$.

$$d_1 = -5578 \equiv 22 \pmod{28}, \quad d_2 \equiv 26 \pmod{28}, \quad d_3 \equiv 17 \pmod{28},$$
$$d_4 \equiv 2 \pmod{28}, \quad d_5 \equiv 16 \pmod{28}, \quad d_6 \equiv 10 \pmod{28},$$
$$d_7 \equiv 0 \pmod{28}, \quad d_8 \equiv 0 \pmod{28}, \quad d_9 \equiv 0 \pmod{28}.$$

Then, by Step 5, we have the block matrix W*:*

$$W = \begin{bmatrix} 22 & 16 & 4 & 6 & 22 & 2 & 24 & 13 & 12 \\ 26 & 11 & 8 & 12 & 22 & 2 & 9 & 24 & 15 \\ 17 & 4 & 23 & 5 & 8 & 4 & 2 & 4 & 17 \\ 2 & 7 & 2 & 11 & 8 & 21 & 12 & 8 & 17 \\ 16 & 11 & 8 & 2 & 18 & 9 & 6 & 8 & 22 \\ 10 & 16 & 18 & 2 & 22 & 6 & 3 & 2 & 2 \\ 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}.$$

Now, we get decoding. Since $n = 3$*, we have*

$$Q_3^3 = \begin{bmatrix} B_{3,5} & -(B_{3,4} + B_{3,3}) & -B_{3,4} \\ B_{3,4} & -(B_{3,3} + B_{3,2}) & -B_{3,3} \\ B_{3,3} & -(B_{3,2} + B_{3,1}) & -B_{3,2} \end{bmatrix} = \begin{bmatrix} 203 & -41 & -35 \\ 35 & -7 & -6 \\ 6 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}.$$

For $1 \leq i \leq 9$, we obtain

$$\begin{array}{llllll}
e_4^1 = 2000, & e_5^1 = -402, & e_6^1 = -344, & e_7^1 = 5749, & e_8^1 = -1157, & e_9^1 = -990, \\
e_4^2 = 3218, & e_5^2 = -648, & e_6^2 = -554, & e_7^2 = 2757, & e_8^2 = -552, & e_9^2 = -474, \\
e_4^3 = 1319, & e_5^3 = -265, & e_6^3 = -227, & e_7^3 = 648, & e_8^3 = -127, & e_9^3 = -111, \\
e_4^4 = 2639, & e_5^4 = -528, & e_6^4 = -454, & e_7^4 = 2818, & e_8^4 = -565, & e_9^4 = -485, \\
e_4^5 = 1090, & e_5^5 = -217, & e_6^5 = -187, & e_7^5 = 1630, & e_8^5 = -324, & e_9^5 = -280, \\
e_4^6 = 1212, & e_5^6 = -242, & e_6^6 = -208, & e_7^6 = 691, & e_8^6 = -139, & e_9^6 = -119, \\
e_4^7 = 732, & e_5^7 = -147, & e_6^7 = -126, & e_7^7 = 732, & e_8^7 = -147, & e_9^7 = -126, \\
e_4^8 = 732, & e_5^8 = -147, & e_6^8 = -126, & e_7^8 = 732, & e_8^8 = -147, & e_9^8 = -126, \\
e_4^9 = 732, & e_5^9 = -147, & e_6^9 = -126, & e_7^9 = 732, & e_8^9 = -147, & e_9^9 = -126.
\end{array}$$

For $1 \leq i \leq 9$, by using

$$\begin{aligned}
d_i = {} & (c_1^i a_1 + c_2^i a_4 + y_i a_7)(e_5^i e_9^i - e_6^i e_8^i) - (c_1^i a_2 + c_2^i a_5 + y_i a_8)(e_4^i e_9^i - e_6^i e_7^i) \\
& + (c_1^i a_3 + c_2^i a_6 + y_i a_9)(e_4^i e_8^i - e_5^i e_7^i),
\end{aligned}$$

we get

$$y_1 = 23,\ y_2 = 16,\ y_3 = 12,\ y_4 = 23,\ y_5 = 2,\ y_6 = 23,\ y_7 = 3,\ y_8 = 3,\ y_9 = 3.$$

We put

$$\begin{aligned}
& y_1 = c_3^1 = 23,\ y_2 = c_3^2 = 16,\ y_3 = c_3^3 = 12,\ y_4 = c_3^4 = 23,\ y_5 = c_3^5 = 2, \\
& y_6 = c_3^6 = 2, \quad y_7 = c_3^7 = 3, \quad y_8 = c_3^8 = 3, \quad y_9 = c_3^9 = 3.
\end{aligned}$$

Thus,

$$C_1 = \begin{bmatrix} 16 & 4 & 3 \\ 6 & 22 & 2 \\ 24 & 23 & 12 \end{bmatrix} = \begin{bmatrix} m & a & t \\ c & s & 0 \\ u & t & i \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 11 & 8 & 16 \\ 12 & 22 & 2 \\ 9 & 24 & 15 \end{bmatrix} = \begin{bmatrix} h & e & m \\ i & s & 0 \\ f & u & l \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 4 & 23 & 12 \\ 5 & 8 & 4 \\ 2 & 4 & 17 \end{bmatrix} = \begin{bmatrix} a & t & i \\ b & e & a \\ 0 & a & n \end{bmatrix}, \qquad C_4 = \begin{bmatrix} 7 & 2 & 23 \\ 11 & 8 & 21 \\ 12 & 8 & 17 \end{bmatrix} = \begin{bmatrix} d & 0 & t \\ h & e & r \\ i & e & n \end{bmatrix},$$

$$C_5 = \begin{bmatrix} 11 & 8 & 2 \\ 2 & 18 & 9 \\ 6 & 8 & 22 \end{bmatrix} = \begin{bmatrix} h & e & 0 \\ 0 & o & f \\ c & e & s \end{bmatrix}, \qquad C_6 = \begin{bmatrix} 16 & 18 & 23 \\ 2 & 22 & 6 \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} m & o & t \\ 0 & s & c \\ . & 0 & 0 \end{bmatrix},$$

$$C_7 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C_8 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_9 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So,

$$P = \begin{bmatrix} m & a & t & h & e & m & a & t & i \\ c & s & 0 & i & s & 0 & b & e & a \\ u & t & i & f & u & l & 0 & a & n \\ d & 0 & t & h & e & 0 & m & o & t \\ h & e & r & 0 & o & f & 0 & s & c \\ i & e & n & c & e & s & . & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Table 1. English Alphabet

a	b	c	d	e	f	g	h	i	j
$n+1$	$n+2$	$n+3$	$n+4$	$n+5$	$n+6$	$n+7$	$n+8$	$n+9$	$n+10$
k	l	m	n	o	p	q	r	s	t
$n+11$	$n+12$	$n+13$	$n+14$	$n+15$	$n+16$	$n+17$	$n+18$	$n+19$	$n+20$
u	v	w	x	y	z	$.$	0	$-$	$-$
$n+21$	$n+22$	$n+23$	$n+24$	$n+25$	$n+26$	$n+27$	n	$-$	$-$

4 Coding and decoding on the generalized balancing matrix

Here, we discuss coding and decoding on the generalized balancing matrix Q_m^n and get its error detection and correction.

For $m \geq 3$, we represent P_m in the form of a square matrix $P_m = (p_{ij})_{m\times m}$, where $i, j = 1, 2, \ldots, m$. We put the matrix Q_m^n as a coding matrix and its inverse matrix Q_m^{-n} as a decoding matrix. We name the transformation $P_m \times Q_m^n = E$ as *generalized balancing matrix coding* and the transformation $E \times Q_m^{-n} = P_m$ as *generalized balancing matrix decoding*.

Also, the matrix E is given as a code matrix. We have

$$\begin{aligned} E &= P_m \times Q_m^n \\ &= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix} \\ &\times \begin{bmatrix} B_{m,n+m-1} & -(B_{m,n}+\cdots+B_{m,n+m-2}) & -(B_{m,n+1}+\cdots+B_{m,n+m-2}) & \cdots & -B_{m,n+m-2} \\ B_{m,n+m-2} & -(B_{m,n-1}+\cdots+B_{m,n+m-3}) & -(B_{m,n}+\cdots+B_{m,n+m-3}) & \cdots & -B_{n+m-3} \\ B_{m,n+m-3} & -(B_{m,n-2}+\cdots+B_{m,n+m-4}) & -(B_{m,n-1}+\cdots+B_{m,n+m-4}) & \cdots & -B_{m,n+m-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m,n} & -(B_{m,n-m+1}+\cdots+B_{m,n-1}) & -(B_{m,n-m}+\cdots+B_{m,n-1}) & \cdots & -B_{m,n-1} \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{13} \\ e_{21} & e_{22} & \cdots & e_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ e_{m1} & e_{m2} & \cdots & e_{mm} \end{bmatrix}.$$

The entries of $E, e_{11}, e_{12}, \ldots, e_{mm}$ and $\det(P_m)$ are sent throught the channel. Assuming that the transmitted sequence is received with no errors, E can be multiplied by Q_m^{-n} to recover the message matrix. Then, we have

$$P_m = E \times Q_m^{-n}$$

$$= \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1m} \\ e_{21} & e_{22} & \cdots & e_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ e_{m1} & e_{m2} & \cdots & e_{mm} \end{bmatrix}$$

$$\times \begin{bmatrix} B_{m,n+m-1} & -(B_{m,n} + \cdots + B_{m,n+m-2}) & -(B_{m,n+1} + \cdots + B_{m,n+m-2}) & \cdots & -B_{m,n+m-2} \\ B_{m,n+m-2} & -(B_{m,n-1} + \cdots + B_{m,n+m-3}) & -(B_{m,n} + \cdots + B_{m,n+m-3}) & \cdots & -B_{n+m-3} \\ B_{m,n+m-3} & -(B_{m,n-2} + \cdots + B_{m,n+m-4}) & -(B_{m,n-1} + \cdots + B_{m,n+m-4}) & \cdots & -B_{m,n+m-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m,n} & -(B_{m,n-m+1} + \cdots + B_{m,n-1}) & -(B_{m,n-m} + \cdots + B_{m,n-1}) & \cdots & -B_{m,n-1} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}.$$

Example 4.1. *Let $m = 3$, $n = 4$ and*

$$P_3 = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 3 \\ 1 & 0 & 2 \end{bmatrix}.$$

We have

$$Q_3^4 = \begin{bmatrix} B_{3,6} & -(B_{3,5} + B_{3,4}) & -B_{3,5} \\ B_{3,5} & -(B_{3,4} + B_{3,3}) & -B_{3,4} \\ B_{3,4} & -(B_{3,3} + B_{3,2}) & -B_{3,3} \end{bmatrix} = \begin{bmatrix} 1177 & -238 & -203 \\ 203 & -41 & -35 \\ 35 & -7 & -6 \end{bmatrix}.$$

By the above notations, we have

$$E = P_3 \times Q_3^4 = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 3 \\ 1 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1177 & -238 & -203 \\ 203 & -41 & -35 \\ 35 & -7 & -6 \end{bmatrix} = \begin{bmatrix} 1723 & -348 & -297 \\ 3271 & -661 & -564 \\ 1274 & -252 & -215 \end{bmatrix}.$$

Also, we obtain

$$P_3 = E \times Q_3^{-4}$$

$$= \begin{bmatrix} 1723 & -348 & -297 \\ 3271 & -661 & -564 \\ 1274 & -252 & -215 \end{bmatrix} \times \begin{bmatrix} 1 & -7 & 7 \\ -7 & 43 & -14 \\ 14 & -91 & 57 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 3 \\ 1 & 0 & 2 \end{bmatrix}.$$

4.1 Relations among the code matrix elements

Here, we obtain the relations among the code matrix element considering the basic property that $\det Q_m^n = (-1)^{nm}$. Then by using Corollary 2.1, we have

$$\det E = \det(P_m \times Q_m^n) = \det P_m \times \det Q_m^n = (-1)^{nm} \times \det P_m.$$

Suppose that $m = 3$ and n is an even positive number. Then we have

$$\begin{aligned} E &= P_3 \times Q_3^n \\ &= \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \times \begin{bmatrix} B_{3,n+2} & -(B_{3,n+1}+B_{3,n}) & -B_{3,n+1} \\ B_{3,n+1} & -(B_{3,n}+B_{3,n-1}) & -B_{3,n} \\ B_{3,n} & -(B_{3,n-1}+B_{3,n-2}) & -B_{3,n-1} \end{bmatrix} \\ &= \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} P_3 &= E \times Q_3^{-n} \\ &= \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \times \begin{bmatrix} B_{3,n+2} & -(B_{3,n+1}+B_{3,n}) & -B_{3,n+1} \\ B_{3,n+1} & -(B_{3,n}+B_{3,n-1}) & -B_{3,n} \\ B_{3,n} & -(B_{3,n-1}+B_{3,n-2}) & -B_{3,n-1} \end{bmatrix}^{-1} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}, \end{aligned}$$

where

$$Q_3^{-n} = \begin{bmatrix} B_{3,n-1}^2 - B_{3,n}B_{3,n-2} & -B_{3,n-1}B_{3,n} + B_{3,n-2}B_{3,n+1} & B_{3,n}^2 - B_{3,n-1}B_{3,n+1} \\ B_{3,n+1}B_{3,n-1} - B_{3,n}^2 & B_{3,n}B_{3,n+1} - B_{3,n+2}B_{3,n-1} & B_{3,n+2}B_{3,n} - B_{3,n+1}^2 \\ a & b & c \end{bmatrix},$$

in which

$$\begin{aligned} a &= B_{3,n}^2 + B_{3,n}B_{3,n-1} - B_{3,n+1}B_{3,n-1} - B_{3,n+1}B_{3,n-2}, \\ b &= B_{3,n+2}B_{3,n-1} + B_{3,n+2}B_{3,n-2} - B_{3,n}B_{3,n+1} - B_{3,n}^2, \\ c &= B_{3,n+1}^2 + B_{3,n}B_{3,n+1} - B_{3,n}B_{3,n+2} - B_{3,n+2}B_{3,n-1}. \end{aligned}$$

Hence, we have

$$\det Q_3^n = B_{3,n+2}B_{3,n-1}^2 - B_{3,n}B_{3,n-2}B_{3,n+2} - 2B_{3,n}B_{3,n-1}B_{3,n+1} + B_{3,n+1}^2B_{3,n-2} + B_{3,n}^3 = 1, \quad (4)$$

and

$$\begin{aligned} p_{11} &= e_{11}(B_{3,n-1}^2 - B_{3,n}B_{3,n-2}) + e_{12}(B_{3,n+1}B_{3,n-1} - B_{3,n}^2) \\ &\quad + e_{13}(B_{3,n}^2 + B_{3,n}B_{3,n-1} - B_{3,n+1}B_{3,n-1} - B_{3,n+1}B_{3,n-2}) \geq 0, \quad (5) \\ p_{12} &= e_{11}(-B_{3,n-1}B_{3,n} + B_{3,n-2}B_{3,n+1}) + e_{12}(B_{3,n}B_{3,n+1} - B_{3,n+2}B_{3,n-1}) \\ &\quad + e_{13}(B_{3,n+2}B_{3,n-1} + B_{3,n+2}B_{3,n-2} - B_{3,n}B_{3,n+1} - B_{3,n}^2) \geq 0, \quad (6) \\ p_{13} &= e_{11}(B_{3,n}^2 - B_{3,n-1}B_{3,n+1}) + e_{12}(B_{3,n+2}B_{3,n} - B_{3,n+1}^2) \\ &\quad + e_{13}(B_{3,n+1}^2 + B_{3,n}B_{3,n+1} - B_{3,n}B_{3,n+2} - B_{3,n+2}B_{3,n-1}) \geq 0, \quad (7) \end{aligned}$$

$$\begin{aligned}
p_{21} &= e_{21}(B^2_{3,n-1} - B_{3,n}B_{3,n-2}) + e_{22}(B_{3,n+1}B_{3,n-1} - B^2_{3,n}) \\
&\quad + e_{23}(B^2_{3,n} + B_{3,n}B_{3,n-1} - B_{3,n+1}B_{3,n-1} - B_{3,n+1}B_{3,n-2}) \geq 0, \quad (8) \\
p_{22} &= e_{21}(-B_{3,n-1}B_{3,n} + B_{3,n-2}B_{3,n+1}) + e_{22}(B_{3,n}B_{3,n+1} - B_{3,n+2}B_{3,n-1}) \\
&\quad + e_{23}(B_{3,n+2}B_{3,n-1} + B_{3,n+2}B_{3,n-2} - B_{3,n}B_{3,n+1} - B^2_{3,n}) \geq 0, \quad (9) \\
p_{23} &= e_{21}(B^2_{3,n} - B_{3,n-1}B_{3,n+1}) + e_{22}(B_{3,n+2}B_{3,n} - B^2_{3,n+1}) \\
&\quad + e_{23}(B^2_{3,n+1} + B_{3,n}B_{3,n+1} - B_{3,n}B_{3,n+2} - B_{3,n+2}B_{3,n-1}) \geq 0, \quad (10) \\
p_{31} &= e_{31}(B^2_{3,n-1} - B_{3,n}B_{3,n-2}) + e_{32}(B_{3,n+1}B_{3,n-1} - B^2_{3,n}) \\
&\quad + e_{33}(B^2_{3,n} + B_{3,n}B_{3,n-1} - B_{3,n+1}B_{3,n-1} - B_{3,n+1}B_{3,n-2}) \geq 0, \quad (11) \\
p_{32} &= e_{31}(-B_{3,n-1}B_{3,n} + B_{3,n-2}B_{3,n+1}) + e_{32}(B_{3,n}B_{3,n+1} - B_{3,n+2}B_{3,n-1}) \\
&\quad + e_{33}(B_{3,n+2}B_{3,n-1} + B_{3,n+2}B_{3,n-2} - B_{3,n}B_{3,n+1} - B^2_{3,n}) \geq 0, \quad (12) \\
p_{33} &= e_{31}(B^2_{3,n} - B_{3,n-1}B_{3,n+1}) + e_{32}(B_{3,n+2}B_{3,n} - B^2_{3,n+1}) \\
&\quad + e_{33}(B^2_{3,n+1} + B_{3,n}B_{3,n+1} - B_{3,n}B_{3,n+2} - B_{3,n+2}B_{3,n-1}) \geq 0. \quad (13)
\end{aligned}$$

Dividing both sides of (5), (6) and (7) by $e_{11} > 0$, we have

$$\begin{aligned}
&\frac{e_{13}}{e_{11}}(B^2_{3,n} + B_{3,n}B_{3,n-1} - B_{3,n+1}B_{3,n-1} - B_{3,n+1}B_{3,n-2}) \geq \\
&\qquad \frac{e_{12}}{e_{11}}(-B_{3,n+1}B_{3,n-1} + B^2_{3,n}) + (-B^2_{3,n-1} + B_{3,n}B_{3,n-2}), \quad (14) \\
&\frac{e_{13}}{e_{11}}(-B_{3,n+2}B_{3,n-1} - B_{3,n+2}B_{3,n-2} + B_{3,n}B_{3,n+1} + B^2_{3,n}) \leq \frac{e_{12}}{e_{11}}(B_{3,n}B_{3,n+1} - B_{3,n+2}B_{3,n-1}) \\
&\qquad + (-B_{3,n-1}B_{3,n} + B_{3,n-2}B_{3,n+1}), \quad (15)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{e_{13}}{e_{11}}(B^2_{3,n+1} + B_{3,n}B_{3,n+1} - B_{3,n}B_{3,n+2} - B_{3,n+2}B_{3,n-1}) \geq \\
&\frac{e_{12}}{e_{11}}(-B_{3,n+2}B_{3,n} + B^2_{3,n+1}) + (-B^2_{3,n} + B_{3,n-1}B_{3,n+1}).
\end{aligned}$$

Let

$$\begin{aligned}
A_1 &= (B^2_{3,n} + B_{3,n}B_{3,n-1} - B_{3,n+1}B_{3,n-1} - B_{3,n+1}B_{3,n-2}), \\
A_2 &= (-B_{3,n+2}B_{3,n-1} - B_{3,n+2}B_{3,n-2} + B_{3,n}B_{3,n+1} + B^2_{3,n}), \\
A_3 &= (B^2_{3,n+1} + B_{3,n}B_{3,n+1} - B_{3,n}B_{3,n+2} - B_{3,n+2}B_{3,n-1}).
\end{aligned}$$

By considering $3^3 = 27$ cases for $A_1 >=< 0,\ A_2 >=< 0$ and $A_3 >=< 0$, we discuss some of 27 cases.

Case 1. $A_1 > 0,\ A_2 > 0,\ A_3 > 0$. Then by using (14), we have

$$\frac{e_{13}}{e_{11}} \geq u, \quad (16)$$

where

$$u = \frac{e_{12}}{e_{11}}(\frac{-B_{3,n+1}B_{3,n-1} + B^2_{3,n}}{A_1}) + (\frac{-B^2_{3,n-1} + B_{3,n}B_{3,n-2}}{A_1}).$$

From (15), we get

$$\frac{e_{13}}{e_{11}} \le v, \tag{17}$$

where

$$v = \frac{e_{12}}{e_{11}}(\frac{B_{3,n}B_{3,n+1} - B_{3,n+2}B_{3,n-1}}{A_2}) + (\frac{-B_{3,n-1}B_{3,n} + B_{3,n-2}B_{3,n+1}}{A_2}).$$

From (16) we have

$$\frac{e_{13}}{e_{11}} \ge w, \tag{18}$$

where

$$w = \frac{e_{12}}{e_{11}}(\frac{-B_{3,n+2}B_{3,n} + B^2_{3,n+1}}{A_3}) + (\frac{-B^2_{3,n} + B_{3,n-1}B_{3,n+1}}{A_3}). \tag{19}$$

By using (4), from (16) and (17), we have

$$\frac{e_{11}}{e_{12}} \ge \min\{\frac{-B_{3,n+2}}{B_{3,n+1} + B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n} + B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1} + B_{3,n-2}}\}. \tag{20}$$

Hence, using (4), from (16) and (18), we have

$$\frac{e_{11}}{e_{12}} \le \max\{\frac{-B_{3,n+2}}{B_{3,n+1} + B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n} + B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1} + B_{3,n-2}}\}. \tag{21}$$

Thus,

$$\begin{aligned} &\min\{\frac{-B_{3,n+2}}{B_{3,n+1} + B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n} + B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1} + B_{3,n-2}}\} \le \frac{e_{11}}{e_{12}} \\ &\le \max\{\frac{-B_{3,n+2}}{B_{3,n+1} + B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n} + B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1} + B_{3,n-2}}\}. \end{aligned} \tag{22}$$

Similarly, we have

$$\begin{aligned} &\min\{\frac{B_{3,n+1} + B_{3,n}}{B_{3,n+1}}, \frac{B_{3,n} + B_{3,n-1}}{B_{3,n}}, \frac{B_{3,n-1} + B_{3,n-2}}{B_{3,n-1}}\} \le \frac{e_{12}}{e_{13}} \\ &\le \max\{\frac{B_{3,n+1} + B_{3,n}}{B_{3,n+1}}, \frac{B_{3,n} + B_{3,n-1}}{B_{3,n}}, \frac{B_{3,n-1} + B_{3,n-2}}{B_{3,n-1}}\}, \end{aligned}$$

and

$$\begin{aligned} &\min\{\frac{-B_{3,n+2}}{B_{3,n+1}}, \frac{-B_{3,n+1}}{B_{3,n}}, \frac{-B_{3,n}}{B_{3,n-1}}\} \le \frac{e_{11}}{e_{13}} \\ &\le \max\{\frac{-B_{3,n+2}}{B_{3,n+1}}, \frac{-B_{3,n+1}}{B_{3,n}}, \frac{-B_{3,n}}{B_{3,n-1}}\}. \end{aligned}$$

<u>Case 2.</u> $A_1 = 0,\ A_2 > 0,\ A_3 > 0$. Since $A_1 = 0$, from (14) we have

$$\frac{e_{11}}{e_{12}} \ge \min\{\frac{-B_{3,n+2}}{B_{3,n+1} + B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n} + B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1} + B_{3,n-2}}\}. \tag{23}$$

Using (4) and $A_1 = 0$, from (15) and (16), we obtain

$$\frac{e_{11}}{e_{12}} \le \max\{\frac{-B_{3,n+2}}{B_{3,n+1} + B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n} + B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1} + B_{3,n-2}}\}. \tag{24}$$

From (23) and (24), we have

$$\min\{\frac{-B_{3,n+2}}{B_{3,n+1}+B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n}+B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1}+B_{3,n-2}}\} \leq \frac{e_{11}}{e_{12}}$$
$$\leq \max\{\frac{-B_{3,n+2}}{B_{3,n+1}+B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n}+B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1}+B_{3,n-2}}\}. \quad (25)$$

<u>Case 3.</u> $A_1 < 0,\ A_2 < 0,\ A_3 < 0$. Then by using (14), we have

$$\frac{e_{13}}{e_{11}} \leq u, \quad (26)$$

where

$$u = \frac{e_{12}}{e_{11}}(\frac{-B_{3,n+1}B_{3,n-1}+B_{3,n}^2}{A_1}) + (\frac{-B_{3,n-1}^2+B_{3,n}B_{3,n-2}}{A_1}).$$

From (15), we get

$$\frac{e_{13}}{e_{11}} \geq v, \quad (27)$$

where

$$v = \frac{e_{12}}{e_{11}}(\frac{B_{3,n}B_{3,n+1}-B_{3,n+2}B_{3,n-1}}{A_2}) + (\frac{-B_{3,n-1}B_{3,n}+B_{3,n-2}B_{3,n+1}}{A_2}).$$

From (16), we have

$$\frac{e_{13}}{e_{11}} \leq w, \quad (28)$$

where

$$w = \frac{e_{12}}{e_{11}}(\frac{-B_{3,n+2}B_{3,n}+B_{3,n+1}^2}{A_3}) + (\frac{-B_{3,n}^2+B_{3,n-1}B_{3,n+1}}{A_3}).$$

By using (4), from (26) and (27), we have

$$\frac{e_{11}}{e_{12}} \leq \max\{\frac{-B_{3,n+2}}{B_{3,n+1}+B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n}+B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1}+B_{3,n-2}}\}. \quad (29)$$

Hence, using (4), from (26) and (28), we have

$$\frac{e_{11}}{e_{12}} \geq \min\{\frac{-B_{3,n+2}}{B_{3,n+1}+B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n}+B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1}+B_{3,n-2}}\}. \quad (30)$$

Thus,

$$\min\{\frac{-B_{3,n+2}}{B_{3,n+1}+B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n}+B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1}+B_{3,n-2}}\} \leq \frac{e_{11}}{e_{12}}$$
$$\leq \max\{\frac{-B_{3,n+2}}{B_{3,n+1}+B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n}+B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1}+B_{3,n-2}}\}. \quad (31)$$

Similarly, we have

$$\min\{\frac{B_{3,n+1}+B_{3,n}}{B_{3,n+1}}, \frac{B_{3,n}+B_{3,n-1}}{B_{3,n}}, \frac{B_{3,n-1}+B_{3,n-2}}{B_{3,n-1}}\} \leq \frac{e_{12}}{e_{13}}$$
$$\leq \max\{\frac{B_{3,n+1}+B_{3,n}}{B_{3,n+1}}, \frac{B_{3,n}+B_{3,n-1}}{B_{3,n}}, \frac{B_{3,n-1}+B_{3,n-2}}{B_{3,n-1}}\},$$

and

$$\min\{\frac{-B_{3,n+2}}{B_{3,n+1}}, \frac{-B_{3,n+1}}{B_{3,n}}, \frac{-B_{3,n}}{B_{3,n-1}}\} \leq \frac{e_{11}}{e_{13}}$$
$$\leq \max\{\frac{-B_{3,n+2}}{B_{3,n+1}}, \frac{-B_{3,n+1}}{B_{3,n}}, \frac{-B_{3,n}}{B_{3,n-1}}\}.$$

Similarly, we can prove the rest of the cases.

Therefore, for $i = 1, 2, 3$, we have

$$\min\{\frac{-B_{3,n+2}}{B_{3,n+1}+B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n}+B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1}+B_{3,n-2}}\} \leq \frac{e_{i1}}{e_{i2}}$$
$$\leq \max\{\frac{-B_{3,n+2}}{B_{3,n+1}+B_{3,n}}, \frac{-B_{3,n+1}}{B_{3,n}+B_{3,n-1}}, \frac{-B_{3,n}}{B_{3,n-1}+B_{3,n-2}}\}. \quad (32)$$

$$\min\{\frac{B_{3,n+1}+B_{3,n}}{B_{3,n+1}}, \frac{B_{3,n}+B_{3,n-1}}{B_{3,n}}, \frac{B_{3,n-1}+B_{3,n-2}}{B_{3,n-1}}\} \leq \frac{e_{i2}}{e_{i3}}$$
$$\leq \max\{\frac{B_{3,n+1}+B_{3,n}}{B_{3,n+1}}, \frac{B_{3,n}+B_{3,n-1}}{B_{3,n}}, \frac{B_{3,n-1}+B_{3,n-2}}{B_{3,n-1}}\}, \quad (33)$$

and

$$\min\{\frac{-B_{3,n+2}}{B_{3,n+1}}, \frac{-B_{3,n+1}}{B_{3,n}}, \frac{-B_{3,n}}{B_{3,n-1}}\} \leq \frac{e_{i1}}{e_{i3}}$$
$$\leq \max\{\frac{-B_{3,n+2}}{B_{3,n+1}}, \frac{-B_{3,n+1}}{B_{3,n}}, \frac{-B_{3,n}}{B_{3,n-1}}\}. \quad (34)$$

Now, for any value of m, the generalized relations among the code matrix elements are

$$\min\{\frac{-B_{m,n+k}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m+1}} : \ k=0,1,\ldots,m-1\} \leq \frac{e_{i1}}{e_{i2}}$$
$$\leq \max\{\frac{-B_{m,n+k}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m+1}} : k=0,1,\ldots,m-1\}, \quad (35)$$

$$\min\{\frac{-B_{m,n+k}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}} : \ k=0,1,\ldots,m-1\} \leq \frac{e_{i1}}{e_{i3}}$$
$$\leq \max\{\frac{-B_{m,n+k}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}} : k=0,1,\ldots,m-1\}, \quad (36)$$

$$\vdots$$

$$\min\{\frac{-B_{m,n+k}}{B_{m,n+k-1}+B_{m,n+k-2}} : \ k=0,1,\ldots,m-1\} \leq \frac{e_{i1}}{e_{i(m-1)}}$$
$$\leq \max\{\frac{-B_{m,n+k}}{B_{m,n+k-1}+B_{m,n+k-2}} : k=0,1,\ldots,m-1\}, \quad (37)$$

$$\min\{\frac{-B_{m,n+k}}{B_{m,n+k-1}} : \ k=0,1,\ldots,m-1\} \leq \frac{e_{i1}}{e_{im}}$$
$$\leq \max\{\frac{-B_{m,n+k}}{B_{m,n+k-1}} : k=0,1,\ldots,m-1\}, \quad (38)$$

$$\min\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m+1}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}} : \ k=0,1,\ldots,m-1\} \leq \frac{e_{i2}}{e_{i3}}$$
$$\leq \max\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m+1}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}} : k=0,1,\cdots,m-1\}, \quad (39)$$

$$\min\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m+1}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m-1}} : \ k=0,1,\ldots,m-1\} \leq \frac{e_{i2}}{e_{i4}}$$
$$\leq \max\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m+1}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m-1}} : k=0,1,\ldots,m-1\}, \quad (40)$$

$$\min\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m+1}}{B_{m,n+k-1}}:\ k=0,1,\ldots,m-1\}\le\frac{e_{i2}}{e_{im}}$$
$$\le\max\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m+1}}{B_{m,n+k-1}}:k=0,1,\ldots,m-1\},\tag{41}$$
$$\min\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m-1}}:\ k=0,1,\ldots,m-1\}\le\frac{e_{i3}}{e_{i4}}$$
$$\le\max\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m-1}}:k=0,1,\ldots,m-1\},\tag{42}$$
$$\min\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m-2}}:\ k=0,1,\ldots,m-1\}\le\frac{e_{i3}}{e_{i5}}$$
$$\le\max\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}}{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m-2}}:k=0,1,\ldots,m-1\},\tag{43}$$
$$\vdots$$
$$\min\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}}{B_{m,n+k-1}}:\ k=0,1,\ldots,m-1\}\le\frac{e_{i3}}{e_{im}}$$
$$\le\max\{\frac{B_{m,n+k-1}+B_{m,n+k-2}+\cdots+B_{m,n+k-m}}{B_{m,n+k-1}}:k=0,1,\ldots,m-1\},\tag{44}$$
$$\vdots$$
$$\min\{\frac{B_{m,n+k-1}+B_{m,n+k-2}}{B_{m,n+k-1}}:\ k=0,1,\ldots,m-1\}\le\frac{e_{i(m-1)}}{e_{im}}$$
$$\le\max\{\frac{B_{m,n+k-1}+B_{m,n+k-2}}{B_{m,n+k-1}}:k=0,1,\ldots,m-1\}.\tag{45}$$

Therefore, it is clear that the determinant of the initial message P_m is connected to the determinant of the code message E. So, we obtain the determinant of the matrix P_m. The $\det(P_m)$ is treated as a controller of entries of the code matrix E received from the communication channel. After receiving the code matrix E and computing the determinant of P_m, we will compute the determinant of E. Then, we will compare them. If $\det(E)=\pm\det(P_m)$, this means that the matrix E has passed through the communication channel without error. Otherwise, according to the order of the matrix E, we have $m\times m$ "single", "double", ..., "m^2-fold" errors. Thus,

$$\binom{m^2}{1}+\binom{m^2}{2}+\cdots+\binom{m^2}{m^2}=2^{m^2}-1.$$

For example, let $m=3$. According to the matrix E of order 3×3, we have "single", "double", ..., "nine-fold" errors. The first assumption is that there exists only one error in the matrix E received from the communication channel. It is clear that there are nine different cases for it as follows

$$(1)\begin{bmatrix}a&e_2&e_3\\e_4&e_5&e_6\\e_7&e_8&e_9\end{bmatrix},\qquad(2)\begin{bmatrix}e_1&b&e_3\\e_4&e_5&e_6\\e_7&e_8&e_9\end{bmatrix},\qquad(3)\begin{bmatrix}e_1&e_2&c\\e_4&e_5&e_6\\e_7&e_8&e_9\end{bmatrix},$$
$$(4)\begin{bmatrix}e_1&e_2&e_3\\d&e_5&e_6\\e_7&e_8&e_9\end{bmatrix},\qquad(5)\begin{bmatrix}e_1&e_2&e_3\\e_4&e&e_6\\e_7&e_8&e_9\end{bmatrix},\qquad(6)\begin{bmatrix}e_1&e_2&e_3\\e_4&e_5&f\\e_7&e_8&e_9\end{bmatrix},$$

$$(7)\begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ g & e_8 & e_9 \end{bmatrix}, \qquad (8)\begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & h & e_9 \end{bmatrix}, \qquad (9)\begin{bmatrix} e_1 & e_2 & e_1 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & i \end{bmatrix},$$

where $a, b, \dots, i$ are the possible "destroyed" entries.

From $\det(E) = (-1)^{nm} \times \det(P_m)$, we have

$$\begin{aligned}
&(1)\ \ a(e_5e_9 - e_6e_8) - e_2(e_4e_9 - e_6e_7) + e_3(e_4e_8 - e_5e_7) = (-1)^{nm} \times \det P_m, \\
&(2)\ \ e_1(e_5e_9 - e_6e_8) - b(e_4e_9 - e_6e_7) + e_3(e_4e_8 - e_5e_7) = (-1)^{nm} \times \det P_m, \\
&\qquad\qquad\qquad\qquad\qquad\vdots \\
&(9)\ \ e_1(e_5i - e_6e_8) - e_2(e_4i - e_6e_7) + e_3(e_4e_8 - e_5e_7) = (-1)^{nm} \times \det P_m.
\end{aligned}$$

In a similar way, we will obtain a double error for the matrix E. For example, we consider a bivariate case for the matrix E as follows

$$\begin{bmatrix} a & b & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix},$$

that possible cases are $\binom{9}{2} = 36$. Similarly, we obtain "triple", "four-fold", ..., "nine-fold" errors, for which the total number of cases is

$$\binom{9}{1} + \binom{9}{2} + \cdots + \binom{9}{9} = 2^9 - 1.$$

Therefore, there are $2^9 - 1 = 511$ errors. By using $\det E = (-1)^{3n} \det P_3$ and the relations (32)–(34), we can correct up to "single", "double", ..., "eight" errors except "nine" errors. Thus, we get that the correcting ability of the method is equal to $\frac{510}{511} = 0.9980 \approx 99.8\%$.

In general, for sufficiently large values of m, by the above method we obtain that the correcting ability of the generalized balancing matrix coding is equal to

$$\frac{2^{m^2} - 2}{2^{m^2} - 1} \approx 1 = 100\%.$$

5 Conclusion

Coding theory is one of the most important and direct applications of the generalized balancing matrix. We obtain the following results.

1. For $m = 3$, the correcting ability of errors is equal to 99.80%.
2. The correcting ability of this method increases with the increasing of m. Then for enough large values of m, the correcting ability is approximately equal to 100%.
3. The generalized balancing matrix coding/decoding is calculated very quickly by computer. Also, the correcting ability and detection ability of this coding method is very high in comparison to a classical algebraic coding-decoding method.

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References

[1] Basu, M., & Prasad, B. (2009). Generalized relations among the code elements for Fibonacci coding theory. *Chaos, Solitons & Fractals*, 41(5), 2517–2525.

[2] Behera, A., & Panda, G. K. (1999). On the square roots of triangular numbers. *The Fibonacci Quarterly*, 37(2), 98–105.

[3] Chen, W. Y. C., & Louck, J. D. (1996). The combinatorial power of the companion matrix. *Linear Algebra and its Applications*, 232, 261–278.

[4] Dutta, U. K., & Ray, P. K. (2019). On arithmetic functions of balancing and Lucas-balancing numbers. *Mathematical Communications*, 24, 77–81.

[5] Esmaeili, M., Esmaeili, M., & Gulliver, T. A. (2017). A new class of Fibonacci sequence based error correcting codes. *Cryptography and Communications*, 9(3), 379–396.

[6] Frontczak, R. (2018). A note on hybrid convolutions involving balancing and Lucas-balancing numbers. *Applied Mathematical Sciences*, 12(25), 1201–1208.

[7] Frontczak, R. (2018). Sums of balancing and Lucas-balancing numbers with binomial coefficients. *International Journal of Mathematical Analysis*, 12(12), 585–594.

[8] Frontczak, R. (2019). On balancing polynomials. *Applied Mathematical Sciences*, 13(2), 57–66.

[9] Gautam, R. (2018). Balancing numbers and application. *Journal of Advanced College of Engineering and Management*, 4, 137–143.

[10] Hashemi, M., & Mehraban, E. (2021). Some new codes on the k-Fibonacci sequence. *Mathematical Problems in Engineering*, Article ID 7660902.

[11] Liptai, K., Luca, F., Pinter, A., & Szalay, L. (2009). Generalized balancing numbers. *Indagationes Mathematicae*, 20(1), 87–100.

[12] Nalli, A., & Ozyilmaz, C. (2015). Third order variations on the Fibonacci universal code. *Journal of Number Theory*, 149, 15–32.

[13] Özkoç, A., & Tekcan, A. (2017). On k-balancing numbers. *Notes on Number Theory and Discrete Mathematics*, 23(3), 38–52.

[14] Prasad, B. (2018). Coding theory on balancing numbers. *International Journal of Open Problems in Computer Science and Mathematics*, 11, 73–85.

[15] Prasad, B. (2019). A new Gaussian Fibonacci matrices and its applications. *Journal of Algebra and Related Topics*, 7, 65–72.

[16] Prasad, B. (2021). Coding theory based on balancing polynomials. *Control and Cybernetics*, 50(2), 335–346.

[17] Ray, P. K. (2015). Balancing and Lucas-balancing sums by matrix methods. *Mathematical Reports*, 17(2), 225–233.

[18] Shannon, A. G., Erdağ, Ö., & Deveci, Ö. (2021). On the connections between Pell numbers and Fibonacci p-numbers. *Notes on Number Theory and Discrete Mathematics*, 21(1), 148–160.

[19] Stakhov, A. P. (2006). Fibonacci matrices, a generalization of the Cassini formula and new coding theory. *Chaos, Solitions & Fractals*, 30(1), 56–66.