Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
2023, Volume 29, Number 3, 474–485
DOI: 10.7546/nntdm.2023.29.3.474-485

Digits of powers of 2 in ternary numeral system

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Received: 23 November 2022 **Revised:** 22 June 2023
Accepted: 27 June 2022 **Online First:** 10 July 2023

Abstract: We study the digits of the powers of 2 in the ternary number system. We propose an algorithm for doubling numbers in ternary numeral system. Using this algorithm, we explain the appearance of "stairs" formed by 0s and 2s when the numbers 2^n $(n = 0,1,2, \dots)$ are written vertically so that for example the last digits are forming one column, the second last digits are forming another column, and so forth. We use the patterns formed by the leftmost digits, and the patterns formed by the rightmost digits to prove that the sizes of these blocks of 0s and 2s are unbounded. We also study how this regularity changes when the digits are taken between the left end and the right end of the numbers.

Keywords: Ternary numeral system, Benford's law, Digits, Zeros, Twos, Powers of two.
2020 Mathematics Subject Classification: Primary: 11A63; Secondary: 11A07, 11B83.

1 Introduction

In 1979, Paul Erdős conjectured that there are only finitely many positive integers n such that 2^n can be written as sum of distinct powers of 3. This is equivalent to saying that its ternary representation does not contain the digit 2 (see p. 67 in [5], p. 80 in [6]). Neil Sloane and Eric Weisstein made similar conjectures for digits 0 and 1 ([13, 16, 17] and [20], p. 28). The digits of powers of 2 in ternary numeral system were also studied in [20], p. 20–25. The encyclopedic entries [13] and [17] contain many references and results related to ternary representations, including connections with Collatz conjecture and cellular automaton. Similar questions for decimal numeral system were asked and answered in [4, 19]. In the current paper we studied the patterns formed by the ternary digits of 2^n, when these numbers are taken together and individually.

Let us write first powers of two $(1,2,4,\dots,2^{18})$ in ternary numeral system so that all their digits in the corresponding place values are aligned along vertical columns (see Table 1.1).

1	1	1	0	2	2	1	2	1	0	0	1	$= 2^{18}$
	2	0	1	2	2	2	1	0	1	1	2	$= 2^{17}$
	1	0	0	2	2	2	2	0	0	2	1	$= 2^{16}$
		1	1	2	2	2	2	1	1	2	2	$= 2^{15}$
			2	1	1	1	1	0	2	1	1	$= 2^{14}$
			1	0	2	0	2	0	1	0	2	$= 2^{13}$
				1	2	1	2	1	2	0	1	$= 2^{12}$
					2	2	1	0	2	1	2	$= 2^{11}$
					1	1	0	1	2	2	1	$= 2^{10}$
						2	0	0	2	2	2	$= 2^{9}$
						1	0	0	1	1	1	$= 2^{8}$
							1	1	2	0	2	$= 2^{7}$
								2	1	0	1	$= 2^{6}$
								1	0	1	2	$= 2^{5}$
									1	2	1	$= 2^{4}$
										2	2	$= 2^{3}$
										1	1	$= 2^{2}$
											2	$= 2^{1}$
											1	$= 2^{0}$

Table 1.1.

1	1	2
0	2	1
1	2	2
2	1	1
1	0	2
2	0	1
2	1	2
2	2	1
2	2	2
1	1	1
2	0	2
1	0	1
0	1	2
1	2	1
0	2	2
0	1	1
0	0	2
0	0	1

Table 1.2.

In the future, we will refer to this infinite list simply as *the table* or *the construction*. We can easily observe several interesting patterns from this table.

I Observation. If we look only at the rightmost k digits of each power of two, then, as we go upwards along the table, we can see that all possible k-digit endings (of course, except those which ends with 0) appear in the table and they repeat periodically (see [20], p. 20–25). For example, if $k = 3$, then there are $2 \cdot 3^{k-1} = 18$ possible 3-digit endings and all of them appear in the rightmost 3 columns of the above table and will appear periodically with period $2 \cdot 3^{k-1} = 18$ (see Table 1.2).

II Observation. If we look only at the leftmost k digits, then we can see that all the possible k-digit headings (of course, excluding those which start with 0) appear but this time not periodically (cf. Exercise 20.3.2 in [10, p. 502]). For example, if $k = 2$, then there are $2 \cdot 3^{k-1} = 6$ possible 2-digit headings $(10, 11, 12, 20, 21, 22)$ and all of them appear in the table, although not with the same frequency. One can notice that the numbers starting with smaller two-digit blocks such as 10 or 11, appear more frequently than larger two-digit blocks, say, 21 or 22 (see Table 1.3).

III Observation. The digits 0 and 2 (alternatively) appear in triangular blocks (see the red and blue parts of Table 1.1, where each step is of height either 1 or 2 digits and of length 1 digit). One such block is shown in Table 1.4 which is formed using some of the digits of $2^{15}, 2^{16}, \ldots, 2^{20}$ (see also Table 2.1 below). The total height of each of these blocks is greater than or equal to its total length, which can be arbitrarily large as n (i.e., the exponent of 2) is free to run over the positive integers. These blocks of twos are not touching each other except diagonally. The same is true for the blocks of zeros.

2	**0**	0	2	2	2
1	**0**	0	1	1	1
	1	**1**	2	0	2
		2	**1**	0	1
		1	**0**	1	2
			1	**2**	1
				2	**2**
				1	**1**
					2
					1

Table 1.3.

The total height is 6.

$\boxed{2}$				← Step of height 1 and length 1.
$\boxed{2}$	$\boxed{2}$			
$\boxed{2}$	$\boxed{2}$			← Step of height 2 and length 1.
$\boxed{2}$	$\boxed{2}$	$\boxed{2}$		← Step of height 1 and length 1.
$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	
$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	← Step of height 2 and length 1.

The total length is 4.

Table 1.4.

We will prove that both observations I and II are in general true. The observations I and II will be expressed as Lemma 3.3 and Lemma 4.1, respectively, which are used to prove Theorem 5.1, the main result of the current paper that generalizes the observation III. Some formulae for the probability of occurrence of certain blocks of digits in between the first and last digits of the number are proved. These formulae are then used to show that the probabilities for different k-digit blocks are different but as these blocks of digits shift in the direction of the right side of the numbers, the probabilities become more and more uniform.

2 The structure of the stairs

We need to determine the rules obeyed by the digits when the numbers are doubled. This will help us to explain the patterns in Observation III above and prove the main result (Theorem 5.1) below. Let us first describe the algorithm for doubling an arbitrary positive integer in ternary numeral system. Consider the substitutions $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} = (0)(12)$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} = (01)(2)$, which we prefer to write as $A = \begin{pmatrix} 0 & 2 & 1 \\ \uparrow & \uparrow & \uparrow \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 2 \\ \uparrow & \uparrow & \uparrow \\ 0 & 1 & 2 \end{pmatrix}$, because we go upwards as we double the numbers and write the digits.

1) Add an extra 0 to the left of the given number. Start with the rightmost digit of the number and apply A. Write the result above this digit.
2) If we obtained 0 or 2, then for the next digit of the number to the left, we use the last used

substitution, otherwise if we obtained 1 then we switch to the other substitution, and apply it for the next digit on the left. Write the result above the digit.

3) Return to Step 2) unless you already reached the extra 0 at the left end.

For example, we can use this algorithm to double $131072 = (20122210112)_3$.

	B	A	B	B	B	B	A	A	B	B	B	A
	1	1	1	0	2	2	1	2	1	0	0	1
Extra zero →	0	2	0	1	2	2	2	1	0	1	1	2

After completion of the algorithm, we erase the leftmost extra zero of the original number and, if there is any, the leftmost extra zero of the resulting number. The proof of this algorithm is self-evident, since each time the digit 1 appears as a result of the algorithm, there is a change in the number carried over the next place value. Let us see how this algorithm is applied to a string of twos in a number. Suppose that we have a string of twos as in the red part of the numbers in the following Table 2.1.

$2^{21} = \cdots$	1	1	2	2	0	2	...
$2^{20} = \cdots$	0	$\boxed{2}$	1	1	0	2	...
$2^{19} = \cdots$	1	$\boxed{2}$	$\boxed{2}$	0	1	2	...
$2^{18} = \cdots$	0	$\boxed{2}$	$\boxed{2}$	1	2	1	...
$2^{17} = \cdots$	1	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	1	0	...
$2^{16} = \cdots$	0	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	0	...
$2^{15} = \cdots$	1	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	1	...
$2^{14} = \cdots$	2	1	1	1	1	0	...

Table 2.1.

According to the described doubling algorithm, the digit 2 can be obtained only from 1 (the substitution A) or from 2 (the substitution B). In both cases, we keep using the same substitution until we run out of twos. Because of this, either

1. all these twos are obtained from only ones (e.g., check above how the red digits of 2^{15} are obtained from 2^{14}),
2. or all these twos are obtained from again the twos (e.g., check above how the red digits of 2^{16} are obtained from 2^{15}).

On the other hand, it is not possible to have three or more vertical 2s or two or more horizontal 2s on one step of the construction. One can put all possible combination of digits instead of asterisks ($*$), which stands for a non-two digit, and, if necessary, instead of dots (...), which stands for any digit, to check that the following two cases are not possible.

...	...	...	...	...	...
...	2	2	2	$*$	...
...	2	2	2	$*$	...
...	2	2	2	$*$	...
...	2	2	2	2	...

...	...	...	...	...	...
...	2	2	$*$	...	...
...	2	2	2	2	...
...	2	2	2	2	...

This shows that each step of the triangular block of twos is of vertical height either 1 or 2 digits and of horizontal length of only 1 digit. This also proves that the total height of each of these blocks is greater than or equal to its total length. The same method works for the triangular blocks of zeros.

3 The last digits

In the present section we prove the main result of this paper by showing that Observation I is valid for any arbitrarily large value of k, thanks to the following pair of lemmas.

Lemma 3.1. For any given $k \in \mathbb{Z}^+$, $3^k | \left(2^{3^{k-1}} + 1\right)$, and $3^{k+1} \nmid \left(2^{3^{k-1}} + 1\right)$.

Proof. Lemma 3.1 can be proved using the method of mathematical induction. Denote $A_k = 2^{3^k} + 1$. For $k = 1$ we have $3^1 | A_0$, but $3^2 \nmid A_0$. Suppose that it is true for $k = n$, that is $3^{n+1} | A_n$, but $3^{n+2} \nmid A_n$. Then $A_{n+1} = 2^{3^{n+1}} + 1 = \left(2^{3^n}\right)^3 + 1 = \left(2^{3^n} + 1\right)\left(2^{2\cdot 3^n} - 2^{3^n} + 1\right)$ $= A_n\left(A_n^2 - 3\cdot(A_n - 1)\right) = A_n(A_n^2 - 3A_n + 3)$. Note that $3|(A_n^2 - 3A_n + 3)$ but $3^2 \nmid (A_n^2 - 3A_n + 3)$. Therefore, $3^{n+2} | A_{n+1}$, $3^{n+3} \nmid A_{n+1}$, and this completes the proof. □

Lemma 3.2. For any given $k \in \mathbb{Z}^+$, $3^k | 2^{2\cdot 3^{k-1}} - 1$ and $3^{k+1} \nmid 2^{2\cdot 3^{k-1}} - 1$.

Proof. The statement easily follows from Lemma 3.1 by observing that

$$2^{2\cdot 3^{k-1}} - 1 = \left(2^{3^{k-1}} - 1\right)\left(2^{3^{k-1}} + 1\right) \text{ and } 3 \nmid \left(2^{3^{k-1}} - 1\right). \qquad \square$$

Note. By invoking Lemma 3.1 and Lemma 3.2, we could also use the special case of Euler's Theorem, which says that $2^{\varphi(3^{n+1})} \equiv 1 \pmod{3^{n+1}}$, and the fact that $\varphi(3^{n+1}) = 2\cdot 3^n$. Here φ is The Euler Phi-Function [14, Sec. 6.3 and 7.1]. Also note that $\mathrm{ord}_{3^{n+1}} 2 = 2\cdot 3^n$, which means that $x = 2\cdot 3^n$ is the least positive integer such that $2^x \equiv 1 (\mathrm{mod}\ 3^{n+1})$. Otherwise, since $\mathrm{ord}_{3^{n+1}} 2 | \varphi(3^{n+1})$ and $\varphi(3^{n+1}) = 2\cdot 3^n$, either (I option) $\mathrm{ord}_{3^{n+1}} 2 = 3^k$ or (II option) $\mathrm{ord}_{3^{n+1}} 2 = 2\cdot 3^k$ for some $0 \le k < n$. But $3 \nmid \left(2^{3^k} - 1\right)$, as mentioned earlier, so, the first option is not possible. The second option is also impossible, because in this case $2^{2\cdot 3^k} \equiv 1 (\mathrm{mod}\ 3^{n+1})$ for some $0 \le k < n$. But as proved in Lemma 3.2 above $3^{k+1} \nmid 2^{2\cdot 3^k} - 1$, therefore, $3^{n+1} \nmid 2^{2\cdot 3^k} - 1$, too. The equality $\mathrm{ord}_{3^{n+1}} 2 = \varphi(3^{n+1})$ that we just proved means that 2 is a primitive root modulo 3^{n+1}. See [14, Sec. 9.1] for the definition of primitive roots, the notation $\mathrm{ord}_m a$ (order of a modulo m) and its properties. The following result generalizes Observation I above.

Lemma 3.3. Except those which ends by zero, any finite sequence of digits can appear infinitely many times, at the end of a ternary numeral system representations of powers of 2.

Proof. If k is a positive integer, then $\left\{1, 2, 2^2, 2^3, \ldots, 2^{\varphi(3^{k+1})-1}\right\}$ gives the set of $\varphi(3^{k+1})$ integers such that each element of the set is relatively prime to 3, and no two different elements of the set are congruent modulo 3^{k+1}, i.e., the set forms a reduced residue set modulo 3^{k+1} (see [14], sec. 6.3). So, the rightmost n digits of the elements of the set $\left\{1, 2, 2^2, 2^3, \ldots, 2^{\varphi(3^{k+1})-1}\right\}$ written in radix-3, go through all the possible k-tuples without repetitions, except those ending with 0 (see [20], p. 20–25). By Lemma 3.2, the last k digits of 2^n are periodic with period $2 \cdot 3^k$, and this completes the proof of Lemma 3.3. □

4 The first digits

Let us now turn our attention to the first digits of the elements of the set $\{1, 2, 2^2, 2^3, \ldots\}$, when they are written in ternary numeral system. The following result generalizes Observation II above.

Lemma 4.1. Except those which start with zero, any finite sequence of digits can appear infinitely many times, at the beginning of a ternary numeral system representation of powers of 2.

Proof. Suppose that the first m digits of 2^n are $(\overline{a_1 a_2 \ldots a_m})_3 = A$, where $a_1 \in \{1,2\}$ and $a_i \in \{0, 1, 2\}$ for $i = 2, \ldots, m$. Then

$$A \cdot 3^k < 2^n < (A+1) \cdot 3^k, \tag{4.1}$$

for some nonnegative integer k. Taking base 3 logarithm of both sides of (4.1) gives

$$k + \log_3 A < n \log_3 2 < k + \log_3(A+1). \tag{4.2}$$

Since $m - 1 \le \log_3 A < m$ and $m - 1 < \log_3(A+1) \le m$, we obtain that

$$k + m - 1 < n \log_3 2 < k + m. \tag{4.3}$$

This means that $k + m - 1$ is simply the integer part of $n \log_3 2$. So,

$$\log_3 A - m + 1 < n \log_3 2 - \lfloor n \log_3 2 \rfloor < \log_3(A+1) - m + 1. \tag{4.4}$$

Note that $[\log_3 A - m + 1, \log_3(A+1) - m + 1] \subseteq [0,1]$. By the well-known result of Bohl [3], Sierpinski [15], and Weyl [21] (see also [11], Chapter 1, Example 2.1; [2, 9, 22]) the sequence $\{n \log_3 2\} (n = 1,2, \ldots)$ is uniformly distributed modulo 1. In particular, this means that there are infinitely many n such that the difference $n \log_3 2 - \lfloor n \log_3 2 \rfloor$ is in the interval $[\log_3 A - m + 1, \log_3(A+1) - m + 1]$. This completes the proof of Lemma 4.1. □

Note. Suppose that m is a positive integer and $n = n_0$ is the least integer such that $2^n > 3^m$. Then, the first m digits of $\{2^n\}_{n=n_0, n_0+1, \ldots}$, give all of possible $2 \cdot 3^{m-1}$ sequences of digits at the beginning of these numbers. The frequency with which $\overline{a_1 a_2 \ldots a_m} = A$ appears at the beginning when $n \to \infty$, is equal to the length of the interval $[\log_3 A - m + 1, \log_3(A+1) - m + 1]$, which is $\log_3 \frac{A+1}{A} = \log_3(A+1) - \log_3 A$. For example, the frequency of 1 and the frequency of 2 as the leftmost digit of 2^n, are $\log_3 \frac{2}{1} = \log_3 2 \approx 0.63$ and $\log_3 \frac{3}{2} = 1 - \log_3 2 \approx 0.37$, respectively. In contrast to the case of rightmost digits described above, where the frequency is the same for all combinations, the frequency of the first digits shows preference for smaller A, when m is fixed. This phenomenon is generally known as Benford's law or The Significant-Digit Phenomenon (see, e.g., [7, 8]). See also Exercise 20.3.2 in [10, p. 502] for a version of this law similar to ours. See [1], the recent paper [4], and the references therein for more details about Benford's law.

5 Main result

We can now use the stated results for the first and last digits in order to prove the main result of the present paper, which generalizes Observation III in Section 1. If the powers of 2 are written so that each next power of 2, in ternary number system notation, is written on top of the previous power of 2, and the digits corresponding to the same place values are all on the same vertical lines, then arbitrarily large triangular blocks of zeros (twos) can appear in this infinite triangular table. In a more formal way this can be expressed in the following way. Our proof strategy arises from the fact that an arbitrary large number of consecutive zeros and twos appear infinitely many times in a ternary representation of 2^n.

Theorem 5.1. Let radix-3 be given. Then, we define the sequence given by $a_n = 2^n$ $(n \geq 0)$, with radix-3 representation $a_n = \left(a_n^{(k_n)} \ldots a_n^{(1)} a_n^{(0)}\right)_3$. Consider numbers $a_m, a_{m+1}, \ldots, a_{m+r}$, such that for some positive integer j, and integers $l_0 \geq l_1 \geq \cdots \geq l_{r-1} \geq l_r = 0$,

$a_m^{(j)}, a_{m+1}^{(j)}, \ldots, a_{m+r}^{(j)} = 2$, $a_m^{(j+1)}, a_{m+1}^{(j+1)}, \ldots, a_{m+r}^{(j+1)} \neq 2$, $a_{m+r+1}^{(j)} = 1$,

$a_{m-1}^{(j)} = a_{m-1}^{(j-1)} = \cdots = a_{m-1}^{(j-l_0)} = 1$, $a_m^{(j-1)} = a_m^{(j-2)} = \cdots = a_m^{(j-l_0)} = 2$,

$a_{m+1}^{(j-1)} = a_{m+1}^{(j-2)} = \cdots = a_{m+1}^{(j-l_1)} = 2, \ldots, a_{m+r-1}^{(j-1)} = a_{m+r-1}^{(j-2)} = \cdots = a_{m+r-1}^{(j-l_r)} = 2$,

$a_m^{(j-l_0-1)}, a_{m+1}^{(j-l_1-1)}, \ldots, a_{m+r-1}^{(j-l_{r-1}-1)}, a_{m+r}^{(j-l_r-1)} \neq 2$ (See Table 5.1).

Then the natural numbers r and l_0 can be made arbitrarily large, provided that m is sufficiently large. The same is true when digit 2 in the above relationships is replaced by digit 0.

2^{m+r+1}	$= \cdots$		$a_{m+r+1}^{(j)}$						$\cdots$
2^{m+r}	$= \cdots$	$a_{m+r}^{(j+1)}$	$\boxed{a_{m+r}^{(j-l_r)}}$	$a_{m+r}^{(j-l_r-1)}$					$\cdots$
$\vdots$	$= \cdots$	$\vdots$	$\vdots$	$\vdots$					$\cdots$
$\vdots$	$= \cdots$	$\vdots$	$\vdots$	$\vdots$					$\cdots$
2^{m+2}	$= \cdots$	$a_{m+2}^{(j+1)}$	$\boxed{a_{m+2}^{(j)}}$	$\boxed{a_{m+2}^{(j-1)}}$	$\cdots$	$\boxed{a_m^{(j-l_2)}}$	$a_m^{(j-l_2-1)}$		$\cdots$
2^{m+1}	$= \cdots$	$a_{m+1}^{(j+1)}$	$\boxed{a_{m+1}^{(j)}}$	$\boxed{a_{m+1}^{(j-1)}}$	$\cdots$	$\boxed{a_{m+1}^{(j-l_1+1)}}$	$\boxed{a_{m+1}^{(j-l_1)}}$	$a_{m+1}^{(j-l_1-1)}$	$\cdots$
2^m	$= \cdots$	$a_m^{(j+1)}$	$\boxed{a_m^{(j)}}$	$\boxed{a_m^{(j-1)}}$	$\cdots$	$\boxed{a_m^{(j-l_0+1)}}$	$\boxed{a_m^{(j-l_0)}}$	$a_m^{(j-l_0-1)}$	$\cdots$
2^{m-1}	$= \cdots$		$a_{m-1}^{(j)}$	$a_{m-1}^{(j-1)}$	$\cdots$	$\cdots$	$a_{m-1}^{(j-l_0)}$		$\cdots$

Table 5.1.

Proof. By Lemma 3.3 and Lemma 4.1, for sufficiently large n, we can obtain any sequence of the digits 0, 1, 2, including arbitrary large number of consecutive zeros $(00 \ldots 0)$ or twos $(22 \ldots 2)$. Because of the doubling algorithm described above, any such block of zeros (or twos) is included in a triangular block of zeros (or twos). This proves that the dimensions (height $r+1$ and width l_0+1) of these blocks are unbounded. The proof is complete. □

Note. It would be interesting to know how frequently such blocks with given dimensions appear in Table 1.1 or how fast the dimensions of the largest blocks of zeros and twos formed by radix-3 representations of $\{2^0, 2^1, \ldots, 2^n\}$ grow as n approaches infinity.

In view of these results, it would also be interesting to study the question of frequency for the intermediate digits of the powers of two and how this frequency changes when the block of digits A shifts from left endpoint, where the frequencies are different and obey Benford's law, to the right endpoint, where all the frequencies are equal. We determined that the probability of an m-digit number A, which cannot start with zero digit, appearing at the beginning (after 0th position from left) of 3-base representations of 2^n, is $p_0(A) = \log_3\left(1+\frac{1}{A}\right)$. Let us now find the probability $p_k(A)$ of an m-digit number A, which can start with zero or zeros now (it can even be only zeros 00…0), appearing after kth position from left of 3-base representations of 2^n. By adding the probabilities of A appearing after 3^k, after 3^k+1, …, after $3^{k+1}-1$, we obtain

$$p_k(A) = \log_3 \prod_{i=3^k}^{3^{k+1}-1}\left(1+\frac{1}{3^m i + A}\right). \tag{5.1}$$

See [10, 11] for the discussion of the case $m = 1$ in decimal number system. For simplicity, we will give examples only about the case $m = 1$. We already mentioned that $p_0(1) = \log_3 2 \approx 0.63$ and $p_0(2) = \log_3 \frac{3}{2} \approx 0.37$. Let us find corresponding probabilities for some $k > 1$. Using the above formula, we calculate that

$$p_1(0) = \log_3\left(\frac{67925}{45927}\right) \approx 0.36,$$
$$p_1(1) = \log_3\left(\frac{2737}{1900}\right) \approx 0.33,$$
$$p_1(2) = \log_3\left(\frac{78732}{55913}\right) \approx 0.31.$$

Similarly, $p_2(0) \approx 0.341$, $p_2(1) \approx 0.333$, $p_2(2) \approx 0.326$ and $p_3(0) \approx 0.336$, $p_3(1) \approx 0.333$, $p_3(2) \approx 0.331$, etc. We can observe that for each k the sum of the probabilities is $p_k(0)+p_k(1)+p_k(2) = 1$ and $p_k(0)$, $p_k(1)$, $p_k(2)$ approach each other. We can prove these in more general m-digit case. For the sum of the probabilities, we can write

$$\begin{aligned}
p_k(0)+p_k(1)+\cdots+p_k(3^m-1) &= \log_3 \prod_{i=3^k}^{3^{k+1}-1} \frac{3^m i+1}{3^m i}\cdot\frac{3^m i+2}{3^m i+1}\cdot\ldots\cdot\frac{3^m i+3^m}{3^m i+3^m-1}\\
&= \log_3 \prod_{i=3^k}^{3^{k+1}-1} \frac{3^m(i+1)}{3^m i}\\
&= \log_3 \prod_{i=3^k}^{3^{k+1}-1} \frac{i+1}{i}\\
&= \log_3 \frac{3^{k+1}}{3^k}\\
&= \log_3 3 = 1.
\end{aligned}$$

For the difference of the probabilities, note that $p_k(A)$ decreases as A increases. So, we will estimate the difference of the largest $p_k(0)$ and smallest $p_k(3^m - 1)$:

$$\begin{aligned}
p_k(0) - p_k(3^m - 1) < p_k(0) - p_k(3^m) & \\
&= \log_3 \prod_{i=3^k}^{3^{k+1}-1} \frac{3^m i + 1}{3^m i} \cdot \frac{3^m(i+1)}{3^m(i+1)+1} \\
&= \log_3 \frac{3^{m+k}+1}{3^k} \cdot \frac{3^{k+1}}{3^{m+k+1}+1} \\
&= \log_3 \frac{3^{m+k+1}+3}{3^{m+k+1}+1} \\
&= \log_3 \frac{1+3^{-m-k}}{1+3^{-m-k-1}} \\
&< \frac{2}{(1+3^{m+k+1})\ln 3}.
\end{aligned}$$

The last inequality can be easily proved by applying The Mean Value Theorem to function $f(x) = \log_3(1+x)$ in the interval $(3^{-m-k-1}, 3^{-m-k})$. Indeed, there is $c \in (3^{-m-k-1}, 3^{-m-k})$ such that

$$\begin{aligned}
f(3^{-m-k}) - f(3^{-m-k-1}) &= \log_3(1+3^{-m-k}) - \log_3(1+3^{-m-k-1}) \\
f'(c)(3^{-m-k} - 3^{-m-k-1}) &= \frac{2}{3^{m+k+1}(1+c)\ln 3}.
\end{aligned}$$

It remains to note that since $c > 3^{-m-k-1}$,

$$\begin{aligned}
\log_3(1+3^{-m-k}) - \log_3(1+3^{-m-k-1}) &< \frac{2}{3^{m+k+1}(1+3^{-m-k-1})\ln 3} \\
&= \frac{2}{(1+3^{m+k+1})\ln 3}.
\end{aligned}$$

This expression approaches zero exponentially if m is fixed and $k \to \infty$. Since $\sum_{A=0}^{3^m-1} p_k(A) = 1$, this proves that (cf. [7], p. 355)

$$\lim_{k\to\infty} p_k(A) = \frac{1}{3^m} \quad (A = 0,1,\dots,3^m-1).$$

We can also show that as k increases $(k \geq 0)$, $p_k(0)$ decreases and $p_k(3^m - 1)$ increases, that is $p_k(0) > p_{k+1}(0)$ and $p_k(3^m - 1) < p_{k+1}(3^m - 1)$ for $m > 0$. For this let us write these probabilities differently:

$$\begin{aligned}
p_k(0) &= \log_3 \prod_{i=3^k}^{3^{k+1}-1} \left(1 + \frac{1}{3^m i}\right), \\
p_{k+1}(0) &= \log_3 \prod_{i=3^{k+1}}^{3^{k+2}-1} \left(1 + \frac{1}{3^m i}\right)
\end{aligned}$$

$$= \log_3 \prod_{i=3^k}^{3^{k+1}-1} \left(1 + \frac{1}{3^m \cdot 3i}\right)\left(1 + \frac{1}{3^m \cdot (3i+1)}\right)\left(1 + \frac{1}{3^m \cdot (3i+2)}\right),$$

$$p_k(3^m - 1) = \log_3 \prod_{i=3^k}^{3^{k+1}-1} \frac{1}{1 - \frac{1}{3^m(i+1)}},$$

$$p_{k+1}(3^m - 1) = \log_3 \prod_{i=3^{k+1}}^{3^{k+2}-1} \frac{1}{1 - \frac{1}{3^m(i+1)}}$$

$$= \log_3 \prod_{i=3^k}^{3^{k+1}-1} \frac{1}{1 - \frac{1}{3^m(3i+1)}} \cdot \frac{1}{1 - \frac{1}{3^m(3i+2)}} \cdot \frac{1}{1 - \frac{1}{3^m(3i+3)}}.$$

For the monotonicity of $p_k(0)$ and $p_k(3^m - 1)$, it sufficient to show that

$$1 + \frac{1}{3^m i} > \left(1 + \frac{1}{3^m \cdot 3i}\right)\left(1 + \frac{1}{3^m \cdot (3i+1)}\right)\left(1 + \frac{1}{3^m \cdot (3i+2)}\right),$$

$$1 - \frac{1}{3^m(i+1)} > \left(1 - \frac{1}{3^m(3i+1)}\right) \cdot \left(1 - \frac{1}{3^m(3i+2)}\right) \cdot \left(1 - \frac{1}{3^m(3i+3)}\right),$$

which can be easily proved. Similarly, if k is fixed and m increases, then the probabilities $p_k(0), p_k(1), \ldots, p_k(3^m - 1)$ become more uniform. In particular, this means that a sequence of m zeros 00…0 is more likely to appear towards the left side of the construction than a sequence of m twos 22…2, but as the block of m digits approach the right side of the construction then the probabilities become closer to each other. It would be interesting to study the effect of this on the probability of appearance and the sizes of triangular blocks of zeros and twos discussed above.

We can now return to the mentioned question asked by Erdős [5, 6]: *How frequently do the powers of* 2 *have ternary expansions that omit the digit* 2*?* He conjectured that this holds only for finitely many powers of 2. See [12] for the detailed discussion of this problem. In view of the results of the current paper, Erdős' conjecture can be interpreted in the following way. There are only finitely many powers of 2 which do not intersect the triangular blocks containing only twos. In Table 1.1 the numbers $2^0, 2^2$ and, 2^8 do not cross any of the regions with digits 2. One could prove, using these elementary methods, that 2 appears in the ternary expansion of 2^n for "almost all" n, in the sense of asymptotic density. But there is a large gap between "almost all" and "all but finitely many", and this is the real difficulty of Erdős' problem. A similar observation can be made about the powers of two which miss the regions of zeros. A better understanding of these structures of zeros and twos may be helpful in the future for attempting to solve Erdős' problem.

6 Conclusion

In the present paper we have focused our attention on the regularities occurring in base 3 representation of powers of 2, so we have shown that (I) every string of ending digits appears infinitely often, by assuming that the string itself is not congruent to 0 modulo 3, (II) every string of starting digits (not beginning with 0) appears infinitely often, (III) if the powers of 2 are all written in base 3 as one column so that the digits of the same place value are on top of each other, then the size of the triangular blocks of zeros and twos grow indefinitely. Part (I) was proved using elementary number theory methods, but it can also be proved using the fact that 2 is a primitive root modulo each power of 3. Part (II) was proved using the fact that for irrational number α, the sequence $x_n = \{n\alpha\}$, where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x, is uniformly distributed in $[0,1]$, but it can also be interpreted in the context of "Benford's law". Part (III), which is the main objective of the current paper, is shown to be a direct consequence of the Parts (I) and (II). Also, the change of the distribution of probabilities of these combination of digits when they are taken in between the left and right endpoints, is studied.

Acknowledgements

I thank the reviewers for their critical comments and suggestions which resulted in a much clearer and rigorous presentation of the results.

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