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Narayana sequence and the Brocard–Ramanujan equation

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Abstract: Let $\{a_n\}_{n\geq 0}$ be the Narayana sequence defined by the recurrence $a_n = a_{n-1} + a_{n-3}$ for all $n \geq 3$ with intital values $a_0 = 0$ and $a_1 = a_2 = 1$. In this paper, we fully characterize the 3-adic valuation of $a_n + 1$ and $a_n - 1$ and then we find all positive integer solutions (u, m) to the Brocard–Ramanujan equation $m! + 1 = u^2$ where u is a Narayana number.
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1 Introduction

Diophantine equations involving factorial numbers have been studied by many mathematicians in the last few years. By Bertrand's postulate, we can prove that $n!$ is a perfect power only when $n=1$. However, one of the most famous among such equations was posed by Brocard [4] in 1876 and independently by Ramanujan [12] in 1913. This Diophantine equation

$$m! + 1 = u^2 \tag{1}$$

is now known as *Brocard–Ramanujan equation*.

The three known solutions $m = 4, 5, 7$ are easy to check, meanwhile, no other solutions exist with $m \leq 10^9$ as it has been proved by Berndt and Galaway in [2]. Although Overholt [11] showed that the equation (1) has only many solutions under a weak version of the ABC conjecture, the Brocard–Ramanujan equation is still an open problem. Grossman and Luca [6] showed that if k is fixed, and F_n is the n-th Fibonacci number, then there are only finitely many positive integers n such that

$$F_n = m_1! + m_2! + \cdots + m_k!$$

holds for some positive integers $m_1, m_2, \ldots, m_k$. Moreover, all the solutions for the case $k \leq 2$ were determined. In 1999, Luca [7] proved that the n-th Fibonacci number F_n is a product of factorials only when $n = 1, 2, 3, 6$ and 12. Furthermore, Luca and Stanica [8] showed that the largest product of distinct Fibonacci numbers which is a product of factorials is

$$F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!\ .$$

In 2012 and 2016, Marques [5,9] proved that $(u, m) = (4, 5)$ is the only solution of Eq. (1) where u is a Fibonacci number and there is no solution of Eq. (1) when u is a Tribonacci number. Let $\{a_n\}_{n\geq 0}$ be the Narayana sequence defined by the recurrence $a_n = a_{n-1} + a_{n-3}$ for all $n \geq 3$ with initial values $a_0 = 0$ and $a_1 = a_2 = 1$. The first terms of this sequence are

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 28, 41, 60, 88, 129, 189, 277.$$

Some properties of the Narayana sequence and its generalizations can be found in [1,3]. We are following the same technique used in [5] by Vinicius Facó and Diego Marques. More precisely, we prove the following theorem.

Theorem 1.1. *There are no positive integer solutions (m, u) with $u = a_n$ for the Brocard–Ramanujan equation (1), where a_n is the n-th member of the Narayana sequence.*

2 Auxiliary results

Before proceeding further, some lemmas will be needed. The next lemma provides a formula for the Narayana numbers.

Lemma 2.1. *For all positive integers m, n, we have*

$$a_{m+n} = a_{m-1}a_{n+2} + a_{m-3}a_{n+1} + a_{m-2}a_n.$$

Proof. We prove this result using induction on n. At $n = 0$, we have $a_{m-1}a_2 + a_{m-3}a_1 + a_{m-2}a_0 = a_m$. So the relation is true at $n = 0$. Now, assume that the relation is true for all $j \leq n$. In particular,

$$a_{m+k} = a_{m-1}a_{k+2} + a_{m-3}a_{k+1} + a_{m-2}a_k$$

and we want to prove this relation at $n = k + 1$.

$$\begin{aligned} a_{m+k+1} &= a_{m+k} + a_{m+k-2} \\ &= a_{m-1}a_{k+2} + a_{m-3}a_{k+1} + a_{m-2}a_k + a_{m-1}a_k + a_{m-3}a_{k-1} + a_{m-2}a_{k-2} \\ &= a_{m-1}a_{k+3} + a_{m-3}a_{k+2} + a_{m-2}a_{k+1}. \end{aligned}$$

So, the relation is true for every positive integer n. □

The following lemma gives the upper and lower bound for the Narayana numbers.

Lemma 2.2. *For all integers $n \geq 1$, we have $\alpha^{n-3} \leq a_n \leq \alpha^{n-1}$, where α is the real root of the characteristic polynomial $f(x) = x^3 - x^2 - 1$ given by*

$$\alpha = \frac{1}{3}\left(1 + \sqrt[3]{\frac{29 - 3\sqrt{93}}{2}} + \sqrt[3]{\frac{29 + 3\sqrt{93}}{2}}\right).$$

Proof. Using induction on n. □

The p-adic order $v_p(k)$ of k is the exponent of the highest power of a prime p, which divides k. The next lemma gives the upper and lower bound of p-adic of factorials.

Lemma 2.3. *For any integer $m \geq 1$ and prime p, we have*

$$\frac{m}{p-1} - \left\lfloor \frac{\log m}{\log p} \right\rfloor - 1 \leq v_p(m!) \leq \frac{m-1}{p-1}.$$

Proof. This formula can be found in [10]. □

Lemma 2.4.

1. *If $i \equiv 16, 21 \bmod 24$, then $a_i \equiv 0 \bmod 9$;*
2. *If $i \equiv 7 \bmod 24$, then $a_i \equiv 0 \bmod 3$.*

Proof. Case (1): $i \equiv 16, 21 \bmod 24$.

- Subcase (1): $i \equiv 16 \bmod 24$. We prove that $a_i \equiv 0 \bmod 9$ using induction. At $k = 16$, we have $a_{16} \equiv 0 \bmod 9$. Now, assume that $a_{24k+16} \equiv 0 \bmod 9$ and we want to prove that $a_{24(k+1)+16} \equiv 0 \bmod 9$. Using Lemma 2.1, we have

$$\begin{aligned} a_{24(k+1)+16} &= a_{23}a_{24k+18} + a_{21}a_{24k+17} + a_{22}a_{24k+16} \\ &\equiv a_{24k+16} \bmod 9 \\ &\equiv 0 \bmod 9. \end{aligned}$$

Subcase (2) and Case (2) can be done in the same way. □

Proposition 2.5. *For all integers s and $n \geq 2$, we have*

$$\begin{aligned} a_{8s3^n} &\equiv 3^{n+3} \cdot 2s + 3^{n+2} \cdot 2s \bmod 3^{n+4}; \\ a_{8s3^n+1} &\equiv 3^{n+2} \cdot 5s + 3^{n+1} \cdot s + 1 \bmod 3^{n+4}; \\ a_{8s3^n+2} &\equiv 3^{n+3} \cdot 2s + 3^{n+2} \cdot 5s + 1 \bmod 3^{n+4}. \end{aligned} \tag{2}$$

Proof. We prove this proposition using induction on n. At $n = 2$ we want to prove the following:

$$\begin{aligned} a_{72s} &\equiv 3^4 \cdot 8s \bmod 3^6; \\ a_{72s+1} &\equiv 3^3 \cdot 16s + 1 \bmod 3^6; \\ a_{72s+2} &\equiv 3^4 \cdot 11s + 1 \bmod 3^6. \end{aligned} \tag{3}$$

We can prove this by using induction on s. At $s = 1$, we have

$$\begin{aligned} 374009739309 = a_{72} &\equiv 648 \bmod 3^6; \\ 548137914373 = a_{73} &\equiv 433 \bmod 3^6; \\ 803335158406 = a_{74} &\equiv 163 \bmod 3^6, \end{aligned}$$

which proves the initial step. Now, assume that the congruences are true at $s - 1$ and we want to prove them at s. Using the inductive hypothesis on $s - 1$, the definition of the Narayana numbers and Lemma 2.1, one can deduce the following:

$$\begin{aligned} a_{72s} &= a_{72+72(s-1)} = a_{71}a_{72(s-1)+2} + a_{69}a_{72(s-1)+1} + a_{70}a_{72(s-1)} \\ &\equiv 459\left(3^5 \cdot 2(s-1) + 3^4 \cdot 5(s-1) + 1\right) + 189\left(3^4 \cdot 5(s-1) + 3^3 \cdot (s-1) + 1\right) \\ &\quad + 514\left(3^5 \cdot 2(s-1) + 3^4 \cdot 2(s-1)\right) \bmod 729 \\ &\equiv 3^4 \cdot 8s \bmod 729. \end{aligned}$$

In the same manner, one can deduce the following:

$$\begin{aligned} a_{72s+1} &\equiv 3^3 \cdot 16s + 1 \bmod 729; \\ a_{72s+2} &\equiv 3^4 \cdot 11s + 1 \bmod 729. \end{aligned}$$

Thus the congruences (3) hold for $s \geq 1$ and $n = 2$. Given $s \geq 1$ and $n \geq 2$, assume the congruences (2) are true for $n-1$ and we want to prove them at n. Using the inductive hypothesis and the definition of the Narayana numbers, one can deduce the following:

$$\begin{aligned} a_{3^{n-1}\cdot 8s} &= 3^{n+2} \cdot 2s + 3^{n+1} \cdot 2s + c_0 \cdot 3^{n+3}; \\ a_{3^{n-1}\cdot 8s+1} &= 3^{n+1} \cdot 5s + 3^n \cdot s + 1 + 3^{n+3} \cdot c_1; \\ a_{3^{n-1}\cdot 8s+2} &= 3^{n+2} \cdot 2s + 3^{n+1} \cdot 5s + 1 + 3^{n+3} \cdot c_2; \\ a_{3^{n-1}\cdot 8s-2} &= -3^{n+2} \cdot s + 3^n \cdot s + 1 + (c_1 - c_0)\, 3^{n+3}; \\ a_{3^{n-1}\cdot 8s-1} &= 3^{n+2} \cdot 2s - 3^n \cdot s + 3^{n+3}(c_2 - c_1). \end{aligned}$$

where c_0, c_1, c_2 are integers. Using Lemma 2.1 and the previous relations, we have

$$\begin{aligned} a_{2(3^{n-1}\cdot 8s)} &= a_{(3^{n-1}\cdot 8s+1)+(3^{n-1}\cdot 8s-1)} \\ &= a_{3^{n-1}\cdot 8s}a_{3^{n-1}\cdot 8s+1} + a_{3^{n-1}\cdot 8s-2}a_{3^{n-1}\cdot 8s} + a_{3^{n-1}\cdot 8s-1}a_{3^{n-1}\cdot 8s-1} \\ &\equiv (3^{n+2} \cdot 4s + 3^{n+3} \cdot 2c_0 + 3^{n+1} \cdot 4s) \bmod 3^{n+4}. \end{aligned}$$

In the same manner, one can deduce the following:

$$\begin{aligned} a_{2(3^{n-1}\cdot 8s)+1} &\equiv 1 + 3^{n+1} \cdot 10s + 3^n \cdot 2s + 3^{n+3} \cdot 2c_1 \bmod 3^{n+4}; \\ a_{2(3^{n-1}\cdot 8s)+2} &\equiv 1 + 3^{n+2} \cdot 4s + 3^{n+1} \cdot 10s + 3^{n+3} \cdot 2c_2 \bmod 3^{n+4}. \end{aligned}$$

Consequently,

$$\begin{aligned}
a_{3^n\cdot 8s} &= a_{3^{n-1}\cdot 8s+2(3^{n-1}\cdot 8s)}\\
&= a_{3^{n-1}\cdot 8s-1}a_{2(3^{n-1}\cdot 8s)+2} + (a_{3^{n-1}\cdot 8s} - a_{3^{n-1}\cdot 8s-1})a_{2(3^{n-1}\cdot 8s)+1} + a_{3^{n-1}\cdot 8s-2}a_{2(3^{n-1}\cdot 8s)}\\
&\equiv \left(3^{n+2}\cdot 2s - 3^n \cdot s + (c_2 - c_1)\, 3^{n+3}\right)\left(1 + 3^{n+2}\cdot 4s + 3^{n+1}\cdot 10s + 3^{n+3}\cdot 2c_2\right)\\
&\quad + \left(3^{n+2}\cdot 2s + 3^{n+1}\cdot 2s + c_0\cdot 3^{n+3} - 3^{n+2}\cdot 2s + 3^n\cdot s + (c_1 - c_2)\, 3^{n+3}\right)\\
&\quad \left(1 + 3^{n+1}\cdot 10s + 3^n\cdot 2s + 2c_1\cdot 3^{n+3}\right) + \left(-3^{n+2}\cdot s + 3^n\cdot s + 1 + (c_1 - c_0)\, 3^{n+3}\right)\\
&\quad \left(3^{n+2}\cdot 4s + 2c_0\cdot 3^{n+3} + 3^{n+1}\cdot 4s\right) \bmod 3^{n+4}\\
&\equiv 3^{n+3}\cdot 2s + 3^{n+2}\cdot 2s \bmod 3^{n+4}.
\end{aligned}$$

In the same manner, one can deduce the following:

$$\begin{aligned}
a_{3^n\cdot 8s+1} &\equiv 3^{n+2}\cdot 5s + 3^{n+1}\cdot s + 1 \bmod 3^{n+4};\\
a_{3^n\cdot 8s+2} &\equiv 3^{n+2}\cdot 5s + 3^{n+3}\cdot 2s + 1 \bmod 3^{n+4}.
\end{aligned}$$ □

Corollary 2.6. *For all integers $s \geq 1$ and $n \geq 1$, we have*

$$\begin{aligned}
a_{8s3^n} &\equiv 3^{n+2}\cdot 2s \bmod 3^{n+3};\\
a_{8s3^n+1} &\equiv 3^{n+2}\cdot 2s + 3^{n+1}\cdot s + 1 \bmod 3^{n+3};\\
a_{8s3^n+2} &\equiv 3^{n+2}\cdot 2s + 1 \bmod 3^{n+3}.
\end{aligned} \tag{4}$$

Proof. The proof is a straightforward consequence of Proposition 2.5. □

Now we fully characterize the 3-adic valuation of $a_i + 1$ and $a_i - 1$.

Theorem 2.7. *For all positive integers i, and $a_i \neq 1$, we have*

$$v_3(a_i - 1) = \begin{cases}
0, & i \equiv 0, 4, 5, 7 \bmod 8;\\
v_3(i-1) + 1, & i \equiv 1 \bmod 8;\\
v_3(i+2) + 1, & i \equiv 6 \bmod 8;\\
v_3(i-2) + 2, & i \equiv 2 \bmod 24;\\
2, & i \equiv 10 \bmod 24;\\
v_3(i+6)(i+30) + 2, & i \equiv 18 \bmod 24;\\
v_3(i-3) + 2, & i \equiv 3 \bmod 24;\\
v_3(i+13) + 2, & i \equiv 11 \bmod 24;\\
v_3(i+5) + 2, & i \equiv 19 \bmod 24.
\end{cases}$$

Proof. Case (1): $i \equiv 0, 4, 5, 7 \bmod 8$.

- Subcase (1): $i \equiv 0 \bmod 8$, then $i = 8k$ for some integer k. We prove that $v_3(a_i - 1) = 0$ using induction. At $k = 0$, we have $a_0 - 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned}
a_{8(k+1)} - 1 &= a_{8k+8} - 1 = a_7 a_{8k+2} + a_5 a_{8k+1} + a_6 a_{8k} - 1\\
&\equiv a_{8k} - 1 \mod 3.
\end{aligned}$$

  Therefore, $a_{8k} - 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)} - 1 \not\equiv 0 \bmod 3$.

- Subcase (2): $i \equiv 4 \bmod 8$, then $i = 8k+4$ for some integer k. We prove that $v_3(a_i - 1) = 0$ using induction. At $k = 0$, we have $a_4 - 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)+4} - 1 &= a_{(8k+4)+8} - 1 = a_7 a_{8k+6} + a_5 a_{8k+5} + a_6 a_{8k+4} - 1 \\ &\equiv a_{8k+4} - 1 \quad \bmod 3. \end{aligned}$$

  Therefore, $a_{8k+4} - 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)+4} - 1 \not\equiv 0 \bmod 3$.

- Subcase (3): $i \equiv 5 \bmod 8$, then $i = 8k+5$ for some integer k. We prove that $v_3(a_i - 1) = 0$ using induction. At $k = 0$, we have $a_5 - 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)+5} - 1 &= a_{(8k+5)+8} - 1 = a_7 a_{8k+7} + a_5 a_{8k+6} + a_6 a_{8k+5} - 1 \\ &\equiv a_{8k+4} - 1 \bmod 3. \end{aligned}$$

  Therefore, $a_{8k+5} - 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)+5} - 1 \not\equiv 0 \bmod 3$.

- Subcase (4): $i \equiv 7 \bmod 8$, then $i = 8k+7$ for some integer k. We prove that $v_3(a_i - 1) = 0$ using induction. At $k = 0$, we have $a_7 - 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)+7} - 1 &= a_{(8k+7)+8} - 1 = a_7 a_{8k+9} + a_5 a_{8k+8} + a_6 a_{8k+7} - 1 \\ &\equiv a_{8k+7} - 1 \quad \bmod 3. \end{aligned}$$

  Therefore, $a_{8k+7} - 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)+7} - 1 \not\equiv 0 \bmod 3$.

Case (2): $i \equiv 1 \bmod 8$. In this case, we have $i - 1 = 3^n \cdot 8s$, where $n \geq 1$ and $3 \not| s$. Using Corollary 2.6, we have

$$\begin{aligned} a_i - 1 &= a_{3^n \cdot 8s+1} - 1 \\ &\equiv 1 + 3^{n+1} \cdot s + 3^{n+2} \cdot 2s - 1 \bmod 3^{n+3} \\ &\equiv 3^{n+1} \cdot s \bmod 3^{n+3}. \end{aligned}$$

Therefore, $v_3(a_i - 1) = n + 1 = v_3(i - 1) + 1$.
Case (3): $i \equiv 6 \bmod 8$. In this case, we have $i + 2 = 3^n \cdot 8s$, where $n \geq 1$ and $3 \not| s$. Using Corollary 2.6, we have

$$\begin{aligned} a_i - 1 &= a_{3^n \cdot 8s-2} - 1 \\ &\equiv 1 + 3^{n+1} \cdot s + 3^{n+2} \cdot 2s - 1 \bmod 3^{n+3} \\ &\equiv 3^{n+1} \cdot s \bmod 3^{n+3}. \end{aligned}$$

Therefore, $v_3(a_i - 1) = n + 1 = v_3(i + 2) + 1$.
Case (4): $i \equiv 2 \bmod 24$. In this case, we have $i - 2 = 3^n \cdot 8s$, where $n \geq 1$ and $3 \not| s$. Using Corollary 2.6, we have

$$\begin{aligned} a_i - 1 &= a_{3^n \cdot 8s+2} - 1 \\ &\equiv 1 + 3^{n+2} \cdot 2s - 1 \bmod 3^{n+3} \\ &\equiv 3^{n+2} \cdot 2s \bmod 3^{n+3}. \end{aligned}$$

Therefore, $v_3(a_i - 1) = n + 2 = v_3(i - 2) + 2$.

Case (5): $i \equiv 10 \bmod 24$, then $i = 24k + 10$ for some integer k. We are going to prove that $v_3(a_i - 1) = 2$ using induction. At $k = 0$, we have $v_3(a_{10} - 1) = 2$. Using Lemma 2.1, we have

$$\begin{aligned} a_{24(k+1)+10} - 1 &= a_{(24k+10)+24} - 1 = a_{23}a_{24k+12} + a_{21}a_{24k+11} + a_{22}a_{24k+10} - 1 \\ &\equiv (a_{24k+10} - 1) \bmod 9. \end{aligned}$$

Therefore, $a_{24k+10} - 1 \equiv 0 \bmod 9$ if and only if $a_{24(k+1)+10} - 1 \equiv 0 \bmod 9$. But, using Lemma 2.4

$$\begin{aligned} a_{24(k+1)+10} - 1 &\equiv 9\left(a_{24k+12} + a_{24k+12} + a_{24k+11} + a_{24k+10}\right) + a_{24k+10} - 1 \bmod 27 \\ &\equiv 9(3a_{24k+11} + 3a_{24k+9} + a_{24k+7}) + a_{24k+10} - 1 \bmod 27 \\ &\equiv a_{24k+10} - 1 \bmod 27. \end{aligned}$$

Therefore, $a_{24k+10} - 1 \not\equiv 0 \bmod 27$ if and only if $a_{24(k+1)+10} - 1 \not\equiv 0 \bmod 27$. Therefore, $v_3(a_i - 1) = 2$.

Case (6): $i \equiv 18 \bmod 24$, then $i = 24k + 18$ for some integer k. We want to prove that:

$$\begin{aligned} v_3(a_{24k+18} - 1) &= v_3\left((24k + 24)(24k + 48)\right) + 2 \\ &= v_3\left(24^2(k + 1)(k + 2)\right) + 2 \\ &= v_3\left((k + 1)(k + 2)\right) + 4. \end{aligned}$$

- Subcase (1): $k \equiv 0 \bmod 3$. We are going to prove that $v_3(a_{24k+18} - 1) = 4$ using induction. At $k = 0$, we have $a_{18} - 1 \equiv 0 \bmod 81$ and $a_{18} - 1 \not\equiv 0 \bmod 243$. We want to prove that $a_{72(k+1)+18} - 1 \equiv 0 \bmod 81$ and $a_{72(k+1)+18} - 1 \not\equiv 0 \bmod 243$. Using Lemma 2.1, we have

$$\begin{aligned} a_{72(k+1)+18} - 1 &= a_{(72k+18)+72} - 1 = a_{71}a_{72k+20} + a_{69}a_{72k+19} + a_{70}a_{72k+18} - 1 \\ &\equiv 27(2a_{72k+20} + a_{72k+19} + a_{72k+18}) + a_{72k+18} - 1 \bmod 81 \\ &\equiv a_{72k+18} - 1 \bmod 81. \end{aligned}$$

$$\begin{aligned} a_{72(k+1)+18} - 1 &= 27\left(8a_{72k+20} + 7a_{72k+19} + a_{72k+18}\right) + a_{72k+18} - 1 \bmod 243 \\ &= 27\left(9a_{72k+20} - a_{72k+20} + 7a_{72k+19} + a_{72k+18}\right) \\ &= 27\left(9a_{72k+20} + 7a_{72k+16} + 9a_{72k+18} - a_{72k+21}\right) + a_{72k+18} - 1 \bmod 243 \\ &\equiv a_{72k+18} - 1 \bmod 243. \end{aligned}$$

  Therefore, $a_{72k+18} - 1 \equiv 0 \bmod 81$ if and only if $a_{72(k+1)+18} - 1 \equiv 0 \bmod 81$ and $a_{72k+18} - 1 \not\equiv 0 \bmod 243$ if and only if $a_{72(k+1)+18} - 1 \not\equiv 0 \bmod 243$. Therefore, $v_3(a_i - 1) = 4$.

- Subcase (2): $k \equiv 1 \bmod 3$. In this case we have $i = 3^n \cdot 8s - 30$ where $n \geq 2$ and $3 \not| s$. Using Lemma 2.1 and Proposition 2.5, we have

$$
\begin{aligned}
a_i - 1 &= a_{3^n\cdot 8s-30} - 1 = a_{3^n\cdot 8s-27} - a_{3^n\cdot 8s-28} - 1 \\
&= a_{3^n\cdot 8s-24} - 2a_{3^n\cdot 8s-25} + a_{3^n\cdot 8s-26} - 1 \\
&= -3a_{3^n\cdot 8s-21} - 2a_{3^n\cdot 8s-22} + 3a_{3^n\cdot 8s-20} - 1 \\
&= -8a_{3^n\cdot 8s-18} + 4a_{3^n\cdot 8s-17} + a_{3^n\cdot 8s-16} - 1 \\
&= -12a_{3^n\cdot 8s-12} - 26a_{3^n\cdot 8s-11} + 21a_{3^n\cdot 8s-10} - 1 \\
&= 73a_{3^n\cdot 8s-6} - 63a_{3^n\cdot 8s-5} + 9a_{3^n\cdot 8s-4} - 1 \\
&= -64a_{3^n\cdot 8s-4} + 136a_{3^n\cdot 8s-3} - 63a_{3^n\cdot 8s-2} - 1 \\
&= a_{3^n\cdot 8s-2} - 200a_{3^n\cdot 8s-1} + 136a_{3^n\cdot 8s} - 1 \\
&= 201a_{3^n\cdot 8s+1} - 200a_{3^n\cdot 8s+2} + 135a_{3^n\cdot 8s} - 1 \\
&\equiv 201\left(3^{n+2}\cdot 5s + 3^{n+1}\cdot s + 1\right) - 200\left(3^{n+3}\cdot 2s + 3^{n+2}\cdot 5s + 1\right) \\
&\quad + 135\left(3^{n+3}2s + 3^{n+2}\cdot 2s\right) - 1 \bmod 3^{n+4} \equiv -3^{n+3}\cdot 130s \bmod 3^{n+4}.
\end{aligned}
$$

Therefore, $v_3(a_i - 1) = n + 3 = v_3(i + 30) + 3$.

Case (7): $i \equiv 3 \bmod 24$. In this case we have $i - 2 = 3^n \cdot 8s$ where $n \geq 1$ and $3 \not| s$. Using Corollary 2.6, we have

$$
\begin{aligned}
a_i - 1 &= a_{3^n\cdot 8s+3} - 1 \\
&= a_{3^n\cdot 8s} + a_{3^n\cdot 8s+2} - 1 \\
&\equiv 3^{n+2}\cdot 2s + 1 + 3^{n+2}\cdot 2s - 1 \bmod 3^{n+3} \\
&\equiv 3^{n+2}\cdot 4s \bmod 3^{n+3}.
\end{aligned}
$$

Therefore, $v_3(a_i - 1) = n + 2 = v_3(i - 3) + 2$.

Case (8): $i \equiv 11 \bmod 24$. In this case we have $i + 13 = 3^n \cdot 8s$ where $n \geq 1$ and $3 \not| s$. Using Corollary 2.6, we have

$$
\begin{aligned}
a_i - 1 &= a_{3^n\cdot 8s-13} - 1 \\
&= a_{3^n\cdot 8s-10} - a_{3^n\cdot 8s-11} - 1 \\
&= a_{3^n\cdot 8s-10} - a_{3^n\cdot 8s-8} - a_{3^n\cdot 8s-9} - 1 \\
&= a_{3^n\cdot 8s-7} - 2a_{3^n\cdot 8s-8} - a_{3^n\cdot 8s-9} - 1 \\
&= 3a_{3^n\cdot 8s-6} - 2a_{3^n\cdot 8s-5} - 1 \\
&= 5a_{3^n\cdot 8s-3} - a_{3^n\cdot 8s-4} - 2a_{3^n\cdot 8s-1} - 1 \\
&= 5a_{3^n\cdot 8s} - 8a_{3^n\cdot 8s-1} + a_{3^n\cdot 8s-2} - 1 \\
&= 4a_{3^n\cdot 8s} + 9a_{3^n\cdot 8s+1} - 8a_{3^n\cdot 8s+2} - 1 \\
&\equiv 4\left(3^{n+2}\cdot 2s\right) + 9\left(3^{n+2}\cdot 2s + 3^{n+1}\cdot s + 1\right) - 8\left(3^{n+2}\cdot 2s + 1\right) - 1 \bmod 3^{n+3} \\
&\equiv -4\left(3^{n+2}\cdot 2s\right) \bmod 3^{n+3}.
\end{aligned}
$$

Therefore, $v_3(a_i - 1) = n + 2 = v_3(i + 13) + 2$.

Case (9): $i \equiv 19 \bmod 24$. In this case we have $i + 5 = 3^n \cdot 8s$ where $n \geq 1$ and $3 \not| s$. Using

Corollary 2.6, we have

$$\begin{aligned} a_i - 1 &= a_{3^n \cdot 8s-5} - 1 \\ &= a_{3^n \cdot 8s-2} - a_{3^n \cdot 8s-3} - 1 \\ &= -2a_{3^n \cdot 8s} + a_{3^n \cdot 8s+2} - 1 \\ &\equiv -3^{n+2} \cdot 2s \bmod 3^{n+3}. \end{aligned}$$

Therefore, $v_3(a_i - 1) = n + 2 = v_3(i + 5) + 2$. □

Theorem 2.8. *For all integers i, we have*

$$v_3(a_i + 1) = \begin{cases} 0, & i \equiv 0, 1, 2, 3, 5, 6, 7 \bmod 8; \\ 1, & i \equiv 4, 12 \bmod 24; \\ v_3(i + 4) + 1, & i \equiv 20 \bmod 24. \end{cases}$$

Proof. Case (1): $i \equiv 0, 1, 2, 3, 5, 6, 7 \bmod 8$.

- Subcase (1): $i \equiv 0 \bmod 8$, then $i = 8k$ for some integer k. We are going to prove that $v_3(a_i + 1) = 0$ using induction. At $k = 0$, we have $a_8 + 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)} + 1 &= a_{8k+8} + 1 = a_7 a_{8k+2} + a_5 a_{8k+1} + a_6 a_{8k} + 1 \\ &\equiv (a_{8k} + 1) \bmod 3. \end{aligned}$$

  Therefore, $a_{8k} + 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)} + 1 \not\equiv 0 \bmod 3$. Therefore, $v_3(a_i + 1) = 0$.

- Subcase (2): $i \equiv 1 \bmod 8$, then $i = 8k + 1$ for some integer k. We prove that $v_3(a_i + 1) = 0$ using induction. At $k = 0$, we have $a_9 + 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)+1} + 1 &= a_{8k+8+1} + 1 = a_7 a_{8k+3} + a_5 a_{8k+2} + a_6 a_{8k+1} + 1 \\ &\equiv (a_{8k+1} + 1) \bmod 3. \end{aligned}$$

  Therefore, $a_{8k+1} + 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)+1} + 1 \not\equiv 0 \bmod 3$. Therefore, $v_3(a_i + 1) = 0$.

- Subcase (3): $i \equiv 2 \bmod 8$, then $i = 8k + 2$ for some integer k. We prove that $v_3(a_i + 1) = 0$ using induction. At $k = 0$, we have $a_2 + 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)+2} + 1 &= a_{8k+8+2} + 1 = a_7 a_{8k+4} + a_5 a_{8k+3} + a_6 a_{8k+2} + 1 \\ &\equiv (a_{8k+2} + 1) \bmod 3 \end{aligned}$$

  Therefore, $a_{8k+2} + 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)+2} + 1 \not\equiv 0 \bmod 3$. Therefore, $v_3(a_i + 1) = 0$.

- Subcase (4): $i \equiv 3 \bmod 8$, then $i = 8k+3$ for some integer k. We prove that $v_3(a_i+1) = 0$ using induction. At $k = 0$, we have $a_3 + 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)+3} + 1 &= a_{8k+8+3} + 1 = a_7 a_{8k+5} + a_5 a_{8k+4} + a_6 a_{8k+3} + 1 \\ &\equiv (a_{8k+3} + 1) \bmod 3. \end{aligned}$$

Therefore, $a_{8k+3} + 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)+3} + 1 \not\equiv 0 \bmod 3$. Therefore, $v_3(a_i + 1) = 0$.

- Subcase (5): $i \equiv 5 \bmod 8$, then $i = 8k+5$ for some integer k. We prove that $v_3(a_i+1) = 0$ using induction. At $k = 0$, we have $a_5 + 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)+5} + 1 &= a_{8k+8+5} + 1 = a_7 a_{8k+7} + a_5 a_{8k+6} + a_6 a_{8k+5} + 1 \\ &\equiv (a_{8k+5} + 1) \bmod 3. \end{aligned}$$

Therefore, $a_{8k+5} + 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)+5} + 1 \not\equiv 0 \bmod 3$. Therefore, $v_3(a_i + 1) = 0$.

- Subcase (6): $i \equiv 6 \bmod 8$, then $i = 8k+6$ for some integer k. We prove that $v_3(a_i+1) = 0$ using induction. At $k = 0$, we have $a_6 + 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)+6} + 1 &= a_{8k+8+6} + 1 = a_7 a_{8k+8} + a_5 a_{8k+7} + a_6 a_{8k+6} + 1 \\ &\equiv (a_{8k+6} + 1) \bmod 3. \end{aligned}$$

Therefore, $a_{8k+6} + 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)+6} + 1 \not\equiv 0 \bmod 3$. Therefore, $v_3(a_i + 1) = 0$.

- Subcase (7): $i \equiv 7 \bmod 8$, then $i = 8k+7$ for some integer k. We prove that $v_3(a_i+1) = 0$ using induction. At $k = 0$, we have $a_7 + 1 \not\equiv 0 \bmod 3$. Using Lemma 2.1, we have

$$\begin{aligned} a_{8(k+1)+7} + 1 &= a_{8k+8+7} + 1 = a_7 a_{8k+9} + a_5 a_{8k+8} + a_6 a_{8k+7} + 1 \\ &\equiv (a_{8k+7} + 1) \bmod 3 \end{aligned}$$

Therefore, $a_{8k+7} + 1 \not\equiv 0 \bmod 3$ if and only if $a_{8(k+1)+7} + 1 \not\equiv 0 \bmod 3$. Therefore, $v_3(a_i + 1) = 0$.

Case (2): $i \equiv 4, 12 \bmod 24$

- Subcase (1): $i \equiv 4 \bmod 24$, then $i = 24k+4$ for some integer k. We are going to prove that $v_3(a_i+1) = 1$ using induction. At $k = 0$, we have $a_4+1 \equiv 0 \bmod 3$ and $a_4+1 \not\equiv 0 \bmod 9$. Using Lemma 2.1, we have

$$\begin{aligned} a_{24(k+1)+4} + 1 &= a_{24k+24+4} + 1 = a_{23} a_{24k+6} + a_{21} a_{24k+5} + a_{22} a_{24k+4} + 1 \\ &\equiv (a_{24k+4} + 1) \bmod 9. \end{aligned}$$

Therefore, $a_{24k+4} - 1 \equiv 0 \bmod 3$ if and only if $a_{24(k+1)+4} - 1 \equiv 0 \bmod 3$, and $a_{24k+4} - 1 \not\equiv 0 \bmod 9$ if and only if $a_{24(k+1)+4} - 1 \not\equiv 0 \bmod 9$. Therefore, $v_3(a_i + 1) = 1$.

- Subcase (2): $i \equiv 12 \bmod 24$, then $i = 24k + 12$ for some integer k. We prove that $v_3(a_i + 1) = 1$ using induction. At $k = 0$, we have $a_{12} + 1 \equiv 0 \bmod 3$ and $\not\equiv 0 \bmod 9$. Using Lemma 2.1, we have

$$\begin{aligned} a_{24(k+1)+12} + 1 &= a_{24k+24+12} + 1 = a_{23}a_{24k+14} + a_{21}a_{24k+13} + a_{22}a_{24k+12} + 1 \\ &\equiv (a_{24k+12} + 1) \bmod 9 \equiv 0 \bmod 3 \not\equiv 0 \bmod 9. \end{aligned}$$

  Therefore, $a_{24k+12} + 1 \equiv 0 \bmod 3$ if and only if $a_{24(k+1)+12} + 1 \equiv 0 \bmod 3$, and $a_{24k+12} + 1 \not\equiv 0 \bmod 9$ if and only if $a_{24(k+1)+12} + 1 \not\equiv 0 \bmod 9$. Therefore, $v_3(a_i + 1) = 1$.

Case (3): $i \equiv 20 \bmod 24$. In this case we have $i = 3^n \cdot 8s - 4$ where $n \geq 1$ and $3 \not| s$. Using Lemma 2.1 and Corollary 2.6. Then, we have

$$\begin{aligned} a_i + 1 &= a_{8s3^n-4} + 1 = a_{8s3^n-1} - a_{8s3^n-2} + 1 \\ &= a_{8s3^n+2} - 2a_{8s3^n+1} + a_{8s3^n} + 1 \\ &\equiv -3^{n+1} \cdot 2s \bmod 3^{n+3}. \end{aligned}$$

Therefore, $v_3(a_i + 1) = n + 1 = v_3(i + 4) + 1 = n + 1$. □

3 Proof of Theorem 1.1

Proof. If $a_n = 1$, there is no solution for equation (1). Now suppose that $a_n \neq 1$ and using that fact

$$\frac{m}{2} - \left\lfloor \frac{\log m}{\log 3} \right\rfloor - 1 \leq v_3(m!);$$

together with Theorem 2.7 and Theorem 2.8, we get

$$\begin{aligned} &\frac{m}{2} - \left\lfloor \frac{\log m}{\log 3} \right\rfloor - 1 \\ &\leq v_3(m!) = v_3(a_n - 1) + v_3(a_n + 1) \\ &\leq v_3((n-1)(n+2)(n-2)(n+6)(n+30)(n-3)(n+13)(n+15)(n+4)) + 16. \end{aligned}$$

Thus,

$$\frac{m}{2} - \left\lfloor \frac{\log m}{\log 3} \right\rfloor - 1 \leq 9v_3(n + w) + 16,$$

where $w \in \{-1, 2, -2, 6, 30, -3, 13, 5, 4\}$. Therefore,

$$3^{\left\lfloor \frac{1}{9}\left(\frac{m}{2} - \left\lfloor \frac{\log m}{\log 3} \right\rfloor - 17\right)\right\rfloor} \leq n + w \leq n + 30.$$

By applying the log function, we obtain

$$\left\lfloor \frac{1}{9}\left(\frac{m}{2} - \left\lfloor \frac{\log m}{\log 3} \right\rfloor - 17\right)\right\rfloor \leq \frac{n + 30}{\log 3}. \tag{5}$$

On the other hand,

$$(1.64)^{2n-6} \leq a_n^2 = m! + 1 < 2\left(\frac{m}{2}\right)^m;$$

So

$$n < 4 + (1.33)m \log\left(\frac{m}{2}\right).$$

Substituting in equation (5), we obtain

$$\left\lfloor \frac{1}{9}\left(\frac{m}{2} - \left\lfloor \frac{\log m}{\log 3} \right\rfloor - 17\right) \right\rfloor \leq \frac{34 + 1.33 \log\left(\frac{m}{2}\right)}{\log 3}.$$

This inequality yields $m \leq 221$. Then $n \leq 1386$. Now, we use a simple routine written in SAGE to get the solutions. The proof is completed. □

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