The mean value of the function $\frac{d(n)}{d^*(n)}$ in arithmetic progressions

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Abstract: Let $d(n)$ and $d^*(n)$ be, respectively, the number of divisors and the number of unitary divisors of an integer $n \geq 1$. A divisor $d$ of an integer is to be said unitary if it is prime over $\frac{n}{d}$. In this paper, we study the mean value of the function $D(n) = \frac{d(n)}{d^*(n)}$ in the arithmetic progressions $\{l + mk \mid m \in \mathbb{N}^* \text{ and } (l, k) = 1\}$; this leads back to the study of the real function $x \mapsto S(x; k, l) = \sum_{n \leq x \atop n \equiv l[k]} D(n)$. We prove that

$$S(x; k, l) = A(k)x + O_k \left( x \exp \left( -\frac{\theta}{2} \sqrt{(2 \ln x)(\ln \ln x)} \right) \right) \quad (0 < \theta < 1),$$

where $A(k) = \frac{c}{k} \prod_{p|k} \left( 1 + \frac{1}{2} \sum_{n=2}^{+\infty} \frac{1}{p^n} \right)^{-1} \left( c = \zeta(2) \prod_p \left( 1 - \frac{1}{2p^2} + \frac{1}{2p^3} \right) \right)$.

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1 Introduction

For an integer \( n \in \mathbb{N}^* \), we denote by \( d(n) \) the number of divisors of \( n \), by \( d^*(n) \) the number of unitary divisors of \( n \). We recall that a divisor \( d \) of \( n \) is unitary if it is prime with the quotient \( \frac{n}{d} \).

\[
d(n) = \sum_{d|n} 1, \quad d^*(n) = \sum_{\substack{d|n \quad \text{gcd}(d, \frac{n}{d}) = 1}} 1.
\]

We put \( D(n) = \frac{d(n)}{d^*(n)} \), it is known that the function \( n \mapsto \frac{\ln 2 \ln n}{3 \ln \ln n} \) \((n \geq 3)\) is a maximum order of the function \( n \mapsto \ln (D(n)) \) (see [4]), however the function \( D(n) \) takes the value 1 on all prime numbers \( p \). This erratic behavior of the function \( D(n) \) motivates to study its mean value in (see [5]), the authors obtained the asymptotic formula \( \sum_{n \leq x} D(n) = c x + O(\sqrt{x}) \) where \( c = \frac{\pi^2}{6} \prod_p \left(1 - \frac{1}{2p^2} + \frac{1}{2p^3}\right) = 1.4276565\ldots \). In this paper, we are interested in the study of the mean value of the function \( D(n) \) in the arithmetic progressions \( \{l + mk \mid m \in \mathbb{N}^*\} \). The latter leads to studying the real function \( \sum_{n \leq x} D(n) \). We obtained the results in the following Theorem.

**Theorem 1.1.** Let \( l \) and \( k \) be two integers, such that \( 1 \leq l \leq k \) and \( (l, k) = 1 \), and let \( S(x; k, l) \) be the real summation function defined by \( S(x; k, l) = \sum_{n \leq x} D(n) \). Uniformly with respect to \( k \geq 2 \), we have

\[
S(x; k, l) = A(k)x + \mathcal{O}_k \left(x \exp \left(-\frac{\theta}{2} \sqrt{(2 \ln x)(\ln \ln x)}\right)\right) \quad (x \to +\infty) \quad (0 < \theta < 1),
\]

where

\[
A(k) = \frac{c}{k} \prod_{p|k} \left(1 + \frac{\Gamma}{2} \sum_{n=2}^{+\infty} \frac{1}{p^n}\right)^{-1} \quad \left( c = \zeta(2) \prod_p \left(1 - \frac{1}{2p^2} + \frac{1}{2p^3}\right) \right).
\]

Our study is mainly based on:

a) The use of an orthogonality relation on the Dirichlet characters modulo \( k \) which has allowed us to write \( S(x; k, l) \) in the form \( S(x; k, l) = \frac{1}{\varphi(k)} \sum_{n \leq x} a(n), \) where \( a(n) \) is the arithmetic function defined by \( a(n) = \sum_{\chi \in \hat{G}(k)} \chi(l) \chi(n) D(n) \) \((n \in \mathbb{N}^*)\), where \( \hat{G}(k) \) denotes the set of Dirichlet characters modulo \( k \).

b) The estimation of the generating function \( \mathcal{Q}(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s} \) \((\Re(s) > 1)\).

2 Dirichlet characters modulo \( k \)

Let \( k \in \mathbb{N}^* \), we denote by \( G(k) \) the multiplicative group of invertible residues modulo \( k \)

\[
\left( G(k) = \left( \mathbb{Z}/k\mathbb{Z} \right)^* \right). \quad \text{The Chinese Remainder Theorem gives the canonical decomposition}
\]

\[
G(k) \cong \left( \mathbb{Z}/p_1^{\alpha_1} \mathbb{Z} \right)^* \times \cdots \times \left( \mathbb{Z}/p_r^{\alpha_r} \mathbb{Z} \right)^* \quad (k = p_1^{\alpha_1} \times \cdots \times p_r^{\alpha_r} \quad \text{where} \quad r \in \mathbb{N}^*, \alpha_i \geq 1 \text{ and } p_i \text{ are primes}).
\]
We know that \( \left( \mathbb{Z}/p^\alpha \mathbb{Z} \right)^* \) is cyclic for \( p \geq 3 \) and \( \alpha \in \mathbb{N}^* \) and \( \left( \mathbb{Z}/2^\alpha \mathbb{Z} \right)^* = \langle -1 \rangle \odot \langle 5 \rangle (\alpha \geq 3) \), where \( \langle -1 \rangle \odot \langle 5 \rangle \) denotes the direct product of the two subgroups of \( \left( \mathbb{Z}/2^\alpha \mathbb{Z} \right)^* \) respectively generated by \( \langle -1 \rangle \) and \( \langle 5 \rangle \), which allows us to explicitly determine the morphisms of \( G(k) \) in \( \mathbb{C}^* \) and hence the Dirichlet characters modulo \( k \). The canonical decomposition of \( \left( \mathbb{Z}/k \mathbb{Z} \right)^* \) implies that \( \chi \) is a Dirichlet character modulo \( k \) if and only if \( \chi = \chi_1 \times \cdots \times \chi_r \), where \( \chi_i \) are Dirichlet characters modulo \( p_i^\alpha_i \). So, we obtain the \( \varphi(k) \) characters modulo \( k \). The characters satisfy several properties, in particular the one we used in our study in this case: Any character \( \chi \) is periodic of period \( k \), \( \varphi \) is completely multiplicative, for any \( n \in \mathbb{Z} \) such that \( (n, k) = 1 \), we have \( \chi(n)\overline{\chi(n)} = \chi(n)\overline{\chi(n)} = |\chi(n)|^2 = 1 \) and the following orthogonality relations : for all \( l \in \mathbb{N} \) such that \( (l, k) = 1 \) and \( 1 \leq l \leq k \)

\[
\sum_{\chi \in \hat{G}(k)} \overline{\chi(l)} \chi(n) = \begin{cases} 
\varphi(k), & \text{if } n \equiv l[k], \\
0, & \text{if } n \not\equiv l[k].
\end{cases}
\]

3 Preparatory lemmas

**Lemma 3.1.** For \( l \) and \( k \) be two integers, with \( 1 \leq l \leq k \) and \( (l, k) = 1 \), we put

\[
S(x; k, l) = \sum_{n \leq x \atop n \equiv l[k]} D(n).
\]

We have \( S(x; k, l) = \frac{1}{\varphi(k)} \sum_{n \leq x} a(n) \), where \( a(n) = \sum_{\chi \in \hat{G}(k)} \overline{\chi(l)} \chi(n) D(n) \).

This is a consequence of the following lemma.

**Lemma 3.2.** For \( l \) and \( k \) two integers such that \( 1 \leq l \leq k \) and \( (l, k) = 1 \), we have

\[
\sum_{\chi \in \hat{G}(k)} \overline{\chi(l)} \chi(n) = \begin{cases} 
\varphi(k), & \text{if } n \equiv l[k], \\
0, & \text{if } n \not\equiv l[k].
\end{cases}
\]

**Proof.** If \( n \equiv l[k] \), then \( \chi(n) = \chi(l + mk) = \chi(l) (m \in \mathbb{Z}) \). So we have

\[
\sum_{\chi \in \hat{G}(k)} \overline{\chi(l)} \chi(n) = \sum_{\chi \in \hat{G}(k)} \overline{\chi(n)} \chi(n) = \sum_{\chi \in \hat{G}(k)} |\chi(n)|^2 = \sum_{\chi \in \hat{G}(k)} 1 = \left| \hat{G}(k) \right| = \varphi(k).
\]

If \( n \not\equiv l[k] \), then \( \chi(n) = 0 \left( \forall \chi \in \hat{G}(k) \right) \). So we have \( \sum_{\chi \in \hat{G}(k)} \overline{\chi(l)} \chi(n) = 0 \).

**Lemma 3.3** ([5], p. 556). In the half-plane \( \Re(s) > \frac{1}{2} \), the following equality takes place:

\[
\prod_p \left( 1 + \frac{1}{2} \sum_{n=2}^{+\infty} \frac{1}{n^{as}} \right) = \zeta(2s) \prod_p \left( 1 - \frac{1}{2p^{2s}} + \frac{1}{2p^{3s}} \right).
\]

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Lemma 3.4 ([5]). The function $s \mapsto \sum_{n=1}^{+\infty} \frac{D(n)}{n^s}$ ($\Re(s) > 1$) extends to the half-plane $\Re(s) > \frac{1}{2}$ by the formula $\sum_{n=1}^{+\infty} \frac{D(n)}{n^s} = \zeta(s) G(s)$, where $G(s) = \zeta(2s) \prod_p \left( 1 - \frac{1}{2p^{2s}} + \frac{1}{2p^{3s}} \right)$, into a meromorphic function admitting a single pole at $s = 1$ in which the residue is $G(1) = \zeta(2) \prod_p \left( 1 - \frac{1}{2p^2} + \frac{1}{2p^3} \right)$.

Lemma 3.5. For any character $\chi \in \hat{G}(k)$, we put $F(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)D(n)}{n^s}$ ($\Re(s) > 1$).

The function $F(s, \chi)$ extends to the half-plane $\Re(s) > \frac{1}{2}$ by the formulas

$$F(s, \chi) = L(s, \chi) G(s, \chi), \quad \text{where} \quad G(s, \chi) = \prod_p \left( 1 + \frac{1}{2} \sum_{n=2}^{+\infty} \frac{(\chi(p))^n}{p^{ns}} \right) \left( \chi \in \hat{G}(k) \right),$$

and

$$F(s, \chi_0) = \zeta(s) G(s) \prod_{p|k} \left( 1 - \frac{1}{p^s} \right) \left( 1 + \frac{1}{2} \sum_{n=2}^{+\infty} \frac{1}{p^{ns}} \right)^{-1},$$

where

$$G(s) = \zeta(2s) \prod_p \left( 1 - \frac{1}{2p^{2s}} + \frac{1}{2p^{3s}} \right).$$

Proof. The arithmetic function $n \mapsto \frac{\chi(n)D(n)}{n^s}$ ($n \in \mathbb{N}^*$) is multiplicative and, for all prime numbers $p$ and $m \in \mathbb{N}$, we have

$$\frac{\chi(p^m)D(p^m)}{p^{ms}} = \begin{cases} 1, & \text{if } m = 0 \\ 0, & \text{if } m \geq 1 \text{ and } p \text{ divides } k \\ \frac{\chi(p)}{p^s}, & \text{if } m = 1 \text{ and } (p, k) = 1 \\ \frac{(m+1)(\chi(p))^m}{2p^{ms}}, & \text{if } m \geq 2 \text{ and } (p, k) = 1. \end{cases}$$

Then in the half-plane $\Re(s) > 1$, the Euler product formula gives

$$F(s, \chi) = \prod_p \left( 1 + \frac{\chi(p)}{p^s} + \sum_{n=2}^{+\infty} \frac{(n+1)(\chi(p))^n}{2p^{ns}} \right) \prod_p \left( 1 - \frac{\chi(p)}{p^s} \right),$$

$$= \prod_p \left( 1 + \frac{\chi(p)}{p^s} + \sum_{n=2}^{+\infty} \frac{(n+1)(\chi(p))^n}{2p^{ns}} \right) \prod_p \left( 1 - \frac{\chi(p)}{p^s} \right).$$
Evaluating the product of the numerator of the last expression gives
\[
\prod_p \left( 1 + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi(p)^n}{p^{ns}} \right) = G(s, \chi),
\]
and we know that
\[
\prod_p \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1} = L(s, \chi).
\]
So we have
\[
F(s, \chi) = L(s, \chi) \prod_p \left( 1 + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi(p)^n}{p^{ns}} \right).
\]
For \( \chi = \chi_0 \) we have
\[
F(s, \chi_0) = L(s, \chi_0) \prod_p \left( 1 + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi_0(p)^n}{p^{ns}} \right).
\]
The expression of \( F(s) \) announced then comes from Lemma 3.3 and the two following equalities:
\[
L(s, \chi_0) = \zeta(s) \prod_{p|k} \left( 1 - \frac{1}{p^n} \right),
\]
and
\[
\prod_p \left( 1 + \frac{1}{2} \sum_{n=2}^{\infty} \frac{\chi_0(p)^n}{p^{ns}} \right) = \prod_p \left( 1 + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{p^{ns}} \right) \prod_{p|k} \left( 1 + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{p^{ns}} \right)^{-1}.
\]
This completes the proof.

\[\square\]

**Lemma 3.6** ([3]). Let \( \delta(t) = 1 - \frac{\ln(\ln |t|)}{\ln |t|} \), where \( |t| \geq T_0 \geq 268 \) and \( 0 < \theta < 1 \). For any complex number \( s = \delta + it \) and \( \delta \geq \delta(t) \), we have

a) \( G(s) = \mathcal{O}(1) \), where \( G(s) = \zeta(2s) \prod_p \left( 1 - \frac{1}{2p^{2s}} + \frac{1}{2p^{3s}} \right) \)

b) \( \zeta(\delta + it) = \mathcal{O} \left( \ln |t|^{1+\theta} \right) \).

**Proof of Theorem 1.1.** We consider the generating function \( Q(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \) (\( \Re(s) > 1 \)). We have
\[
Q(s) = F(s) + K(s),
\]
where
\[
F(s) = F(s, \chi_0) = \vartheta(s) \zeta(s) G(s) \left( \vartheta(s) = \prod_{p|k} \left( 1 - \frac{1}{p^n} \right) \left( 1 + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{p^{ns}} \right)^{-1} \right),
\]
and
\[
K(s) = \sum_{\chi \neq \chi_0} \chi(l) F(s, \chi)
\]
\([F(s, \chi) from Lemma 3.5.].\)
By Lemma 3.5, the function \( s \mapsto Q(s) \) is meromorphic in the half-plane \( \Re(s) > \frac{1}{2} \) and except the simple pole at the point \( s = 1 \), where the residue

\[
c(k) = \zeta(2) \prod_p \left( 1 - \frac{1}{2p^2} + \frac{1}{2p^3} \right) \prod_{p \mid k} \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{2} \sum_{n=2}^{+\infty} \frac{1}{p^n} \right)^{-1}.
\]

For \( k \geq 2 \), we have

\[
\varphi(k) = k \prod_{p \mid k} \left( 1 - \frac{1}{p} \right).
\]

With this last equality and Lemma 3.4, we can write

\[
c(k) = c\left( \frac{\varphi(k)}{k} \right) \prod_{p \mid k} \left( 1 + \frac{1}{2} \sum_{n=2}^{+\infty} \frac{1}{p^n} \right)^{-1}.
\]

Thus \( Q(s) - \frac{c(k)}{s-1} \) is analytic in the half-plane \( \Re(s) > \frac{1}{2} \). According to Ikehara’s theorem in [6], p. 332, we have

\[
\sum_{n \leq x} a(n) \sim c(k)x \quad (x \to +\infty),
\]

we obtain

\[
S(x; k, l) \sim A(k)x \quad (x \to +\infty) \left( A(k) = \frac{c(k)}{\varphi(k)} \right).
\]

We put

\[
\Phi(x; k, l) = \sum_{n \leq x} a(n), \quad \text{where } (a(n) = \sum_{\chi \in G(k)} \chi(n)D(n) (n \in \mathbb{N}^*)).
\]

Applying the Perron formula (see [2], p. 242), we have for \( b > 1 \) and \( \Re(s) > \frac{1}{2} \)

\[
\int_{1}^{x} \Phi(u; k, l) du = \frac{1}{2\pi i} \lim_{T \to +\infty} \left( \int_{b-iT}^{b+iT} x^{s+1} \frac{\vartheta(s)\zeta(s)G(s)}{s(s+1)} ds + \int_{b-iT}^{b+iT} x^{s+1} \frac{K(s)}{s(s+1)} ds \right) \quad (\ast)
\]

For \( \chi \neq \chi_0 \) we have

\[
K(s) = O(1),
\]

and, for any complex number \( s = \delta + it \) with \( \delta \geq \delta(t) > \frac{1}{2} \), we have \( \vartheta(s) = O(1) \). Thus, from Lemma 3.6, we obtain the estimate

\[
\frac{\vartheta(s)\zeta(s)G(s) + K(s)}{s(s+1)} = O\left( \frac{1}{|t|^{(1-\theta)/2}} \right).
\]

We put

\[
\delta(t) = 1 - \theta \frac{\ln \left( \ln \max (|t|, T_0) \right)}{\ln \max (|t|, T_0)} + it, \quad \text{where } T_0 \geq 268 \text{ and } 0 < \theta < 1.
\]

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For $|t| > T_0$, we move the line of integration from (*) to the closed contour $C_{T_0, \theta} = \bigcup_{i=1}^{i=6} (\gamma_i)$ (see Figure 1), where

\[
(\gamma_1) = [b - iT, b + iT],
\]

\[
(\gamma_2) : \delta \mapsto \gamma_2(t) = \delta + iT \quad (b \geq \delta \geq \delta(T)),
\]

\[
(\gamma_3) : t \mapsto \gamma_3(t) = \delta(t) + it \quad (T \geq t \geq T_0),
\]

\[
(\gamma_4) : t \mapsto \gamma_4(t) = \delta_0 + it \quad (T_0 \geq t \geq -T_0 \text{ and } \delta_0 = \delta(T_0)),
\]

\[
(\gamma_5) : t \mapsto \gamma_5(t) = \delta(t) + it \quad (-T_0 \geq t \geq -T),
\]

\[
(\gamma_6) : \delta \mapsto \gamma_6(t) = \delta - iT \quad (\delta(T) \geq \delta \geq \delta(T)).
\]

Figure 1. The closed contour $C_{T_0, \theta}$

Knowing that the function $K(s)$ is analytic in the half-plane $\Re(s) > \frac{1}{2}$ for every $\chi \neq \chi_0$ modulo $k$, therefore using the Cauchy theorem gives us

\[
\int_{C_{T_0, \theta}} x^{s+1} K(s) \frac{ds}{s(s+1)} = 0.
\]
The residue theorem then gives us,

$$
\int_{1}^{x} \Phi(u; k, l) du - \frac{c(k)x^2}{2} = \frac{1}{2\pi i} \int_{C_{\epsilon, \infty}} x^{s+1} F(s) \frac{ds}{s(s+1)}
$$

$$
= \frac{1}{2\pi} \int_{-T_0}^{T_0} h(\delta_0 + it) dt + \frac{1}{2\pi} \lim_{T \to +\infty} \int_{T_0}^{T} h(\delta(t) + it)(\delta'(t) + i) dt
$$

$$
- \frac{1}{2\pi} \lim_{T \to +\infty} \int_{T_0}^{T} h(\delta(t) - it)(\delta'(t) - i) dt
$$

$$
\left( \delta_0 = 1 - \theta \frac{\ln \ln T_0}{\ln T_0} \right), \text{ where}
$$

$$
h(s) = \frac{F(s)}{s(s+1)} = \frac{\vartheta(s)\zeta(s)G(s)}{s(s+1)},
$$

The integrals on horizontal lines (γ_2) and (γ_6) are zero when T tends to +∞.

Then we have

$$
\left| \int_{1}^{x} \Phi(u; k, l) du - \frac{c(k)x^2}{2} \right| \leq \frac{1}{2\pi} \int_{T_0}^{T_0} h(\delta_0 + it) dt + \frac{1}{\pi} \sqrt{1 + (\delta'(T_0))^2} \int_{T_0}^{+\infty} |h(\delta(t) + it)| dt.
$$

Since $h(s) = \vartheta(s)H(s)$, where $H(s)$ is defined in [3] by $H(s) = \zeta(s)G(s)/s(s+1)$, and $\vartheta(s) = \mathcal{O}(1)$, then

$$
h(s) = \mathcal{O}(H(s)).
$$

Applying results from [3] gives

$$
\Phi(x; k, l) = c(k)x + \mathcal{O}\left( x \exp\left( -\frac{\theta}{2} \sqrt{2\ln x}(\ln \ln x) \right) \right)
$$

and we have the estimation of the theorem, i.e.

$$
S(x; k, l) = A(k)x + \mathcal{O}\left( \frac{x}{\varphi(k)} \exp\left( -\frac{\theta}{2} \sqrt{2\ln x}(\ln \ln x) \right) \right).
$$

The improvement of the last estimation of the remainder term to $\mathcal{O}(\sqrt{x})$ will be one of the main targets of our future research.

References


