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A note on generating primitive Pythagorean triples using matrices

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Abstract: We present matrices that generate families of primitive Pythagorean triples that arise from generalized Fibonacci sequences. We then present similar results for generalized Lucas sequences and primitive Pythagorean triples.
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1 Introduction and preliminaries

A *Pythagorean triple* (PT) is an ordered triple of positive integers, (a, b, c) such that $a^2 + b^2 = c^2$. A PT is called *primitive* provided $\gcd(a, b, c) = 1$. Any primitive PT (PPT) can be written in the form $(s^2 - t^2, 2st, s^2 + t^2)$ for positive integers s and t with $s > t$ and $\gcd(s, t) = 1$ (see, e.g., [2], p. 248). A *Pythagorean triple preserving matrix (PTPM)* is a 3×3 matrix that transforms any given PT into another when the PTs are expressed as column vectors (see, e.g., [4]). The familiar Fibonacci sequence is defined as $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Leyendekkers and Shannon constructed families of PPTs using generalized Fibonacci sequences of the following form:

$$F_{n+1}(b) = F_n(b) + bF_{n-1}(b), n \geq 3, \tag{1}$$

where b ranges over the positive integers, $F_1(b) = 1$ and $F_2(b) = 1$ [3]. For example, $F_n(1)$ is the Fibonacci sequence and $F_n(2)$ is the Jacobsthal sequence. (It should be mentioned that our notation differs slightly from that in [3].) PPTs are constructed using these sequences by choosing $s = F_{n+1}(b)$ and $t = F_n(b)$ when $\gcd(F_{n+1}(b), F_n(b)) = 1$, and by choosing

$$s = \frac{F_{n+1}(b) + F_n(b)}{2},\ t = \frac{F_{n+1}(b) - F_n(b)}{2}, \tag{2}$$

when $\gcd(F_{n+1}(b), F_n(b)) \neq 1$. In other words, PPTs can be generated using consecutive terms of the sequence $F_n(b)$ for some positive integer value of b. Alternatively, the PTs generated by (2) can be obtained by choosing $s = F_{n+1}(b)$ and $t = F_n(b)$ and dividing each term of the resulting non-primitive PT by 2 [3].

It should first be noted that although it is claimed in [3] that every PPT can be generated in this way, there are counterexamples to this claim. For example, consider the PPT $(119, 120, 169)$. 169 has a unique representation as a sum of nonzero integer squares, namely $169 = 12^2 + 5^2$. To generate this PPT, we therefore must find positive integers b and n such that either $F_{n+1}(b) = 12$ and $F_n(b) = 5$ or $F_{n+1}(b) = 17$ and $F_n(b) = 7$.

Since $F_1(b) = 1$ and $F_2(b) = 1$ for all possible values of b, n must be at least 3. If $b > 9$, $F_n(b) > 17$ for all $n \geq 4$. Therefore, our search is restricted to the cases where $b \leq 8$. We may refer to the values in Table 1 and conclude that the necessary generating values do not occur as consecutive terms in any of the $F_n(b)$ sequences.

Table 1. Values of generalized Fibonacci sequences

Sequence	**Sequence values less than 20**
$F_n(1)$	$1, 1, 2, 3, 5, 8, 13$
$F_n(2)$	$1, 1, 3, 5, 11$
$F_n(3)$	$1, 1, 4, 7, 19$
$F_n(4)$	$1, 1, 5, 9$
$F_n(5)$	$1, 1, 6, 11$
$F_n(6)$	$1, 1, 7, 13$
$F_n(7)$	$1, 1, 8, 15$
$F_n(8)$	$1, 1, 9, 17$
$F_n(9)$	$1, 1, 10, 19$

Similarly, the PPT $(95, 168, 193)$ requires generating values of $F_{n+1}(b) = 12$ and $F_n(b) = 7$ or $F_{n+1}(b) = 19$ and $F_n(b) = 5$. Table 1 once again shows that we cannot find the necessary generating values as consecutive terms in any of the $F_n(b)$ sequences.

Although some PPTs are missing from them, the families of PPTs generated by generalized Fibonacci sequences are nevertheless noteworthy. We will generate these families using PTPMs in the next section.

2 Generating the families of PPTs

The following matrix is a PTPM and, in particular, transforms any PT generated by $F_{n+1}(b)$ and $F_n(b)$ to the PT generated by $s = F_{n+2}(b)$ and $t = F_{n+1}(b)$ (proved in [1]):

$$M_b = \begin{bmatrix} \frac{-b^2}{2} & b & \frac{b^2}{2} \\ 1 & b & 1 \\ 1 - \frac{b^2}{2} & b & 1 + \frac{b^2}{2} \end{bmatrix}. \tag{3}$$

For example, M_1 transforms a PT generated by consecutive Fibonacci numbers to the next such PT and, therefore, each such PT can be obtained using powers of M_1. For readability, we have written the PTs as ordered triples in Table 2 rather than column vectors.

Table 2. PTs generated by consecutive Fibonacci numbers

PT	Generating values	Matrix computation
$(3, 4, 5)$	$s = 3, t = 2$	$M_1[\,0\ 2\ 2\,]^T$
$(5, 12, 13)$	$s = 3, t = 2$	$M_1^2[\,0\ 2\ 2\,]^T$
$(16, 30, 34)$	$s = 5, t = 3$	$M_1^3[\,0\ 2\ 2\,]^T$
$(39, 80, 89)$	$s = 8, t = 5$	$M_1^4[\,0\ 2\ 2\,]^T$
$(105, 208, 233)$	$s = 13, t = 8$	$M_1^5[\,0\ 2\ 2\,]^T$
$(272, 546, 610)$	$s = 21, t = 13$	$M_1^6[\,0\ 2\ 2\,]^T$

The vector $\begin{bmatrix}0 & 2 & 2\end{bmatrix}^T$ appears in the matrix computations because $(0, 2, 2)$ is the ordered triple generated by $s = 1$ and $t = 1$. The triple $(0, 2, 2)$ is not a PT, but is generated by the first two (nonzero) terms of the Fibonacci sequence. As illustrated in Table 2, the non-primitive PTs appear when the power of M_1 is a multiple of 3. Therefore, every PPT generated by consecutive Fibonacci numbers is of one of these two forms:

$$\frac{1}{2}M_1^k \begin{bmatrix}0 & 2 & 2\end{bmatrix}^T, \quad k \equiv 0 \bmod 3,$$
$$M_1^k \begin{bmatrix}0 & 2 & 2\end{bmatrix}^T, \quad k \not\equiv 0 \bmod 3.$$

As another example, we list several PTs generated when $b = 2$ (by consecutive Jacobsthal numbers) in Table 3.

Table 3. PTs generated by consecutive Jacobsthal numbers

PT	Generating values	Matrix Computation
$(8, 6, 10)$	$s = 3, t = 1$	$M_2[\,0\ 2\ 2\,]^T$
$(16, 30, 34)$	$s = 5, t = 3$	$M_2^2[\,0\ 2\ 2\,]^T$
$(96, 110, 146)$	$s = 11, t = 5$	$M_2^3[\,0\ 2\ 2\,]^T$
$(320, 462, 562)$	$s = 21, t = 11$	$M_2^4[\,0\ 2\ 2\,]^T$

Since every term of the Jacobsthal sequence is odd, consecutive terms cannot generate a PPT and we must divide each term of the resulting PT by 2. Therefore, every PPT generated in this case is of the form $\frac{1}{2}M_2^k\begin{bmatrix}0 & 2 & 2\end{bmatrix}^T$. $(0,2,2)$ is once again the ordered triple generated by the first two (nonzero) terms of the sequence.

The parity of the terms in $F_n(b)$ for other values of b follows the same pattern as the Fibonacci sequence when b is odd and the Jacobsthal sequence when b is even. This means that every PPT generated by consecutive terms of the $F_n(b)$ sequences can be expressed using one of the following forms:

$$\begin{aligned}
\frac{1}{2}M_b^k\begin{bmatrix}0 & 2 & 2\end{bmatrix}^T&, \text{ where } b \text{ is odd and } k \equiv 0 \bmod 3,\\
\frac{1}{2}M_b^k\begin{bmatrix}0 & 2 & 2\end{bmatrix}^T&, \text{ where } b \text{ is even}\\
M_b^k\begin{bmatrix}0 & 2 & 2\end{bmatrix}^T&, \text{ where } b \text{ is odd and } k \not\equiv 0 \bmod 3.
\end{aligned}$$

3 Generalized Lucas sequences

We may also consider generalized Lucas sequences that obey the same recurrence relation as $F_n(b)$, namely

$$L_{n+1}(b) = L_n(b) + bL_{n-1}(b),$$

but $L_1(b) = 1$ and $L_2(b) = 2b+1$, [3]. The M_b matrices also transform a PT generated by $L_{n+1}(b)$ and $L_n(b)$ to the PT generated by $L_{n+2}(b)$ and $L_{n+1}(b)$. In other words, if we do not change the recurrence relation, M_b will transform PTs in the same way regardless of the two initial terms of the sequence [1]. As before, we obtain several families of PPTs. We need only replace $\begin{bmatrix}0 & 2 & 2\end{bmatrix}^T$ with the ordered (non-PT) triple generated by $L_2(b) = 2b+1$ and $L_1(b) = 1$:

$$\begin{aligned}
\frac{1}{2}M_b^k\begin{bmatrix}4b(b+1) & 2(2b+1) & 4b(b+1)+2\end{bmatrix}^T&, \text{ where } b \text{ is odd and } k \equiv 0 \bmod 3,\\
\frac{1}{2}M_b^k\begin{bmatrix}4b(b+1) & 2(2b+1) & 4b(b+1)+2\end{bmatrix}^T&, \text{ where } b \text{ is even},\\
M_b^k\begin{bmatrix}4b(b+1) & 2(2b+1) & 4b(b+1)+2\end{bmatrix}^T&, \text{ where } b \text{ is odd and } k \not\equiv 0 \bmod 3.
\end{aligned}$$

As with the families of PPTs generated by the $F_n(b)$ sequences, certain PPTs cannot be expressed in this form. The reader may verify that while $(119, 120, 169)$ can be generated in this way (setting $b = 2$ and $k = 2$), $(95, 168, 193)$ once again cannot.

4 Conclusion

In this paper, we have shown how to generate primitive Pythagorean triples arising from consecutive terms of generalized Fibonacci and generalized Lucas sequences using Pythagorean triple preserving matrices. We have also shown that some primitive Pythagorean triples cannot be generated using consecutive terms of such sequences.

There are several avenues of further research the interested reader may wish to pursue. First, the PPTs that cannot be generated using generalized Fibonacci or generalized Lucas sequences have yet to be enumerated. It may be helpful to find a closed form for M_b^k in the pursuit of this aim. Second, the matrix forms of the PTs discussed here may help one catalog which PPTs appear in multiple families. For example, $(8, 15, 17)$ is generated by both the Fibonacci and Jacobsthal sequences. Finally, since each matrix M_b transforms any PT to another, the reader may wish to explore how the M_b-matrices transform other collections of PTs: for example, PTs with a leg difference of 1.

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