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Hyperbolic Horadam hybrid functions

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Abstract: The aim of this paper is to introduce the hybrid form of the hyperbolic Horadam function and to investigate some of its properties such as the generating function. Another aim is to define hyperbolic Horadam hybrid sine and cosine functions and their symmetrical forms. For newly defined functions, some properties such as the recursive relations, derivatives, Cassini and De Moivre type identities are examined. In addition, the quasi-sine Horadam hybrid function and three-dimensional Horadam hybrid spiral are defined.
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1 Introduction

Number sequences are among the significant subjects of literature on mathematics and many other sciences [4, 5, 8, 12, 29, 30]. One of these number sequences is the Horadam number sequence which attracts the attention of researchers because it can be reduced to many famous sequences. The Horadam number sequence $W_n(a, b; p, q)$ $(n \geq 0)$ is defined by the recursive relation for fixed $a = W_0, b = W_1$ and nonzero real numbers p and q [7]

$$W_{n+1} = pW_n + qW_{n-1}. \tag{1}$$

The characteristic equation of the Horadam number sequence W_n is

$$t^2 - pt - q = 0, \tag{2}$$

and the roots of equation (2) are [7]

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2} \quad \text{and} \quad \beta = \frac{p - \sqrt{p^2 + 4q}}{2}. \tag{3}$$

The α and β satisfy the equalities [3]:

$$\alpha^2 = p\alpha + q, \qquad \beta^2 = p\beta + q, \tag{4}$$

$$2q + \alpha^2 + q^2\alpha^{-2} = p^2 + 4q, \qquad 1 - \frac{p}{\alpha} = \frac{q}{\alpha^2}. \tag{5}$$

The Binet formula for the Horadam number sequence is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{6}$$

where $A = b - a\beta$ and $B = b - a\alpha$ [7]. Considering the equality $\alpha\beta = -q$, the Binet formula in equation (6) is rewritten as [3]

$$W_n = \frac{A\alpha^n - (-1)^n Bq^n\alpha^{-n}}{\alpha - \beta} = \begin{cases} \dfrac{A\alpha^n + Bq^n\alpha^{-n}}{\alpha - \beta}, & n \quad \text{is odd} \\ \dfrac{A\alpha^n - Bq^n\alpha^{-n}}{\alpha - \beta}, & n \quad \text{is even.} \end{cases} \tag{7}$$

The generalized Fibonacci and Lucas number sequences are some of the famous number sequences which can be reduced from the Horadam number sequence W_n as [3]

$$U_n = W_n(0, 1; p, q) = \begin{cases} \dfrac{\alpha^n + q^n\alpha^{-n}}{\sqrt{p^2 + 4q}}, & n \quad \text{is odd} \\ \dfrac{\alpha^n - q^n\alpha^{-n}}{\sqrt{p^2 + 4q}}, & n \quad \text{is even} \end{cases} \tag{8}$$

and

$$V_n = W_n(2, p; p, q) = \begin{cases} \alpha^n + q^n\alpha^{-n}, & n \quad \text{is odd} \\ \alpha^n - q^n\alpha^{-n}, & n \quad \text{is even.} \end{cases} \tag{9}$$

Hybrid numbers have recently become an increasingly significant subject in the discipline of mathematics [1, 2, 9, 10, 14, 16, 23–28]. The hybrid number system which is a generalization of complex, hyperbolic and dual numbers was introduced by Özdemir [13] as

$$\mathbb{K} = \{a + bi + c\epsilon + dh : a, b, c, d \in \mathbb{R}, i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = hi = \epsilon + i\}. \tag{10}$$

Szynal-Liana [23] described the n-th Horadam hybrid number H_n, whose components are from Horadam numbers as

$$H_n = W_n + iW_{n+1} + \epsilon W_{n+2} + hW_{n+3}. \tag{11}$$

The Binet formula for H_n is

$$H_n = \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha - \beta}, \tag{12}$$

where $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 + h\alpha^3$ and $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$ [23]. Considering equation (7), the Binet formula in equation (12) can be rewritten as

$$H_n = \frac{A\underline{\alpha}\alpha^n - (-1)^n B\underline{\beta}q^n\alpha^{-n}}{\alpha - \beta} = \begin{cases} \dfrac{A\underline{\alpha}\alpha^n + B\underline{\beta}q^n\alpha^{-n}}{\alpha - \beta}, & n \quad \text{is odd} \\ \dfrac{A\underline{\alpha}\alpha^n - B\underline{\beta}q^n\alpha^{-n}}{\alpha - \beta}, & n \quad \text{is even.} \end{cases} \tag{13}$$

Şentürk et al. [16] investigated the exponentiel generating function, Poisson generating function, Vajda, Catalan, Cassini, and d'Ocagne identities for the Horadam hybrid numbers. Kilic [9] introduced k-Horadam hybrid numbers with the help of the k-Horadam numbers, and presented some of their properties. The author investigated some applications in matrices related to the k-Horadam hybrid numbers. Kızılateş [10] defined the Horadam hybrid polynomials and obtained some of their special cases. Also some of their properties such as the recurrence relation, generating function, Binet formula, Catalan, Cassini and d'Ocagne identities are examined in [10].

Another subject that attracts the attention of researchers in many branches of science is hyperbolic functions. It is clear that the new theory of hyperbolic functions will bring important new results to mathematics, physics, and other sciences. The classical hyperbolic functions are

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \tag{14}$$

where $x \in \mathbb{R}$. We encounter many generalizations of the hyperbolic functions [6, 11, 15, 18–22]. Pandir and Ulusoy [15] defined generalized hyperbolic functions by the formulas

$$\sinh_a(\xi) = \frac{sa^{k\xi} - ta^{-k\xi}}{2} \quad \text{and} \quad \cosh_a(\xi) = \frac{sa^{k\xi} + ta^{-k\xi}}{2}, \tag{15}$$

where s, t, and k are any constants and ξ is a variable quantity. Hyperbolic Fibonacci and Lucas functions are defined by Stakhov and Tkachenko [19], and their symmetrical forms are introduced by Stakhov and Rozin [20]. Koçer et al. [11] described two hyperbolic functions by using the equations (8) and (9) as

$$sU(x) = \frac{\alpha^{2x} - q^{2x}\alpha^{-2x}}{\sqrt{p^2 + 4q}} \quad \text{and} \quad cU(x) = \frac{\alpha^{2x+1} - q^{2x+1}\alpha^{-2x-1}}{\sqrt{p^2 + 4q}}, \tag{16}$$

where $x \in \mathbb{R}$. Furthermore, the authors defined the symmetrical forms of $sU(x)$ and $cU(x)$ by the formulas

$$sUs(x) = \frac{\alpha^x - q^x\alpha^{-x}}{\sqrt{p^2 + 4q}} \quad \text{and} \quad cUs(x) = \frac{\alpha^x - q^x\alpha^{-x}}{\sqrt{p^2 + 4q}}. \tag{17}$$

Considering the similarity between the Binet formulas in equation (7) and generalized hyperbolic functions in equation (15), Bahşi and Solak [3] defined the hyperbolic Horadam functions which are called hyperbolic Horadam sine and cosine functions as

$$sW(x) = \frac{A\alpha^{2x} - Bq^{2x}\alpha^{-2x}}{\sqrt{p^2 + 4q}} \quad \text{and} \quad cW(x) = \frac{A\alpha^{2x+1} - Bq^{2x+1}\alpha^{-2x-1}}{\sqrt{p^2 + 4q}}, \tag{18}$$

where α is as equation (3) and $x \in \mathbb{R}$. They also introduced the symmetrical hyperbolic Horadam sine and cosine functions by the formulas

$$sWs\,(x) = \frac{A\alpha^x - Bq^x\alpha^{-x}}{\sqrt{p^2+4q}} \quad \text{and} \quad cWs\,(x) = \frac{A\alpha^x + Bq^x\alpha^{-x}}{\sqrt{p^2+4q}}. \tag{19}$$

Motivated by the above papers, we first define hyperbolic Horadam hybrid function and examine some of its properties, such as the generating function. Next, we introduce the hyperbolic Horadam hybrid sine and cosine functions and their symmetrical forms. We also investigate some of their properties such as the recursive relations, derivatives, Cassini and De Moivre type identities. Finally, we describe the hybrid forms of the quasi-sine Horadam function and three-dimensional Horadam spiral.

2 Main results

Definition 2.1. *Hyperbolic Horadam hybrid function is defined as*

$$HW\,(x) = \frac{A\underline{\alpha}\alpha^x - (-1)^x\,B\underline{\beta}q^x\alpha^{-x}}{\sqrt{p^2+4q}}, \tag{20}$$

where $x \in \mathbb{R}$, $\underline{\alpha} = 1 + i\alpha + \epsilon\alpha^2 + h\alpha^3$, $\underline{\beta} = 1 + i\beta + \epsilon\beta^2 + h\beta^3$, α *and* β *are as in equation* (3).

Note that, α, β, $\underline{\alpha}$ and $\underline{\beta}$ will be used as in Definition 2.1 through out the paper. If the hyperbolic Horadam function in [3] is expressed as

$$W\,(x) = \frac{A\alpha^x - (-1)^x\,Bq^x\alpha^{-x}}{\sqrt{p^2+4q}}, \tag{21}$$

where $x \in \mathbb{R}$, then it is clear that the following relation is true

$$HW\,(x) = W\,(x) + iW\,(x+1) + \epsilon W\,(x+2) + hW\,(x+3)\,. \tag{22}$$

Moreover, the recurrence relation

$$HW\,(x) = pHW\,(x-1) + qHW\,(x-2) \tag{23}$$

is valid. Hyperbolic Horadam hybrid function is reduced to some functions such as generalized hyperbolic Fibonacci hybrid functions

$$HU\,(x) = \frac{\underline{\alpha}\alpha^x - (-1)^x\,\underline{\beta}q^x\alpha^{-x}}{\sqrt{p^2+4q}} \tag{24}$$

and generalized hyperbolic Lucas hybrid functions

$$HV\,(x) = \underline{\alpha}\alpha^x + (-1)^x\,\underline{\beta}q^x\alpha^{-x}. \tag{25}$$

Theorem 2.1. *The generating function for the hyperbolic Horadam hybrid function is*

$$G\,(t) = \sum_{x=0}^{\infty} HW\,(x)\,t^x = \frac{HW\,(0) + t\,(HW\,(1) - HW\,(0))}{1 - pt - qt^2}. \tag{26}$$

Proof. Considering equation (23), we have

$$\begin{aligned}
\left(1-pt-qt^{2}\right)G\left(t\right) &= \left(1-pt-qt^{2}\right)\left(HW\left(0\right)+HW\left(1\right)t+HW\left(0\right)t^{2}+\ldots\right)\\
&= HW\left(0\right)+HW\left(1\right)t+HW\left(2\right)t^{2}+HW\left(3\right)t^{3}+\ldots\\
&\quad -pHW\left(0\right)t-pHW\left(1\right)t^{2}-pHW\left(2\right)t^{3}-\ldots\\
&\quad -qHW\left(0\right)t^{2}-qHW\left(1\right)t^{3}-qHW\left(2\right)t^{4}-\ldots\\
&= HW\left(0\right)+HW\left(1\right)t+HW\left(2\right)t^{2}+HW\left(3\right)t^{3}+\ldots\\
&\quad -pHW\left(0\right)t-\left(pHW\left(1\right)+qHW\left(0\right)\right)t^{2}\\
&\quad -\left(pHW\left(2\right)+qHW\left(1\right)\right)t^{3}-\ldots\\
&= HW\left(0\right)+HW\left(1\right)t-pHW\left(0\right)t\\
&= HW\left(0\right)+t\left(HW\left(1\right)-pHW\left(0\right)\right).
\end{aligned}$$

□

Definition 2.2. *Hyperbolic Horadam hybrid sine and cosine functions are respectively defined as*

$$sHW\left(x\right)=\frac{A\underline{\alpha}\alpha^{2x}-B\underline{\beta}q^{2x}\alpha^{-2x}}{\sqrt{p^{2}+4q}} \tag{27}$$

and

$$cHW\left(x\right)=\frac{A\underline{\alpha}\alpha^{2x+1}+B\underline{\beta}q^{2x+1}\alpha^{-2x-1}}{\sqrt{p^{2}+4q}}, \tag{28}$$

where $x\in\mathbb{R}$.

Now, we define the symmetrical forms of the functions defined in Definition 2.2.

Definition 2.3. *Symmetrical hyperbolic Horadam hybrid sine and cosine functions are respectively defined as*

$$sHWs\left(x\right)=\frac{A\underline{\alpha}\alpha^{x}-B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^{2}+4q}} \tag{29}$$

and

$$cHWs\left(x\right)=\frac{A\underline{\alpha}\alpha^{x}+B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^{2}+4q}}, \tag{30}$$

where $x\in\mathbb{R}$.

The relation between the hyperbolic Horadam hybrid function and symmetrical hyperbolic Horadam hybrid sine and cosine functions is

$$HW\left(x\right)=\begin{cases} cHWs\left(x\right) & x \text{ is odd}\\ sHWs\left(x\right) & x \text{ is even.}\end{cases}$$

Furthermore, there are the following relations between the symmetrical hyperbolic Horadam sine and cosine functions and their hybrid forms

$$sHWs(x)=sWs(x)+isWs(x+1)+\epsilon sWs(x+2)+hsWs(x+3), \tag{31}$$

$$cHWs(x)=cWs(x)+icWs(x+1)+\epsilon cWs(x+2)+hcWs(x+3). \tag{32}$$

Theorem 2.2. *The recursive relations for the symmetrical hyperbolic Horadam hybrid sine and cosine functions are*

(i) $sHWs\,(x+2) = pcHWs\,(x+1) + qsHWs\,(x)$,

(ii) $cHWs\,(x+2) = psHWs\,(x+1) + qcHWs\,(x)$.

Proof. Considering equations (4) and (5), we have

(i)

$$\begin{aligned} pcHWs\,(x+1) + qsHWs\,(x) &= p\left(\frac{A\underline{\alpha}\alpha^{x+1} + B\underline{\beta}q^{x+1}\alpha^{-x-1}}{\sqrt{p^2+4q}}\right) + q\left(\frac{A\underline{\alpha}\alpha^{x} - B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^2+4q}}\right) \\ &= \frac{A\underline{\alpha}\alpha^{x}\,(p\alpha+q) + B\underline{\beta}q^{x+1}\alpha^{-x}\left(\frac{p}{\alpha}-1\right)}{\sqrt{p^2+4q}} \\ &= \frac{A\underline{\alpha}\alpha^{x+2} - B\underline{\beta}q^{x+2}\alpha^{-x-2}}{\sqrt{p^2+4q}} \\ &= sHWs\,(x+2)\,. \end{aligned}$$

(ii) The proof is similar to the proof of (i). □

Theorem 2.3. *The Cassini identities for the symmetrical hyperbolic Horadam hybrid sine and cosine functions are*

(i) $(sHWs\,(x))^2 - cHWs\,(x+1)\,cHWs\,(x-1) = -AB\underline{\alpha}\underline{\beta}q^{x-1}$,

(ii) $(cHWs\,(x))^2 - sHWs\,(x+1)\,sHWs\,(x-1) = AB\underline{\alpha}\underline{\beta}q^{x-1}$.

Proof. (i)

$$\begin{aligned} &(sHWs\,(x))^2 - cHWs\,(x+1)\,cHWs\,(x-1) \\ &= \frac{\left(A\underline{\alpha}\alpha^{x} - B\underline{\beta}q^{x}\alpha^{-x}\right)^2}{p^2+4q} \\ &\qquad - \frac{\left(A\underline{\alpha}\alpha^{x+1} + B\underline{\beta}q^{x+1}\alpha^{-x-1}\right)\left(A\underline{\alpha}\alpha^{x-1} + B\underline{\beta}q^{x-1}\alpha^{-x+1}\right)}{p^2+4q} \\ &= \frac{-AB\underline{\alpha}\underline{\beta}q^{x-1}\,(2q + q^2\alpha^{-2} + \alpha^2)}{p^2+4q} \\ &= -AB\underline{\alpha}\underline{\beta}q^{x-1}. \end{aligned}$$

(ii) The proof is similar to the proof of (i). □

Theorem 2.4. *The Pythagorean Theorem for the symmetrical hyperbolic Horadam hybrid sine and cosine functions is*

$$(cHWs\,(x))^2 - (sHWs\,(x))^2 = \frac{4ABq^x\underline{\alpha}\underline{\beta}}{p^2+4q}. \tag{33}$$

Proof. The proof is clear from Definition 2.3. □

Theorem 2.5. *The De Moivre type identities for the symmetrical hyperbolic Horadam hybrid sine and cosine functions are*

(i) $(cHWs\,(x) + sHWs\,(x))^n = \left(\frac{2A\underline{\alpha}}{\sqrt{p^+4q}}\right)^{n-1}(cHWs\,(nx) + sHWs\,(nx)),$

(ii) $(cHWs\,(x) - sHWs\,(x))^n = \left(\frac{2B\underline{\beta}}{\sqrt{p^+4q}}\right)^{n-1}(cHWs\,(nx) - sHWs\,(nx)).$

Proof. (i)

$$
\begin{aligned}
&(cHWs\,(x) + sHWs\,(x))^n \\
&= \left(\frac{A\underline{\alpha}\alpha^x + B\underline{\beta}q^x\alpha^{-x}}{\sqrt{p^2+4q}} + \frac{A\underline{\alpha}\alpha^x - B\underline{\beta}q^x\alpha^{-x}}{\sqrt{p^2+4q}}\right)^n \\
&= \left(\frac{2A\underline{\alpha}\alpha^x}{\sqrt{p^2+4q}}\right)^n \\
&= \left(\frac{2A\underline{\alpha}}{\sqrt{p^2+4q}}\right)^{n-1}\left(\frac{2A\underline{\alpha}\alpha^{nx}}{\sqrt{p^2+4q}}\right) \\
&= \left(\frac{2A\underline{\alpha}}{\sqrt{p^2+4q}}\right)^{n-1}\left(\frac{A\underline{\alpha}\alpha^{nx} + B\underline{\beta}q^{nx}\alpha^{-nx}}{\sqrt{p^2+4q}} + \frac{A\underline{\alpha}\alpha^{nx} - B\underline{\beta}q^{nx}\alpha^{-nx}}{\sqrt{p^2+4q}}\right) \\
&= \left(\frac{2A\underline{\alpha}}{\sqrt{p^2+4q}}\right)^{n-1}(cHWs\,(nx) + sHWs\,(nx))\,.
\end{aligned}
$$

(ii) The proof is similar to the proof of (i). □

Theorem 2.6. *The n-th derivatives of the symmetrical hyperbolic Horadam hybrid sine and cosine functions are*

(i) $$(sHWs\,(x))^{(n)} = \begin{cases} (\ln(\alpha))^n\, cHWs\,(x) - \dfrac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^n + (\ln(\alpha))^n}{\sqrt{p^2+4q}}B\underline{\beta}q^x\alpha^{-x}, & n \text{ is odd} \\ (\ln(\alpha))^n\, sHWs\,(x) - \dfrac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^n - (\ln(\alpha))^n}{\sqrt{p^2+4q}}B\underline{\beta}q^x\alpha^{-x}, & n \text{ is even,} \end{cases}$$

(ii) $$(cHWs(x))^{(n)} = \begin{cases} ((\ln(\alpha))^n\, sHWs(x) + \dfrac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^n + (\ln(\alpha))^n}{\sqrt{p^2+4q}}B\underline{\beta}q^x\alpha^{-x}, & n \text{ is odd} \\ ((\ln(\alpha))^n\, cHWs(x) + \dfrac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^n - (\ln(\alpha))^n}{\sqrt{p^2+4q}}B\underline{\beta}q^x\alpha^{-x}, & n \text{ is even.} \end{cases}$$

Proof. (i) We use the induction method on n. Since

$$\begin{aligned}(sHWs(x))' &= \left(\frac{A\underline{\alpha}\alpha^{x} - B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^2+4q}}\right)' \\ &= \frac{A\underline{\alpha}\alpha^{x}\ln(\alpha) - B\underline{\beta}q^{x}\alpha^{-x}(\ln(q)-\ln(\alpha))}{\sqrt{p^2+4q}} \\ &= \ln(\alpha)\left(\frac{A\underline{\alpha}\alpha^{x} + B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^2+4q}}\right) - \frac{B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^2+4q}}\ln(q) \\ &= \ln(\alpha)\, cHWs(x) - \frac{\ln\left(\frac{q}{\alpha}\right)+\ln(\alpha)}{\sqrt{p^2+4q}}B\underline{\beta}q^{x}\alpha^{-x}\end{aligned}$$

and

$$\begin{aligned}(sHWs(x))'' &= \left(\ln(\alpha)\frac{A\underline{\alpha}\alpha^{x} + B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^2+4q}} - \frac{\ln\left(\frac{q}{\alpha}\right)+\ln(\alpha)}{\sqrt{p^2+4q}}B\underline{\beta}q^{x}\alpha^{-x}\right)' \\ &= \ln(\alpha)\frac{A\underline{\alpha}\alpha^{x}\ln(\alpha) + B\underline{\beta}q^{x}\alpha^{-x}(\ln(q)-\ln(\alpha))}{\sqrt{p^2+4q}} \\ &\quad - \frac{\ln\left(\frac{q}{\alpha}\right)+\ln(\alpha)}{\sqrt{p^2+4q}}B\underline{\beta}q^{x}\alpha^{-x}(\ln(q)-\ln(\alpha)) \\ &= (\ln(\alpha))^2\left(\frac{A\underline{\alpha}\alpha^{x} - B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^2+4q}}\right) + \frac{B\underline{\beta}q^{x}\alpha^{-x}(\ln(q)\ln(\alpha))}{\sqrt{p^2+4q}} \\ &\quad - \frac{B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^2+4q}}\left(\ln\left(\frac{q}{\alpha}\right)^2 + \ln(q)\ln(\alpha) - (\ln(\alpha))^2\right) \\ &= (\ln(\alpha))^2\, sHWs(x) - \frac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^2 - (\ln(\alpha))^2}{\sqrt{p^2+4q}}B\underline{\beta}q^{x}\alpha^{-x},\end{aligned}$$

the result is true for $n=1$ and $n=2$. Suppose that k is an odd number and the result is true for $n=k$. For the even number $n=k+1$:

$$
\begin{aligned}
\left((sHWs(x))^{(k)}\right)' &= \left((\ln(\alpha))^k\, cHWs(x) - \frac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^k + (\ln(\alpha))^k}{\sqrt{p^2+4q}} B\underline{\beta} q^x \alpha^{-x} \right)' \\
&= (\ln(\alpha))^k \frac{A\underline{\alpha}\alpha^x \ln(\alpha) + B\underline{\beta} q^x \alpha^{-x} (\ln(q) - \ln(\alpha))}{\sqrt{p^2+4q}} \\
&\quad - \frac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^k + (\ln(\alpha))^k}{\sqrt{p^2+4q}} B\underline{\beta} q^x \alpha^{-x} (\ln(q) - \ln(\alpha)) \\
&= (\ln(\alpha))^{k+1} \frac{A\underline{\alpha}\alpha^x - B\underline{\beta} q^x \alpha^{-x}}{\sqrt{p^2+4q}} \\
&\quad - \frac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^{k+1} - (\ln(\alpha))^{k+1}}{\sqrt{p^2+4q}} B\underline{\beta} q^x \alpha^{-x}.
\end{aligned}
$$

Now, suppose that k is an even number and the result is true for $n = k$. Finally, we must show that the result is true for the odd number $n = k + 1$:

$$
\begin{aligned}
\left((sHWs(x))^{(k)}\right)' &= \left((\ln(\alpha))^k\, sHWs(x) - \frac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^k - (\ln(\alpha))^k}{\sqrt{p^2+4q}} B\underline{\beta} q^x \alpha^{-x} \right)' \\
&= (\ln(\alpha))^k \frac{A\underline{\alpha}\alpha^x \ln(\alpha) - B\underline{\beta} q^x \alpha^{-x} (\ln(q) - \ln(\alpha))}{\sqrt{p^2+4q}} \\
&\quad - \frac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^k - (\ln(\alpha))^k}{\sqrt{p^2+4q}} B\underline{\beta} q^x \alpha^{-x} (\ln(q) - \ln(\alpha)) \\
&= (\ln(\alpha))^{k+1} \frac{A\underline{\alpha}\alpha^x + B\underline{\beta} q^x \alpha^{-x}}{\sqrt{p^2+4q}} \\
&\quad - \frac{\left(\ln\left(\frac{q}{\alpha}\right)\right)^{k+1} + (\ln(\alpha))^{k+1}}{\sqrt{p^2+4q}} B\underline{\beta} q^x \alpha^{-x}.
\end{aligned}
$$

□

3 The quasi-sine Horadam hybrid function and the three-dimensional Horadam hybrid spiral

Bahşi and Solak [3] defined the quasi-sine Horadam function as

$$
WW(x) = \frac{A\alpha^x - \cos(\pi x)\, B q^x \alpha^{-x}}{\sqrt{p^2+4q}}, \tag{34}
$$

for $x \in \mathbb{R}$, by using the equality $(-1)^x = \cos(\pi x)$.

The three-dimensional Horadam spiral is defined by the formula

$$\begin{aligned} CWW(x) &= \frac{A\alpha^x - \cos(\pi x) Bq^x\alpha^{-x}}{\sqrt{p^2+4q}} + i\frac{\sin(\pi x) Bq^x\alpha^{-x}}{\sqrt{p^2+4q}} \\ &= \frac{A\alpha^x + ie^{i\pi\left(\frac{1}{2}-x\right)} Bq^x\alpha^{-x}}{\sqrt{p^2+4q}} \end{aligned}$$

for $x \in \mathbb{R}$ and $i^2 = -1$, [3].

Now, we give the definitions of the hybrid forms of the quasi-sine Horadam function and three-dimensional Horadam spiral, respectively.

Definition 3.1. *The function*

$$HWW(x) = \frac{A\underline{\alpha}\alpha^x - \cos(\pi x) B\underline{\beta}q^x\alpha^{-x}}{\sqrt{p^2+4q}} \tag{35}$$

is called quasi-sine Horadam hybrid function, where $x \in \mathbb{R}$.

Considering equation (13), we note that $HWW(n) = H_n$ for all integer n.

The quasi-sine Horadam hybrid functions have some relations such as

$$HWW(x+2) = pHWW(x+1) + qHWW(x) \tag{36}$$

and

$$(HWW(x))^2 - HWW(x+1)HWW(x-1) = -AB\underline{\alpha}\underline{\beta}q^{x-1}\cos(\pi x). \tag{37}$$

Definition 3.2. *The three-dimensional Horadam hybrid spiral is defined as*

$$\begin{aligned} CHWW(x) &= \frac{A\underline{\alpha}\alpha^x - \cos(\pi x) B\underline{\beta}q^x\alpha^{-x}}{\sqrt{p^2+4q}} + i\frac{\sin(\pi x) B\underline{\beta}q^x\alpha^{-x}}{\sqrt{p^2+4q}} \\ &= \frac{A\underline{\alpha}\alpha^x + ie^{i\pi\left(\frac{1}{2}-x\right)} B\underline{\beta}q^x\alpha^{-x}}{\sqrt{p^2+4q}}, \end{aligned}$$

where $x \in \mathbb{R}$ and $i^2 = -1$.

Note that the real part of the three-dimensional Horadam hybrid spiral equals to quasi-sine Horadam hybrid function.

Theorem 3.1. *The recurrence relation for the three-dimensional Horadam hybrid spiral is*

$$CHWW(x+2) = pCHWW(x+1) + qCHWW(x). \tag{38}$$

Proof.

$$
\begin{aligned}
&pCHWW(x+1)+qCHWW(x)\\
&=p\frac{A\underline{\alpha}\alpha^{x}+ie^{i\pi\left(\frac{1}{2}-x-1\right)}B\underline{\beta}q^{x+1}\alpha^{-x-1}}{\sqrt{p^{2}+4q}}+q\frac{A\underline{\alpha}\alpha^{x}+ie^{i\pi\left(\frac{1}{2}-x\right)}B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^{2}+4q}}\\
&=\frac{A\underline{\alpha}\alpha^{x}(p\alpha+q)+ie^{-i\pi x}B\underline{\beta}q^{x+1}\alpha^{-x-1}\left(pe^{-i\pi\frac{1}{2}}+e^{i\pi\frac{1}{2}}\alpha\right)}{\sqrt{p^{2}+4q}}\\
&=\frac{A\underline{\alpha}\alpha^{x}(p\alpha+q)+ie^{-i\pi x}B\underline{\beta}q^{x+1}\alpha^{-x-1}(\alpha-p)i}{\sqrt{p^{2}+4q}}\\
&=\frac{A\underline{\alpha}\alpha^{x+2}+ie^{-i\pi x-\frac{3\pi}{2}i}B\underline{\beta}q^{x+2}\alpha^{-x-2}}{\sqrt{p^{2}+4q}}\\
&=\frac{A\underline{\alpha}\alpha^{x+2}+ie^{i\pi\left(\frac{1}{2}-x-2\right)}B\underline{\beta}q^{x+2}\alpha^{-x-2}}{\sqrt{p^{2}+4q}}\\
&=CHWW(x+2).
\end{aligned}
$$

□

Additionally, we give the system of equations depend on the three-dimensional Horadam hybrid spiral as

$$y(x)-\frac{A\underline{\alpha}\alpha^{x}}{\sqrt{p^{2}+4q}}=\frac{-\cos(\pi x)B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^{2}+4q}} \tag{39}$$

and

$$z(x)=\frac{\sin(\pi x)B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^{2}+4q}} \tag{40}$$

for the real and imaginary axes Y and Z, respectively. Then,

$$\left(y(x)-\frac{A\underline{\alpha}\alpha^{x}}{\sqrt{p^{2}+4q}}\right)^{2}+z(x)^{2}=\left(\frac{B\underline{\beta}q^{x}\alpha^{-x}}{\sqrt{p^{2}+4q}}\right)^{2}. \tag{41}$$

We note that equation (41) corresponds to the hybrid form of the Metallic Shofar in [17].

4 Conclusion

In this paper, we first defined a special type of hybrid numbers whose components are from the hyperbolic Horadam functions and gave some of its properties. Next, we introduced hyperbolic Horadam hybrid sine and cosine functions and their symmetrical forms. We examined some properties of these functions such as the recursive relations, derivatives, Pythagorean Theorem, Cassini and De Moivre type identities. Additionally, we described the hybrid forms of the quasi-sine Horadam function and three-dimensional Horadam spiral.

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